# Inverse limit spaces I

Introduction, basic properties, inverse limit map.

## Basic definitions

Let $\{X_j\}_{j\in\mathbb{N}}$ be a sequence of top. spaces.

$\{f_j\}_{j\in\mathbb{N}}$ sequence of maps

$f_j : X_{j+1} \to X_j$ cont., surjective.

Then $(X_j, f_j)_{j\in\mathbb{N}}$ is an inverse limit sequence.

$f_j$ are called the bonding maps.

$$\dots \longrightarrow X_3 \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0$$

$\prod_i X_j = \dots \times X_2 \times X_1 \times X_0$

product space, with product topology.

The inverse limit space of $(X_j, f_j)_{j\in\mathbb{N}}$ is $\varprojlim (X_j, f_j) := \{ x = (\dots, x_2, x_1, x_0) \in \prod X_j \mid f_{j+1}(x_{j+1}) = x_j \ \forall j \in \mathbb{N} \}$.

$$\varprojlim (X_j, f_j) = \{ x = (\ldots, x_2, x_1, x_0) \in \prod X_j \mid f_{j+1}(x_{j+1}) = x_j \ \forall j \in \mathbb{N} \}.$$

$$\begin{array}{ccccccccc} \cdots \longrightarrow & x_3 & \xrightarrow{f_2} & x_2 & \xrightarrow{f_1} & x_1 & \xrightarrow{f_0} & x_0 \\ & \in X_3 & & \in X_2 & & \in X_1 & & \in X_0 \end{array}$$

$\varprojlim (X_j, f_j) \subset \prod X_j$
inherits product topology

Often we have more restrictions on $(X_j, f_j)_{j \in \mathbb{N}}$:

- $X_j$ is a compact metric space $\forall j$
- $X_j = X$, $f_j = f$ $\forall j$
  (spaces / maps are all the same)

then we are in the realm of dynamics

$x = (\ldots, x_2, x_1, x_0)$

$$\cdots \longrightarrow x_3 \xrightarrow{f} x_2 \xrightarrow{f} x_1 \xrightarrow{f} x_0$$

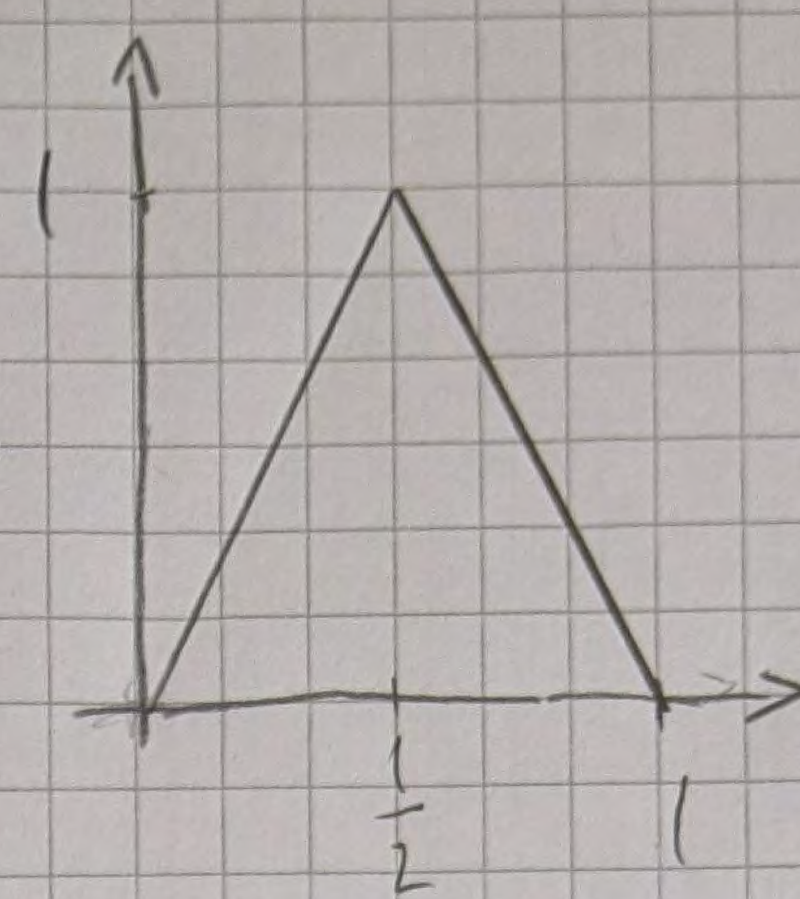
Each point $x \in \varprojlim (X, f)$ represents a point $x_0 \in X$ with its past.

