Planar Inverse Quotients and Planar Metric Spaces

Sylvester Eriksson-Bique

Jyväslylä University
Boseman Summer School

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Overview

1. Daniel: Inverse limits from Dynamic systems, and Browns theorem giving a condition for planarity.
2. Hrant: Inverse limits with Poincaré inequality
3. *Today: How to make inverse limits with Poincaré planar, and relationships. Motivation.
4. *Today: Claytors Theorem on planar metric spaces
5. *Today: Explanation of context/interest of planar setting.

References:

1. Thin Loewner carpets and their quasisymmetric embeddings in S^2, Cheeger, SEB
2. Unpublished work by Kleiner
3. Cheeger, Kleiner: Inverse limit spaces satisfying a Poincare inequality, Analysis and Geometry in Metric Spaces, 2015
4. P. Haissinski: Empilements de cercles et modules combinatoires Ann. Inst. Fourier (Grenoble), (2009) 59(6), p. 2175–2222.
5. S. Claytor: Peanian continua not imbeddable in a spherical surface, Ann. of Math., (1937), p. 631–646.
6. J Heinonen, P Koskela: Quasiconformal maps in metric spaces with controlled geometry, Acta Mathematica, 1998
7. Reinhard Diestel: Graph Theory

Why consider? Sierpiński Carpet

1. Interested in if there is an Ahlfors metric space Y and a Quasisymmetry $f : S_3 \to Y$, so that Y is Loewner.
2. The space Y would be planar, Loewner.
3. Haissinski: Equivalent to attaining the "conformal dimension"
4. Problem: Examples?

Scheme

$$B = B(x, r)$$

$$\int_B |f - f_B| \, d\mu \leq Cr \int_B g \, d\mu$$

1. Ahlfors regular and $(1,1)$-Poincaré sufficient for Loewner. (Heinonen-Koskela: Almost necessary.)
2. Cheeger-Kleiner Inverse Limits can be made Ahlfors regular and with $(1,1)$-Poincaré.
3. **What about planar?**

Cheeger-Kleiner Inverse Limits

$$X_1 \xleftarrow{\pi_1} X_2 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_{n-1}} X_n \xleftarrow{\pi_n} \cdots \xleftarrow{\pi_n^\infty} X_\infty$$

1. **Combinatorial irreducible objects:** Each graph has a measure μ which is a multiple of the Lebesgue measure on each edge. X_i connected graphs equipped with path metric.
2. **Compatibility with metric:** π_i 1-Lipschitz, surjections, isometry on edges
3. **Compatibility with measure:** $\pi_i^*(\mu_{i+1}) = \mu_i$
4. Bounded geometry: X_i each vertex degree $\leq \Delta$ and length $\prod_{n=1}^{i} m_n^{-1}$, $\mu(e) \sim \mu(f)$ for adjacent edges.

Axioms to focus on

1. π_i is open
2. $\mathrm{diam}(\pi_i(x)^{-1}) \leq \theta m^{-(i+1)}$
3. **Balancing:** $\pi_i^*(\mu_{i+1}|_{\mathrm{Str}_v}) = c_i \mu_i|_{\mathrm{Str}_{\pi_i(v)}}$

Two notable examples: Laakso space

Ahlfors regular but not planar $m_i = 2^{-1}$

Two notable examples: Laakso Diamond

Planar but not Ahlfors regular

General way to get Ahlfors regular

1. X_i^j copies of X_i, subdivided by m_{i+1}
2. $X_{i+1} = \bigsqcup_{j=1}^{N_k} X_i^j$ identified at some subdivision points
3. Equally distribute measure on copies.
4. For any edge e_n in X_n we have

$$\mu_n(e_n) = \prod_{k=1}^{n}(N_k)^{-1}\prod_{k=1}^{n}(m_k)^{-1}$$

5. Note

$$\mathrm{Diam}\ (e_n) = \prod_{k=1}^{n}(m_k)^{-1}$$

6. By choosing N_k, m_k constant, or varying within a given range, then we can ensure

$$\mu_n(e_n) \sim \mathrm{Diam}\ (e_n)^Q$$

Example $N_k = 2, m_k = 4$ gives $\mathrm{Diam}\ (e_n) = 2^{-2n}$, and $\mu_n(e_n) = 2^{-3n} = \mathrm{Diam}\ (e_n)^{3/2}$.

