| 0 | 1 | 2 | 3 | 4 | 5 | 6 | |
|---|---|---|---|---|---|---|---|
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# Math 221 Second Exam (1 April 2011)

**Show all work (unless instructed otherwise).** NAME:
**Good Luck!**

0. Circle **True** or **False** without explanation:

( **T** or **F** ) Any subspace in $\mathbf{R}^3$ is the column space of some matrix.

( **T** or **F** ) There is a $3 \times 3$ matrix $A$ with $N(A) = C(A)$.

( **T** or **F** ) A subspace $S$ and its orthogonal complement $S^\perp$ intersect only at zero.

( **T** or **F** ) If $v_3 \neq v_1 + v_2$ then the vectors $v_1, v_2, v_3$ are independent.

( **T** or **F** ) If $A$ and $AB$ are invertible then $B$ is also invertible*.

1. Consider a $3 \times 3$ matrix $A$ given by the following (partly obliterated) product

$$A = \begin{bmatrix} 1 & \dots & \dots \\ 2 & \dots & \dots \\ 3 & \dots & \dots \end{bmatrix} = \begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix} \begin{bmatrix} 1, & 2, & 3 \end{bmatrix}$$

a) Complete the missing entries above.

b) The rank of $A$ must be ... (You do not need a) for this!)

c) A basis of the column space $C(A)$ is ...

2. Find the dimension and a basis of the subspace that consists of all $2 \times 2$ matrices of the form $A = \begin{bmatrix} a & b \\ -b & c \end{bmatrix}$ ($a, b, c$ are "free" scalars.)

3. You are given $A$ and its row reduced form $R$

$$A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 2 & 1 & -1 & 3 \\ -1 & 1 & 2 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Complete the blanks below.

a) The rank of $A$ is ...

b) The column space $C(A)$ has dimension ... and a basis ...

c) The row space $C(A^T)$ has dimension ... and a basis ...

d) Compute a basis for the null space $N(A)$.
*Show Work Here:*

e) The dimension of $N(A^T)$ is ........ and its basis is $(-3, 4, \ldots)$
*Show Work:*

f) State the condition on $b = (b_1, b_2, b_3)$ guaranteeing that $Ax = b$ has a solution. (Use e).)

4. Let $S$ be the plane spanned by the two vectors $a = (1, 1, 1)^T$ and $b = (1, 1, 0)^T$.
a) Use the Gramm-Schmidt process to find an orthonormal basis of $S$.

b) Find the QR-decomposition of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$

5. Consider $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 4 & 2 & 4 \end{bmatrix}$
a) Find the $x_r$ in the row space $C(A^T)$ solving $Ax_r = b$ for $b = (7, 6, 26)^T$.

b) Given that $(1, 0, -1)^T$ is a basis of $N(A)$, write out the complete solution to $Ax = b$ (for the $b$ as in a) above).

do not loose this 1/2 here !

6. Take $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$. Its $QR$-decomposition has $Q = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$, $R = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$.

a) Compute the projection of $b = (1, 1, 1, 0)^T$ onto $C(A)$.

b) Use $Q$ and $R$ to find the least squares solution to $Ax = b$.

#0: T, F, T, F, T

<u>Grading</u>: 10 pts each except for #2 to the total of 80 pts.

#1 $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [1\ 2\ 3]$ ; rank = 1 ; basis $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

#2 $\begin{bmatrix} a & b \\ -b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ These three matrices (without the coeff. in front) constitute a basis. Dim is 3.

#3. rank = # of pivots = 2; dim C(A) = 2 with basis $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. $C(A^T)$'s basis $[1\ 0\ -1\ 1]$, $[0\ 1\ 1\ 1]$ and dim is 2.

dim N(A) = 4 − 2 = 2 $\quad \begin{matrix} x_1 - x_3 + x_4 = 0 \\ x_2 + x_3 + x_4 = 0 \end{matrix}$ ($x_3$, $x_4$ free) gives $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ (from $x_3 = 1$, $x_4 = 0$) and $\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ (from $x_4 = 1$, $x_3 = 0$) which is the basis

dim $N(A^T) = 3 - 2 = 1$ and $-3\,\text{row}_1 + 4\,\text{row}_2 = (5, -5, -10, 0) = -5\,\text{row}_3$ so $(-3, 4, 5)$ is a basis vector.

$Ax = b$ has sol iff $b \perp N(A^T)$ which is $-3b_1 + 4b_2 + 5b_3 = 0$.

#4 $q_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; $q_2 = \text{normalized}\,(b - q_1^T b \cdot q_1) = \text{norm.}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{\sqrt{3}}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \text{norm.}\begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix} \qquad R = \begin{bmatrix} \sqrt{3} & 2/\sqrt{3} \\ 0 & 2/\sqrt{6} \end{bmatrix}$

#6 $p = QQ^Tb = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1 \\ 1/2 \end{bmatrix}$

$QR\hat{x} = b$ so $\hat{x} = R^{-1}Q^Tb = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$

#5: $x_r = t[1,1,1] + s[1,0,1] = [t+s, t, t+s]$ plugged into $Ax_r = b$ gives $\begin{matrix} (t+s) + t + (t+s) = 7 \\ (t+s) + 0 + (t+s) = 6 \\ \cdots\cdots = 26 \end{matrix}$

We got $\begin{matrix} 3t + 2s = 7 \\ 2t + 2s = 6 \end{matrix}$ so $t = 1$ so $s = 2$ giving $x_r = [3, 1, 3]$.

$x = x_r + x_n = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} + a \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ where a is a "free" scalar.