## Examples:

$X_j = X = [0,1]$

$f_j = f$ = tent map

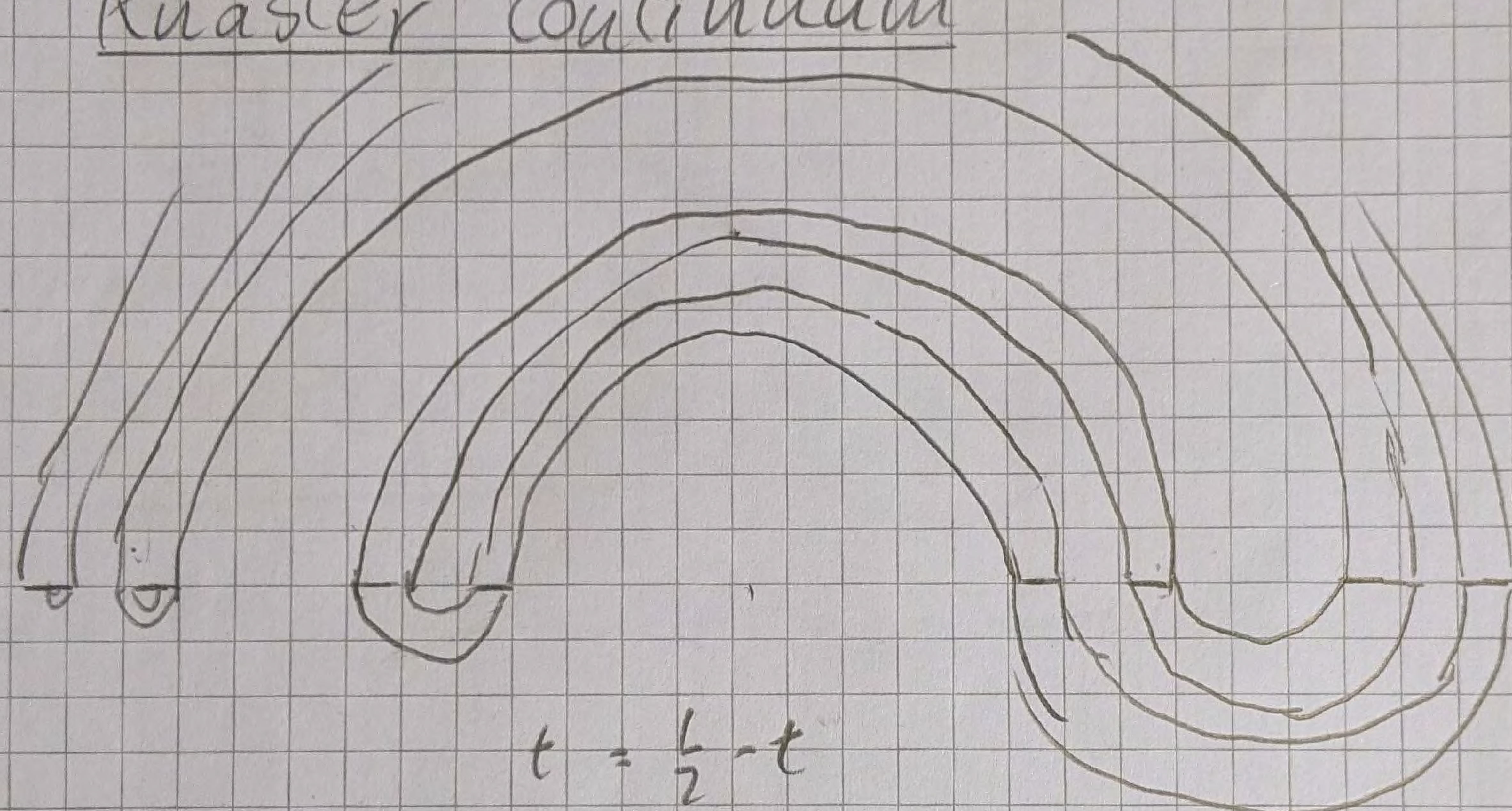
$$f(t) = \begin{cases} 2t, & 0 \le t \le \frac{1}{2} \\ 2-2t, & \frac{1}{2} \le t \le 1 \end{cases}$$