Wormhole graph: Captures connectivity and diameter

For each edge $e \in X_i$, define a graph G_e with edges e_j for each copy of X_i in X_{i+1}, $j = 1, dots, N_j$, and edges (e_j, e_k) if some point is identified in the copies.

Basic Lemma: If Wormhole graph is connected and m_j, N_j constant, then the inverse limit space arising will satisfy all properties and the limit is Q-Ahlfors regular and satisfies $1, 1$-Poincaré.

Ahlfors regular and planar problematic

Planar graphs, Kuratowski: A graph is planar if and only if it does not contain the forbidden subgraphs $K_{3,3}$ and K_5 (as a **minor**).

$K_{3,3}$:

K_5

Proof of non-planarity

1. No double edges.
2. If G is a planar graph with drawing, then $v - e + f = 2$.
3. Each edge belongs to exactly two faces, but each face is counted by at least three edges $2e \geq 3f$. So $f \leq \frac{2}{3}e$, and $2 = v - e + f \leq v - \frac{1}{3}e$
4. $e \leq 3v - 6$.
5. In K_5: $3v - 6 = 9$, but $e = 10$
6. In $K_{3,3}$: every face has at least four edges, so $e \leq 2v - 4$, but $12 - 4 < 9$.

Kuratowski Proof

1. Induction on size of graph, K_4 ok.
2. Hardest case 3-connected.
3. Contract an edge xy so that resulting graph is 3-connected. Possible unless K_4.

Ahlfors regular Cheeger-Inverse limits never planar

Consider a degree four vertex v, it will be copied infinitely often, thus eventually there will be an embedded K_5-minor.

Modify by quotienting:Explicit example, Substitution rule

If one quotients the vertex neighborhoods in the vicinity of a vertex. **But, how to prove Poincaré?**

Organizational Structure: Quotiented Inverse limit

$$
\begin{tikzcd}
X_1^1 & X_1^2 \arrow[l, "\pi_1^1"'] \arrow[d, "q_2^2"] & \cdots \arrow[l, "\pi_1^2"'] & X_1^n \arrow[l, "\pi_1^{n-1}"'] \arrow[d, "q_2^n"] & \cdots \arrow[l, "\pi_1^n"'] & X_1^\infty \arrow[l] \arrow[d, "q_2^\infty"] \\
 & X_2^2 & \cdots \arrow[l, "\pi_2^2"'] & X_2^n \arrow[l, "\pi_2^{n-1}"'] \arrow[d, "q_3^n"] & \cdots \arrow[l, "\pi_2^n"'] & X_2^\infty \arrow[l] \arrow[d, "q_3^\infty"] \\
 & & \vdots & \ddots \arrow[d, "q_4^n"] & \vdots & \vdots \arrow[d, "q_n^\infty"] \\
 & & & X_n^n & \cdots \arrow[l, "\pi_n^n"'] & X_n^\infty \arrow[l] \arrow[d, "q_{n+1}^\infty"] \\
 & & & & \ddots & \vdots \arrow[d] \\
 & & & & & X_\infty^\infty
\end{tikzcd}
$$

Axioms

1. Edge lengths of X_m^n, $s_n = \prod_{k=1}^n$
2. Rows, inverse limits
3. q_i^j preserve measure $(q_i^j)^*(\mu_{i-1}^j) = \mu_i^j$
4. q_j^i isometries on edges, need not be open
5. There are maps $h_i^j : X_i^j \to \frac{1}{2\pi} S^1$ that commute with the diagram.
6. **Doesn't distort too much:** $q_{lk}^i = q_k^i \circ \cdots \circ q_{l+1}^i$

$$(q_{lj}^i)^{-1}(B(x, \delta s_l)) \leq C s_l$$

Proof of Doubling/Ahlfors Regularity

Sufficient to prove uniform bounds for the graphs, and then Measure preserving + pull to an appropriate scale. Choose l so that $r \sim s_l$

$$B(y, Cs_l) \subset (q^i_{lj})^{-1}(B(x, r)) \subset B(y, Cs_l).$$

Poincaré inequality

Sufficient to find c so that for say edgewise linear functions

$$\int_B |f - f_B| d\mu \lesssim \int_B |f - c| d\mu \leq Cr \int_{CB} |\nabla f| \, d\mu.$$

1. By Cheeger-Kleiner + doubling, we can get the inequality on X_i^n at scales comparable to s_i.
2. So, question about bigger scales.
3. Pull back.

