(M221) Review for Test II

① Suppose $\vec{b}_1, \dots, \vec{b}_n$ belong to some subspace V and $\vec{a}$ is their linear combination. Explain why $\vec{a}$ is also in V.

② Suppose $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

Show that $C(AB)$ is contained in $C(A)$.

Hint: See that columns of AB are lin. comb. of columns of A.

③ For each of the following conditions construct a 3×3 A, if possible!

a) $\dim C(A) = 2$; b) $\dim N(A) = 2$;

c) both $N(A)$ and $C(A)$ contain $(1,1,1)$;

d) $Ax = b$ has ∞-many solutions for every 3×1 vector b.

e) $N(A) = C(A)$.

NOTE: e) can't be done for 3×3 but can be done for 2×2 A, Why?

④ You know that a 3×4 matrix A has row reduced form $R = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

a) Which of the following can you determine (the bases of)

$N(A), C(A), N(A^T), C(A^T)$?

b) Comple the $A = \begin{bmatrix} 2 & 0 & & \\ 1 & 3 & & \\ 1 & 1 & & \end{bmatrix}$

c) For the completed A, determine all the four spaces.

⑤ (a) Complete: $Ax = b$ has a solution iff b belongs to ________,

which is the same as b being perpendicular to ________.

b) For $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, use a) above to write an equation involving $b = (b_1, b_2, b_3)$ that guarantees that $Ax = b$ can be solved.

⑥ a) Project $b = (3,3,3)$ onto $C(A)$. Warning: The columns of A are not indep.

Use $A_{\text{better}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. Why?

(M221) Review for Test II (cont.)

Cont. of 6 We still use A from 5 and $b = (3,3,3)$.

b) Find the general solution of $Ax = p$ where p is the projection from a) (It should be $p = (4,2,2)$!)

$$x = x_p + x_n$$

c) Find the unique solution x_r in the row space of A, to $Ax_r = p$.

d) Find the "least squares solution" to $A\hat{x} = b$.

7 a) Find the QR-decomposition of A from 5

b) Decompose p (from #6 c) into projections onto columns of Q.

c) Solve #6 d) by using the QR-decomposition of A.

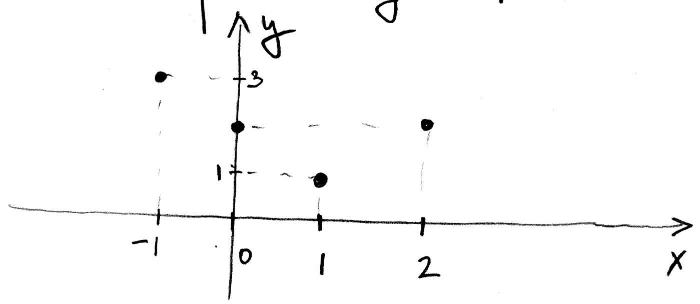
8 a) Let V be the subspace of all 2×2 symmetric matrices that satisfy $A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$. Find the dimension and basis of V.

b) Do the same for antisymmetric 3×3 matrices with $A\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

9 a) Find a basis of $S^\perp$ where S consists of $\begin{pmatrix}1\\2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}3\\2\\3\\2\end{pmatrix}$

b) Find a basis of the space spanned be the three vectors above.

10 Fit a parabola $y = a_1x^2 + a_2x + a_3$ to the data points



Some solutions to Review for Test II

#3 a) $\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$ b) $\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$ c) $\begin{bmatrix}1&-1&0\\1&-1&0\\1&-1&0\end{bmatrix}$ d) Can't be e) Can't be

(since $\dim(N(A)) + \dim C(A) = 3$)

#4 a) $C(A^T)$ has basis $(1,0,1,2)$, $(0,1,2,3)$ (dim = 2)

$N(A)$: $x_3 = 1, x_4 = 0$ give $x_2 = -2, x_1 = -1$ so $(-1,-2,1,0)$

$x_3 = 0, x_4 = 1$ give $x_2 = -3, x_1 = -2$ so $(-2,-3,0,1)$ basis dim = 2

b) Rows of A must be $a_1 = 2 \cdot r_1$ ← rows of R

$a_2 = r_1 + 3r_2$ so $\begin{bmatrix}2&0&2&4\\1&3&7&11\\1&1&3&5\end{bmatrix} = A$

$a_3 = r_1 + r_2$

c) $C(A)$ basis: $\begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}0\\3\\1\end{bmatrix}$, $N(A^T)$ has dim $= 3-2 = 1$ and basis $(1, 1, -3)$ since $a_1 + a_2 - 3a_3 = 0$.

#5 a) b belongs to $C(A)$ and is perpendicular to $N(A^T)$

b) Since rows $a_1 = (1,2,3)$ and $a_2 = (0,1,1)$ are independent and row $a_3 = a_1 + a_2$, rank $= \dim C(A^T) = 2$.