$\varprojlim (X_j, f_j)$ homeomorphic to

Knaster continuum

$K$

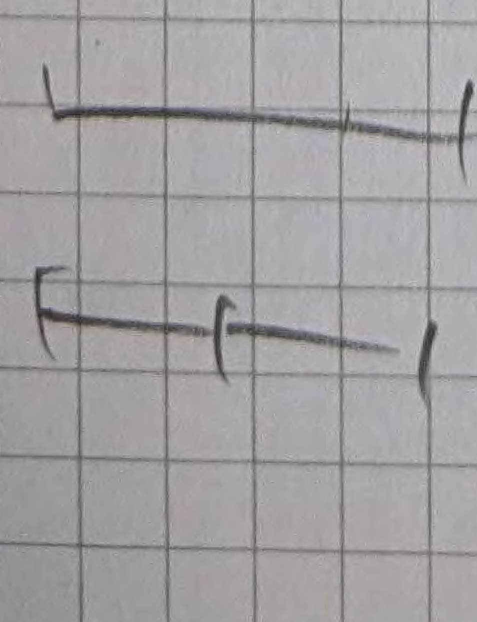


$t = \frac{1}{2} - t$

Indecomposable continuum

$\underline{K} = \underline{A} \cup \underline{B}$, $\underline{A}$, $\underline{B}$ continua

$\Rightarrow$ $A = \underline{K}$ or $B = \underline{K}$.