Verifying Quotient condition: in specific example

1. Fix x and $B(x, \delta s_l)$ in X_k^n.
2. Push to X_k^k.
3. Lift along diagonal.
4. Lifting a connected set A from X_n^n to X_{n-1}^{n-1} increases diameter of $h(A)$ by at most $s_{n-1}/4$.
5. Diameter increases to at most $2\delta s_l + s_l(1/4 + 1/4^2 \cdots) < s_l$. Only one vertex!!

Why limit Planar?

Every X_n^n is planar. X_∞^∞ is Gromov-Hausdorff limit of these. Is it planar?

Not automatically!

Planarity theorem

Theorem

Suppose X_n are a sequence of planar graphs without cut points which converge to an X_∞ which has no cut points. Then X_∞ is planar.

Proof, due to Kleiner

Theorem

(Claytor) If X is a locally connected continuum (Peanian continuum) without cut points, then it is planar if and only if it does not contain any embedded copy of $K_{3,3}$ or K_5.

Proof involved. Somewhat cleaner proof in Moise, E. E., Remarks on the Claytor embedding theorem, Duke Math. Journal, vol. 19 (1952) p. 199-202. Uses Brick partitions, approximation by graphs and Moore's theorem.

Proof of planarity theorem

1. Suppose X_∞ not planar, i.e. has an embedded copy of $K_{3,3}$ or K_5. Lift to X_n.
2. Lift approximately to X_n
3. Connect locally.
4. Quotient to form a forbidden minor.

Further comments

1. Planar embedding can be chosen to be a quasisymmetry.
2. There are uniquely defined peripheral components, which are uniformly relatively separates and uniform quasicircles (for us, bi-Lipschitz to circles).
3. The second is necessary, and can be used to prove general embedding results.

If time, No cut points and Loewner and Poincaré

1. Tools: Modulus, Chained balls

Theorem

If Q-Ahlfors regular X satisfies a $(1,1)$-Poincaré inequality, then it is Loewner.

Theorem

If X is Q-Loewner then it is locally connected and does not have local cut points.

Loewner, Poincaré and cut points

Γ a family of curves, often $\Gamma(E, F)$ connecting curves from E to F.
$\mathrm{Mod}_Q(\Gamma)$ defined as:

Definition Loewner: X is Q-AR and Q-Loewner: For all connected, non-degenerate, disjoint continua E, F
$\Delta(E, F) = \frac{d(E,F)}{\min\{\mathrm{~Diam~}(E), \mathrm{~Diam~}(F)\}}$

No cut points.

Theorem

If X is Q-Loewner then it is locally connected and does not have local cut points.

1. Modulus through a point null: $\rho = \frac{1}{nd(p,\cdot)} 1_{B(x,1)\setminus B(x,2^{-n})}$.

Poincaré inequality

X satisfies a $(1,1)$-Poincaré inequality if for every $f : X \to \mathbb{R}$ Lipschitz and with ρ an upper gradient

$$\forall\gamma : |f(\gamma(0)) - f(\gamma(1))| \leq \int_\gamma \rho \, ds.$$

Heinonen-Koskela: $(1,1)$-PI implies Loewner

Let ρ be admissible for $\mathrm{Mod}_Q(E,F)$.
Define $u(x) = \inf_\gamma \int_\gamma \rho$. Upper gradient bound:

Heinonen-Koskela: $(1,1)$-PI implies Loewner

Let ρ be admissible for $\mathrm{Mod}_Q(E,F)$. Consider first only E,F at unit scale (an iteration for other sets)

1. Define $u(x) = \inf_\gamma \int_\gamma \rho$.
2. Construct cover of E or F Depending on average.
3. $\int_{B_i} g^Q \, d\mu \geq r_i^{Q(\alpha-1)+Q}$
4. $\alpha = \frac{1}{Q}$