Near homeomorphisms

(X, d_1) , (Y, d_2) compact metric spaces

$C(X, Y)$ = set of cont. maps $f : X \to Y$

equipped with metric
$$\|f - g\| = \sup_{x \in X} d_2(f(x), g(x))$$

complete metric space.

$H(X, Y)$ = set of homeos $f : X \to Y$

$f : X \to Y$ is a near-homeomorphism if it is the uniform limit of homeomorphisms

$f \in \overline{H(X, Y)}$

then f surjective

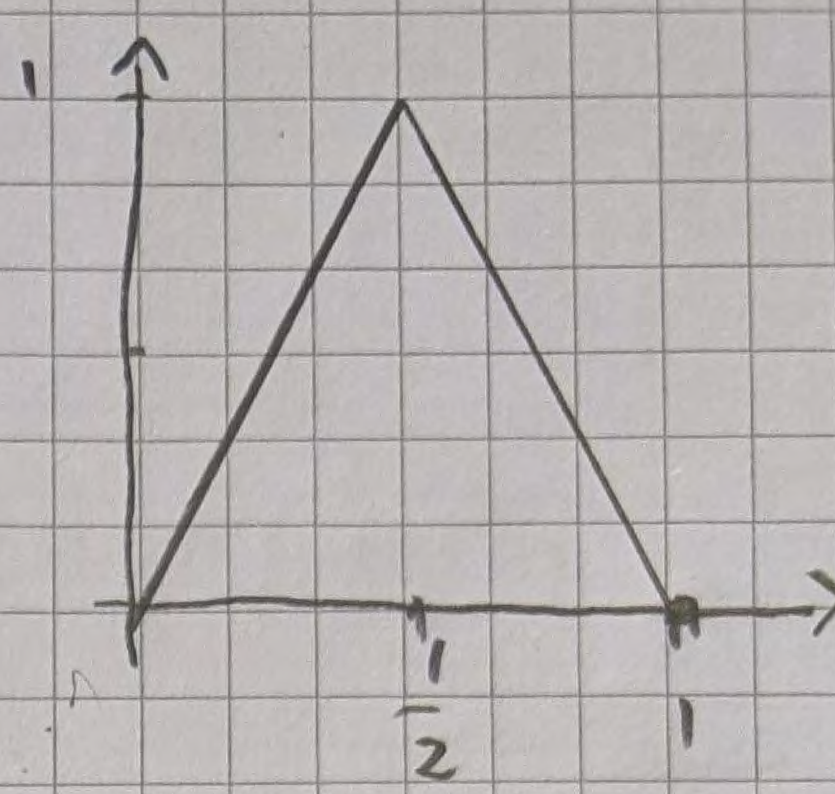
Examples

Many maps $f: X \to X$ have near-homeomorphic extensions.

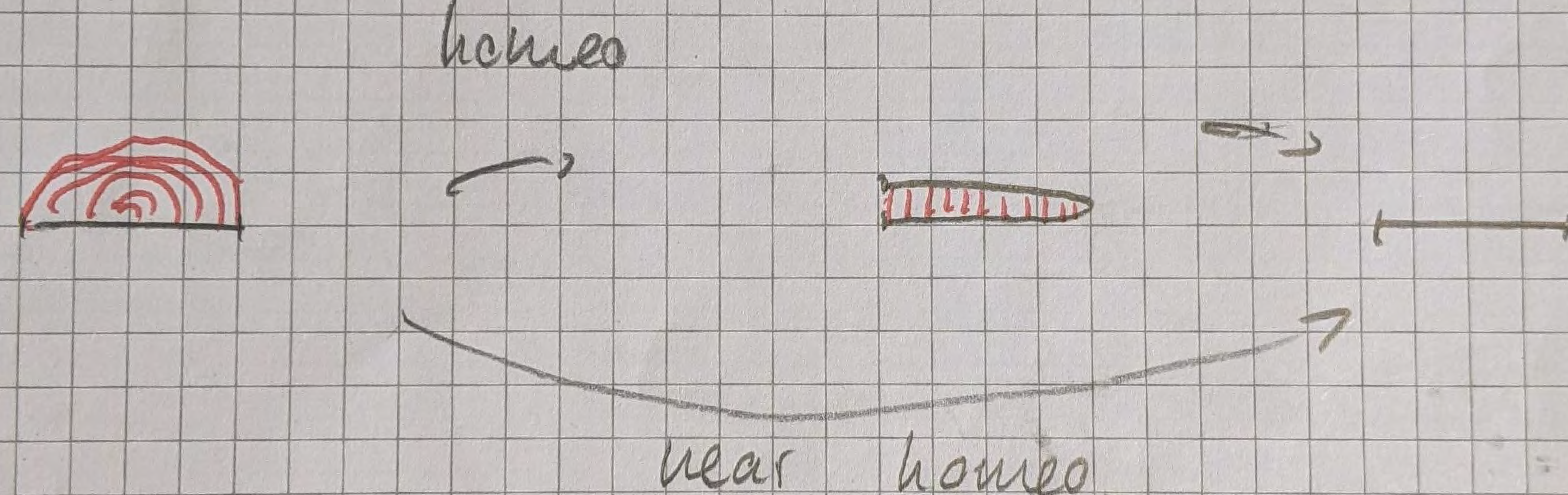
Tent map:

$f: [0,1] \to [0,1]$

$$f(t) = \begin{cases} 2t, & 0 \le t \le \frac{1}{2} \\ 2-2t, & \frac{1}{2} \le t \le 1 \end{cases}$$



In S^2



Rational map: no obvious near-homeomorphic extension.

But: real rational map with crit. values in $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

Ex: $\qquad f(z) = 1 - \dfrac{2}{z^2}$

$$0 \xrightarrow{2:1} \underbrace{\infty \xrightarrow{2:1} 1 \to -1}_{} \circlearrowleft$$

$\mathrm{post}(f) = \{-1, 1, \infty\}$.

Then f descends to quotient

$S = \hat{\mathbb{C}} / \{z \sim \bar{z}\}$. $\tilde{f} : S \to S$

S = "sheet of paper"

$\tilde{f}$ "folds this sheet" of paper, maps it to itself.

there is a near homeomorphism of S^3 (or $\mathbb{R}^3$)

that restricts to $\tilde{f}$.

$$f(\bar{z}) = \overline{f(z)}$$

$$\begin{tikzcd}
\hat{\mathbb{C}} \arrow[r, "f"] \arrow[d, "\bar{z}"'] & \hat{\mathbb{C}} \arrow[d, "\bar{z}"] \\
\hat{\mathbb{C}} \arrow[r, "f"] & \hat{\mathbb{C}}
\end{tikzcd}$$

Morton Brown's theorem '60

Inverse sequence of compact metric spaces

$$\cdots \longrightarrow X_3 \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0$$

$(X_j, f_j)_{j \in \mathbb{N}}$

X_j compact metric space

$f_j : X_{j+1} \to X_j$ cont. surj.
near homeomorphism.

Thm: In the setting as above, each

$\pi_k : \hat{X} \longrightarrow X_k$ is a near-homeomorphism.

In particular $\hat{X}$ homeomorphic to X_k.

Following Ancel '86

Some preparation:

Lemma 1:
X, Y, Z compact metric spaces

Then composition
$$(f, g) \mapsto g \circ f : C(X,Y) \times C(Y,Z) \to C(X,Z)$$
is continuous.

ε - maps

let $\varepsilon > 0$
cont. map $f: X \to Y$
is **ε - map** if $\operatorname{diam}(f^{-1}(y)) < \varepsilon \ \forall y \in Y$.

Note: if map f is an ε - map f.a $\varepsilon > 0$
f is injective.

$C_\varepsilon(X, Y)$ = set of ε - maps $f: X \to Y$.

Lemma 2:
$C_\varepsilon(X, Y)$ open subset of $C(X, Y)$ $\forall \varepsilon > 0$

Proof:
let $f \in C_\varepsilon(X, Y)$
set $\delta = \frac{1}{2} \inf\{ d_2(f(x), f(z)) \mid x, z \in X$
$d_1(x, z) \geq \varepsilon \}$
> 0
let $g \in C(X, Y)$ with $\|f - g\| < \delta$
(g is in δ - Ball around f)

let $x, z \in X$ and $g(x) = g(z)$
Have $d_2(f(x), f(z)) \leq \underbrace{d_2(f(x), g(x)) + d_2(g(z), f(z))}_{< 2\delta}$

$\Rightarrow d_1(x, z) < \varepsilon \Rightarrow g \in C_\varepsilon(X, Y) \Rightarrow C_\varepsilon(X, Y)$ open.