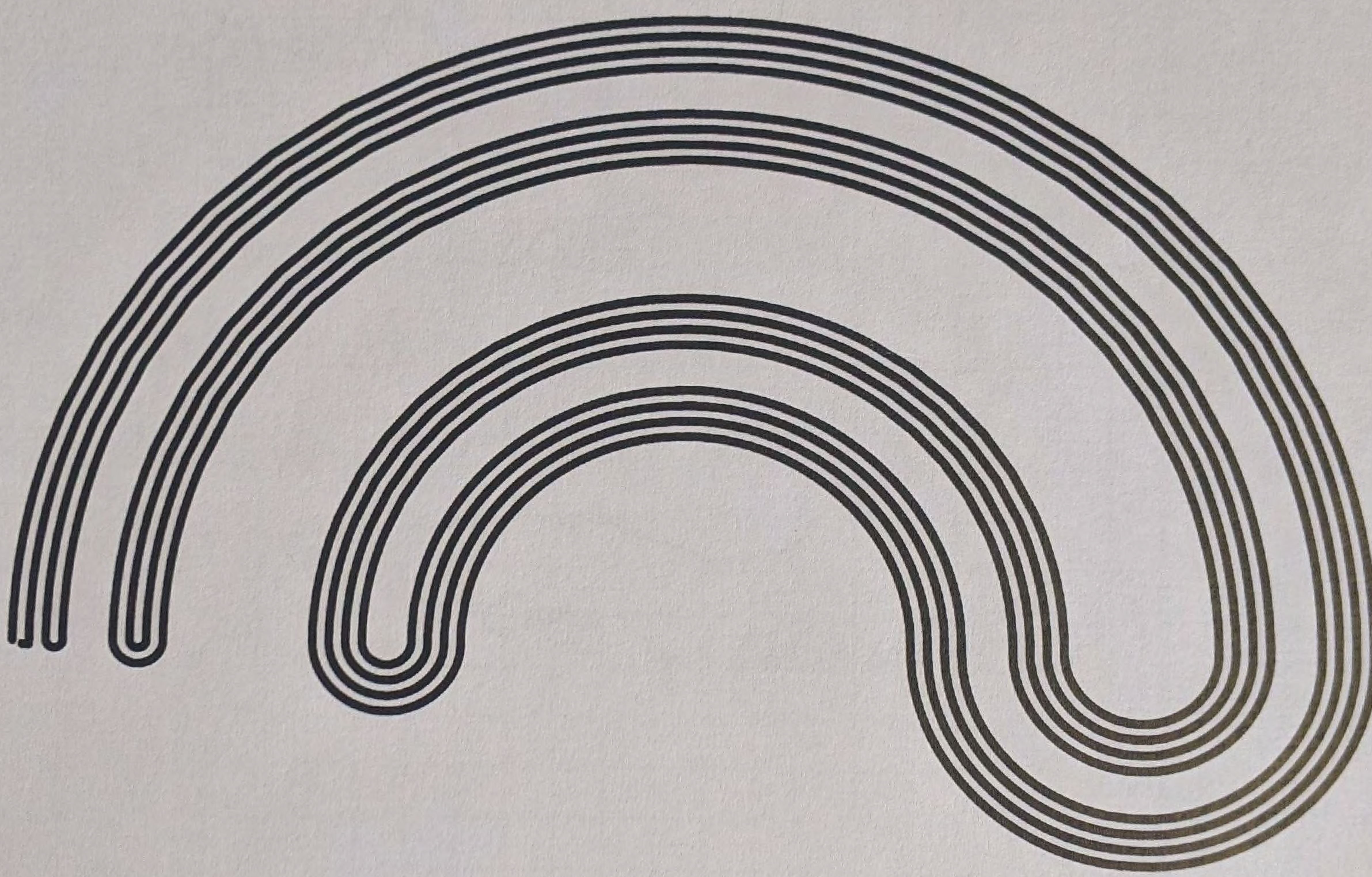


**Fig. 1.5** *A partial picture of the continuum from Example 22*

Example: the solenoid

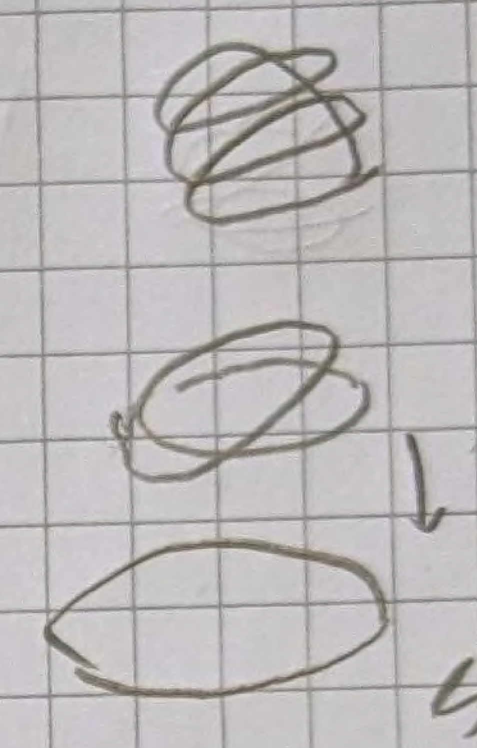
Let $X_j = S^1 = \partial \mathbb{D} \quad \forall j$