Hence $N(A^T)$ is 1 dim with basis vector $(1,-1,-1)$.

b perpendicular to $N(A^T)$ means exactly that $b \cdot (1,-1,-1) = b_1 - b_2 - b_3 = 0$

#6 a) $C(A) = C(B)$ where $B = \begin{bmatrix}1&2\\0&1\\1&1\end{bmatrix}$ is obtained by discarding redundant columns from A

are independent!

The projection matrix is then: $P = B(B^TB)^{-1}B^T = \dots = \frac{1}{3}\begin{bmatrix}2&1&1\\1&2&-1\\1&-1&2\end{bmatrix}$

So $p = Pb = \dots = \begin{bmatrix}4\\2\\2\end{bmatrix}$

b) $\begin{bmatrix}1&2&3\\0&1&1\\1&1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}4\\2\\2\end{bmatrix}$ is equivalent to (since eq3 = eq1 + eq2) $\begin{bmatrix}1&2&3\\0&1&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}4\\2\\0\end{bmatrix}$

free

twice col. 2

particular solution: $x_p = (0,2,0)$ as can be observed or solved by setting $x_3 = 0$.

null space $N(A)$ is $3-2=1$ dim with basis $(1,1,-1)$ so $x_n = x_3(1,1,-1)$

$x = x_p + x_n = (0,2,0) + x_3(1,1,-1)$

#6 c) $C(A^T)$ basis can be taken as any two rows, say $(0,1,1),(1,1,2)$
or even $(0,1,1),(1,0,1)$ ← difference of

We are then looking for $x_r = t(0,1,1)+s(1,0,1) = (s,t,s+t)$

such that $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} s \\ t \\ s+t \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$ so
$$\begin{aligned} &\dots \\ &t+s+t=2 \\ - \ &s+t+s+t=2 \\ \hline &\text{so } s=0,\ t=1 \end{aligned}$$

Got $x_r = (0,1,1)$ ←

NOTE: This is a particular sol. different than x_p we found before.

d) We actually just did it! The least squares solution to $A\hat{x}=b$ is the solution to $Ax = p$ as given by #6 b)!

Rk: Note that there are ∞-many least squares solutions because the columns of A were not independent!
Hence, $(A^TA)^{-1}$ does not exist and cannot be used to solve $A^TAx = A^Tb$.
That is why we used $P = B(B^TB)^{-1}B^T$ for the projection!
Of course, you could solve $A^TA\hat{x} = A^Tb$ from scratch, but that would have been more work. (The projection would still be $p = A\hat{x}$.)

#7. 1^{st} column $a_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$ normalizes to $q_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}$

2^{nd} column $a_2 = \begin{pmatrix}2\\1\\1\end{pmatrix}$ minus the projection $q_1^Ta_2 \cdot q_1$ is $\begin{pmatrix}2\\1\\1\end{pmatrix} - \frac{1}{2}\cdot 3\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1/2\\1\\-1/2\end{pmatrix}$

which normalizes to $\frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\-1\end{pmatrix} = q_2$

Thus: $a_1 = \sqrt{2}\cdot q_1$
$a_2 = \frac{3}{2}\sqrt{2}\,q_1 + \frac{\sqrt{6}}{2}q_2$ and

3^{rd} dependent column → $a_3 = a_1 + a_2 = \frac{5}{2}\sqrt{2}\,q_1 + \frac{\sqrt{6}}{2}q_2$

so $A = \underbrace{[q_1|q_2]}_{Q}\cdot\underbrace{\begin{bmatrix} \sqrt{2} & \frac{3\sqrt{2}}{2} & \frac{5\sqrt{2}}{2} \\ 0 & \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{2} \end{bmatrix}}_{R}$ where $Q = [q_1, q_2] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$

Again: Because the columns of A were not dependent the matrix Q is not square (so not exactly orthogonal) and R is not square (but still upper triangular).

#10

@ $x=-1$: $a_1 - a_2 + a_3 = 3$

@ $x=0$: $a_3 = 2$

@ $x=1$: $a_1 + a_2 + a_3 = 1$

@ $x=2$: $4a_1 + 2a_2 + a_3 = 2$

to be solved for $\vec{a}$

so $\underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}}_{A} \vec{a} = \underbrace{\begin{bmatrix} 3 \\ 2 \\ 1 \\ 2 \end{bmatrix}}_{\vec{b}}$

least squares eq. $A^T A \vec{a} = A^T \vec{b}$

$$A^T A = \begin{bmatrix} 18 & 9 & 6 \\ 9 & 6 & 3 \\ 6 & 3 & 4 \end{bmatrix} \qquad A^T \vec{b} = \begin{bmatrix} 12 \\ 2 \\ 8 \end{bmatrix}$$

I computed this by doing dot products of columns of A (all six, not 9, of them!)

dot products of columns of A with $\vec{b}$

$$\vec{a} = (A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} 5/18 & -1/3 & -1/6 \\ -1/3 & 2/3 & 0 \\ -1/6 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 12 \\ 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -8/3 \\ 2 \end{bmatrix}$$

Inverse computed by using Gauss-Jordan

<u>Answer</u> : $y = \frac{4}{3}x^2 - \frac{8}{3}x + 2$