Proof of Brown's theorem

$$\hat{X} = \varprojlim (X_j, f_j)$$

$$\ldots, x_3, x_2, x_1, x_0) \in X^{\mathbb{N}} \quad , \quad f_j(x_{j+1}) = x_j \ \forall j$$

Let d_j metric on X_j with

$$\operatorname{diam}(X_j) < \frac{1}{j}.$$

Then

$$\hat{d}(x, y) = \sup\{ d_j(x_j, z_j) \mid j \geq 0 \}$$

is metric on $\hat{X}$ that induces top. def. earlier.

Each projection $\pi_j : \hat{X} \to X_j$ is a $\frac{1}{j}$-map

$$\begin{tikzcd} x_j \arrow[d, "\pi_j"'] & \cdots \arrow[r] & x_1 \arrow[r] & x_0 & = x \in \hat{X} \\ x_j \end{tikzcd}$$

$$\pi_j(x) = \pi_j(y)$$

$$x_j = y_j \quad , \quad f_{j-1}(x_j) = x_{j-1} = y_{j-1}$$

$$\vdots$$

$$x_0 = y_0$$

so $\hat{d}(x, y) < \frac{1}{j+1} < \frac{1}{j}$

Prove

$\pi_0 : \hat{X} \to X_0$ is a near-homeo.

let $\mathcal{F}$ = closure in $C(\hat{X}, X_0)$
of all sets of the form

$h \circ \pi_k$, where $h : X_k \to X_0$ homeo

$h \circ \pi_k : \hat{X} \to X_0$ $\qquad k \geq 0$

$$\begin{tikzcd}
(x_k, \dots, x_2, x_1, x_0) \in \hat{X} \arrow[d, "\pi_k"] & \\
x_k \arrow[r, "h"] & x_0
\end{tikzcd}$$

Have $\pi_0 \in \mathcal{F}$

$(\mathcal{F}, \|f-g\|)$ complete metric space.

Want to show $\mathcal{F}$ has a dense subset
of homeomorphisms.

Note π_k surj.
$h \circ \pi_k$ surj.
uniform limit of surj maps surj.

Each $f \in \mathcal{F}$ surj.

Let $\varepsilon > 0$.

set $\mathcal{F}(\varepsilon) = \mathcal{F} \cap C_\varepsilon(X, Y)$

<u>open</u> subset of $\mathcal{F}$ by lemma 2.

<u>$\mathcal{F}(\varepsilon) \subset \mathcal{F}$ dense</u>

consider $h \circ \pi_k \quad , k \geq 0$

$h : X_k \to X_0$

$\delta > 0$.

Need to show δ-nbhd of $h \circ \pi_k$ contains element of $\mathcal{F}(\varepsilon)$.

Let $j > k$ s.t. $\frac{1}{j} < \varepsilon$.

$$\begin{tikzcd} X_j & X_k \arrow[d] & \cdots & X_4 & X_3, & X_2, & X_1 & X_0 \\ & X_k & & & & & \end{tikzcd}$$

$$h \circ \pi_k = h \circ \underbrace{f_k \circ \dots \circ f_{j-1}} \circ \pi_j$$

$$\begin{tikzcd} X_j \cdots, \arrow[d, "\pi_j"] & X_k, \cdots, X_1, X_0 \arrow[d, "\pi_k"] & \\ X_j \arrow[r, "f_k \circ \dots \circ f_{j-1}"', "g"'] & X_k \arrow[r, "h"] & X_0 \end{tikzcd}$$

Have $f_k \circ \dots \circ f_{j-1}$ near homeo

and have $g : X_j \to X_k$ homeo close to $f_k \circ \dots \circ f_{j-1}$

s.t. $\| h \circ \pi_k - h \circ g \circ \pi_j \| < \delta$ by lemma 2.

Have π_j is $\frac{1}{j}$-map so

$h \circ g \circ \pi_j \in \mathcal{F}(\varepsilon)$

so $\mathcal{F}(\varepsilon)$ is dense.

Have $H := \bigcap_k \mathcal{F}(\tfrac{1}{k})$

count. intersection of open dense subsets

each $h \in H$ surj., inj., cont.

$\Rightarrow h$ homeo. $\square$

$h : \hat{\underline{X}} \to X_0$

Baire's category thm

(X, d) complete metric space

$\{U_i\}_{i \in \mathbb{N}}$ count. family of open dense subsets of X.

Then

$\bigcap_{i \in \mathbb{N}} U_i$ is dense.

<u>References</u>

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