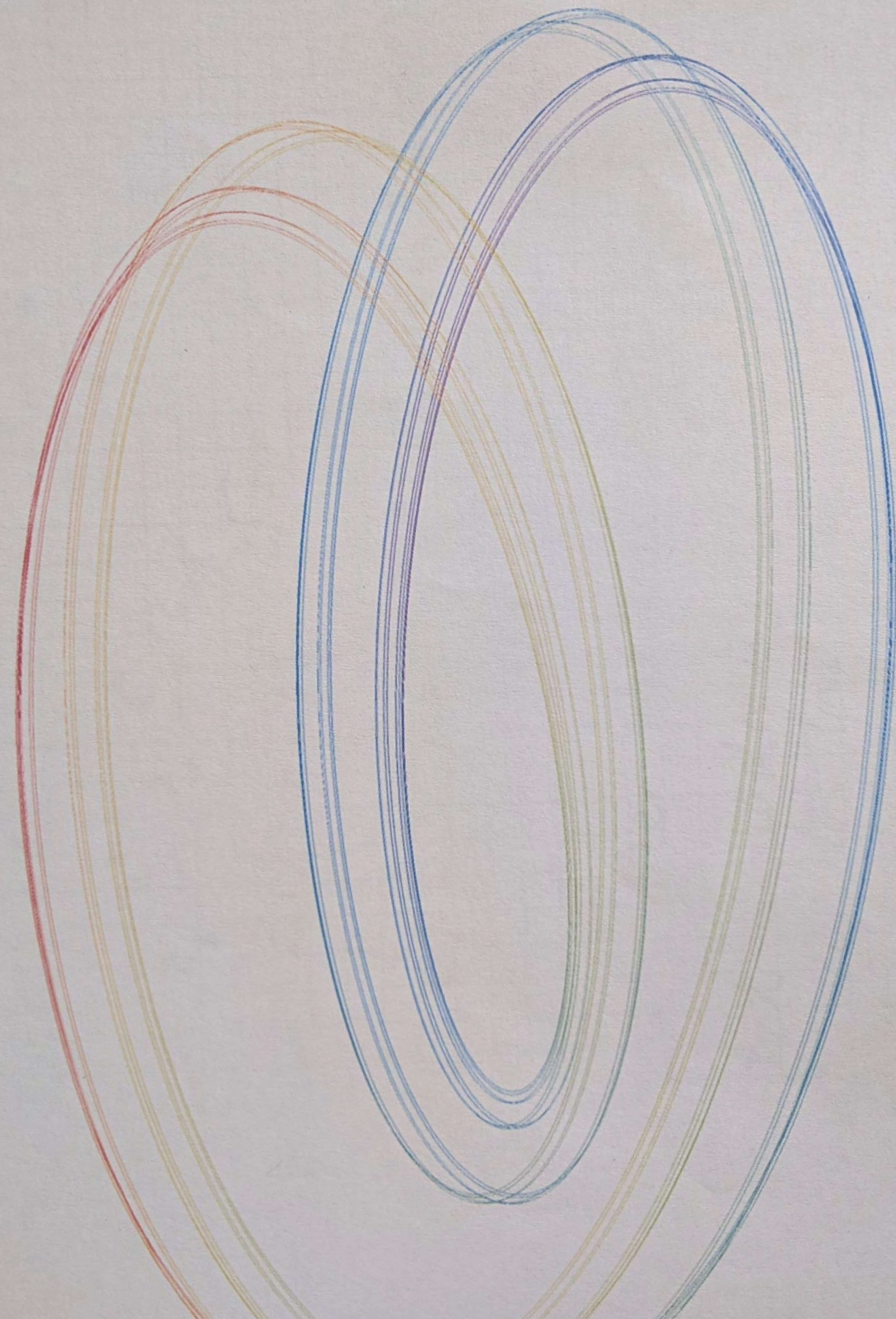
$f_j(z) = z^2 \quad \forall j$

$\varprojlim (S^1, z^2)$ solenoid

indecomposable
topological group
($\Rightarrow$ homogeneous)
each non-degenerate
subcontinuum is an arc

$\iff$ p-solenoid





<u>Example</u>: limits of Riemann surfaces

$$\{(z_1, z_0) \in \hat{\mathbb{C}}^2 \mid z_1^2 = z_0\}$$

Riemann surface of $\sqrt{z}$.

$$z_1 \xrightarrow{z^2} z_0$$

$$\{(z_2, z_1, z_0) \in \hat{\mathbb{C}}^3 \mid z_2^2 = z_1 ,\; z_1^2 = z_0\}$$

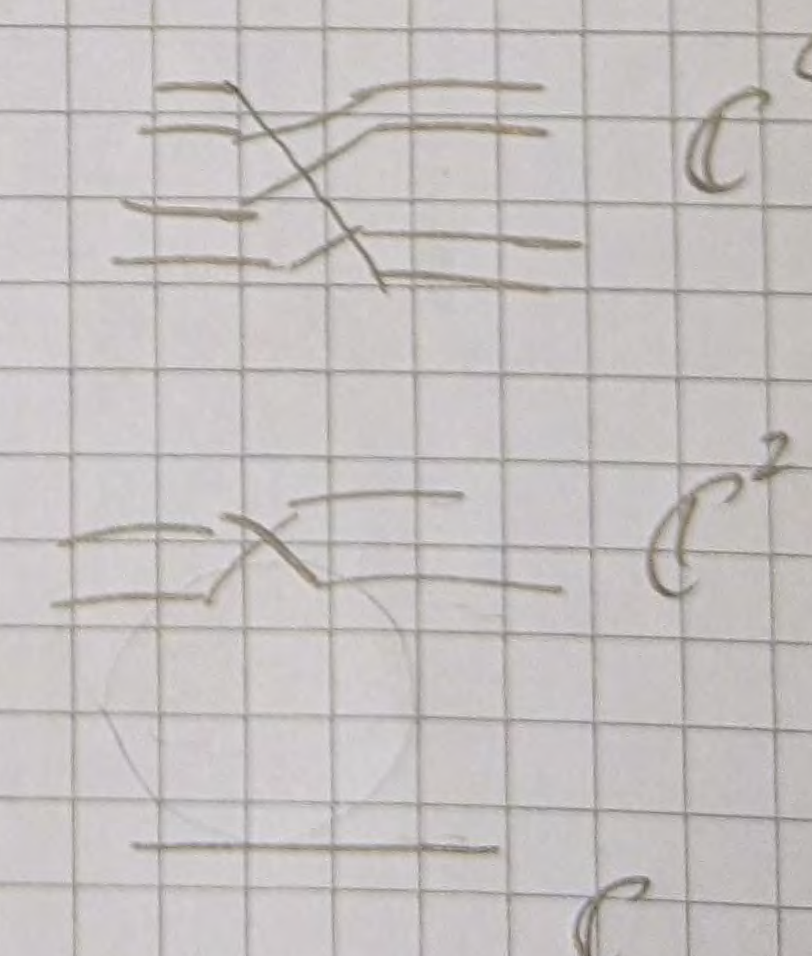
$$z_2 \xrightarrow{z^2} z_1 \xrightarrow{z^2} z_0$$

Riemann surface of $\sqrt[4]{z}$.

$\varprojlim(\hat{\mathbb{C}}, z^2)$ "limit" of Riemann surfaces $\sqrt[\underleftarrow{2^n}]{z}$.

Can use other rational maps.

iterated monodromy groups encodes how different sheets are connected.




(Meiyu Su, Lyubich - Minsky)

# Review: the product topology

$I$ non-empty index set,

$X_i$ non-empty top. space $\forall i \in I$

$$\prod_{i \in I} X_i := \{ \{x_i\}_{i \in I} \mid x_i \in X_i \quad \forall i \in I \}$$

is the **product** of the spaces $X_i$.

Note: axiom of choice

$\Leftrightarrow \prod X_i \neq \emptyset$.

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$x_j$ $j$-th coordinate

$\pi_j : \prod X_i \to X_j$, $\pi_j(\{x_i\}) = x_j$

projection.

The product topology on $\prod X_i$ is the coarsest / smallest (with fewest open sets)

s.t. $\pi_j$ continuous $\forall j$.

# Review: product topology

So $U_j \subset X_j$ open

$$\Rightarrow \pi_j^{-1}(U_j) = \prod \begin{pmatrix} U_j & , j = i \\ X_i & , j \neq i \end{pmatrix} \quad (*)$$

(in $j$-th coordinate have $U_i$
all other coordinates are whole space)

Set is open in $\prod X_i$ iff it is a

union of finite intersections
of sets as in $(*)$

sets in $(*)$ form a <u>subbasis</u> of product top.

<u>Note</u>: $U \subset \prod X_i$ open

$\pi_j(U) \neq X_j$ only in finitely many coordinates $j$.

<u>Tychonoff's thm</u>

Product of compact spaces is compact.

# Topology on inverse limit space

Now $X_j = X$

$f_j = f : X \to X$ cont. surj. $\forall j.$

Notation: $X^{\mathbb{N}} = \prod_{i \in \mathbb{N}} X$

$$\hat{X} = \underleftarrow{\lim} (X, f)$$

$x = (\ldots, x_2, x_1, x_0) \in X^{\mathbb{N}}$

$x_n \in X \quad \forall n$

$U_n \subset X$ then

$\ldots \times U_n \times U_{n-1} \times \ldots \times U_0 = (\ldots, U_n, U_{n-1}, \ldots, U_0).$

— sets:

$(\ldots, X, U, X, \ldots, X) \cap \hat{X}$ for $U \subset X$ open

$$\underbrace{\text{sets form subbasis of } X^{\mathbb{N}}}_{\text{sets form subbasis of top of } \hat{X}.}$$

$(\ldots, X, U, X, \ldots, X) \cap \hat{X}$

$\uparrow$ n-th coordinate

$(\ldots, f^{-2}(U), f^{-1}(U), U, f(U), \ldots, f^{n}(U))$

subbasis of top on $\hat{X}$.

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**Fact**: $f : X \to X$ homeomorphism

$\Rightarrow \hat{X} \simeq X$.

In this case all open sets are of the form $(\ldots, f^{-2}(U), f^{-1}(U), U)$ $U \subset X$ open

$\pi_0 : \hat{X} \to X$ is homeo.

## The inverse limit map

Define $\hat{f} : \hat{X} \to \hat{X}$ by

$$\hat{f}(\{x_n\}) := \{f(x_n)\}$$

$$\hat{f}(\dots, x_2, x_1, x_0) = (\dots, f(x_2), f(x_1), x_0)$$
$$= (\dots, x_1, x_0, f(x_0))$$

• **$\hat{f}$ has $f$ as a factor** :

$$\begin{tikzcd}
\hat{X} \arrow[r, "\hat{f}"] \arrow[d, "\pi_0"'] & \hat{X} \arrow[d, "\pi_0"] \\
X \arrow[r, "f"] & X
\end{tikzcd}$$

commutes.

• **$\hat{f}$ is invertible**

$$\hat{f}^{-1}(\dots, x_1, x_0, x_0') = (\dots, x_1, x_0)$$

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Consider open set $\hat{U} \subset \hat{X}$ of the form

$$(\dots, f^{-2}(U), f^{-1}(U), U, f(U), \dots, f^{n}(U))$$

n-th coord.

$$\hat{f}(\hat{U}) = (\dots, f^{-1}(U), U, f(U), \dots, f^{n+1}(U))$$

open

$$\hat{f}^{-1}(\hat{U}) = (\dots, f^{-3}(U), f^{-2}(U), f^{-1}(U), U, \dots, f^{n-1}(U))$$

$\hat{f} : \hat{X} \to \hat{X}$ is a homeomorphism.

<u>Theorem</u>: $\hat{f}$ is minimal homeomorphic extension of $f$.

Precise formulation:

<u>Theorem</u>: Let $g : Y \to Y$ be a <u>homeomorphism</u> of a top space $Y$ that has $f$ as a factor, i.e.,

$\exists\, H : Y \to X$ with

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "H"'] & Y \arrow[d, "H"] \\ X \arrow[r, "f"] & X \end{tikzcd} \qquad f \circ H = H \circ g.$$

Then there $\exists\, h : Y \to \hat{X}$ continuous with

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "h"'] \arrow[dd, bend right=60, "H"'] & Y \arrow[d, "h"] \arrow[dd, bend left=60, "H"] \\ \hat{X} \arrow[r, "\hat{f}"] \arrow[d, "\pi_0"'] & \hat{X} \arrow[d, "\pi_0"] \\ X \arrow[r, "f"] & X \end{tikzcd} \qquad \begin{aligned} H &= \pi_0 \circ h \\ \hat{f} \circ h &= h \circ g \end{aligned}$$

"Every homeomorphic extension of $f$ fibers through $\hat{f}$."

<u>Proof:</u> For $y \in Y$ need to define

$h_0(y) := H(y)$ (0-th coordinate)

$h_1(y) := H(g^{-1}(y))$ (1-st coordinate)

$\vdots$

$h_n(y) := H(g^{-n}(y))$ (n-th coordinate)

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<u>i) Have $H = \pi_0 \circ h$</u>

$$\pi_0 \circ h(y) = \pi_0(\ldots, h_1(y), h_0(y))$$
$$= h_0(y) = H(y). \qquad \square$$

<u>ii) $\hat{f} \circ h = h \circ g$</u>

($\hat{f}$ is a factor of $g$)

Have

$$\begin{aligned}\hat{f} \circ h(y) &= \hat{f}(\ldots, h_1(y), h_0(y)) \\ &= (\ldots, f \circ h_1(y), f \circ h_0(y)) \\ &= (\ldots, f \circ H \circ g^{-1}(y), f \circ H(y)) \\ &= (\ldots, H \circ g(g^{-1}(y)), H \circ g(y)) \\ &= (\ldots, H \circ g^{-1}(g(y)), H(g(y))) \\ &= (\ldots, h_1(g(y)), h_0(g(y))) \\ &= h \circ g(y).\end{aligned}$$

$\forall y \in Y$ $\square$

## iii) $h$ is continuous

Want $\hat{U} \subset \hat{X}$ open

$$\Rightarrow h^{-1}(\hat{U}) = h_0^{-1}(\hat{U}) \cap h_1^{-1}(\hat{U}) \cap \dots$$

$$\subset Y \text{ open.}$$

Enough to show for $\hat{U}$ of the form

$$\hat{U} = (\dots, f^{-1}(U), U, f(U), \dots, f^{n}(U)) \subset \hat{X},$$

where $U \subset X$ open.

sets form subbasis of topology.

Case 1

$$\hat{U} = (\dots, f^{-2}(U), f^{-1}(U), U) \subset \hat{X}, \quad U \subset X \text{ open}$$

$y \in h^{-1}(\hat{U})$

$\Leftrightarrow \quad h_0(y) = H(y) \in U \quad \Leftrightarrow \quad \underline{y \in H^{-1}(U)}$

and $h_1(y) = H(g^{-1}(y)) \in f^{-1}(U)$

$\Leftrightarrow g^{-1}(y) \in H^{-1}(f^{-1}(U))$

$\Leftrightarrow \quad \underline{y} \in g \circ (f \circ H)^{-1}(U)$

$= g \circ (H \circ g)^{-1}(U)$

$= g \circ g^{-1}(H^{-1}(U))$

$= \underline{H^{-1}(U)}$

Similar for $n$-th coordinate.

For sets as in Case 1, the conditions coming from each coordinate are the same

or $h^{-1}(\hat{U}) = H^{-1}(U)$ open.

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Case 2

Consider
$$\hat{f}(\hat{U}) = (\ldots, f^{-1}(U), U, f(U))$$

Want: $h^{-1}(\hat{f}(\hat{U})) \subseteq Y$ open

Recall: $h \circ g = \hat{f} \circ h$

or $\hat{f}^{-1} \circ h = h \circ g^{-1}$

so
$$h^{-1}(\hat{f}(\hat{U})) = [\hat{f}^{-1} \circ h]^{-1}(\hat{U})$$
$$= [h \circ g^{-1}]^{-1}(\hat{U})$$
$$= \underbrace{g \circ \underbrace{h^{-1}(\hat{U})}_{\text{open Case 1}}}_{\text{open since } g \text{ homeo.}}$$

General case $\hat{U} = (\ldots, f^{-1}(U), U, \ldots, f^{n}(U))$ by induction, or using $\hat{f}^{-n} \circ h = h \circ g^{-n}$.

$\square$