									max
10	15	20	15	15	20	15	15	15	140

Math 221 Final (6 May 2010)

NAME:

Avg 96 ≈ 69
Median 101 ≈ 72

Show all work (unless instructed otherwise). Good Luck!

0.[10pts] Indicate whether the statements are true or false.

([✓ T] F) Eigenvectors with different eigenvalues are independent.
([✓ T] F) $\det(AB) = \det(A)\det(B)$.
([✓ T] F) A 5×3 matrix cannot be of rank 4.
(T [✓ F]) If $Ax = b$ can be solved for some b then it can be solved for all b.
(T [✓ F]) $(AB)^T = A^T B^T$.

1.[15pts] Let V be the subspace spanned by $v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $v_2 = \begin{pmatrix}1\\2\\2\end{pmatrix}$, $v_3 = \begin{pmatrix}0\\1\\1\end{pmatrix}$.

5 a) For what a does $(a, a, 1)$ belong to V?

$$x_1\begin{pmatrix}1\\1\\1\end{pmatrix} + x_2\begin{pmatrix}1\\2\\2\end{pmatrix} + x_3\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}x_1+x_2\\x_1+2x_2+x_3\\x_1+2x_2+x_3\end{pmatrix} = \begin{pmatrix}a\\a\\1\end{pmatrix} \qquad a = 1$$

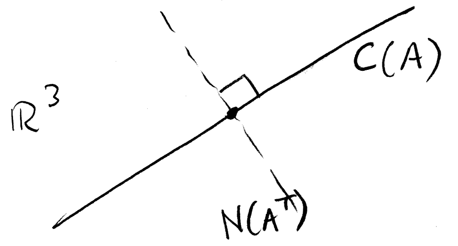
same so

When $a=1$: $x_1 + x_2 = 1$, $x_1 + 2x_2 + x_3 = 1$ is solved by, say, $x_1 = 1$, $x_2 = 0$, $x_3 = 0$ simply $v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$

should have just looked at it first!

5 b) Find a basis of V. $v_1 + v_3 = v_2$ so v_1, v_3 form a basis.

5 c) The subspace $V^{\perp}$ orthogonal to V is the null space of some matrix A. Write out A. (Do not find $N(A)$.)



$$A = \begin{bmatrix}1 & 1 & 1\\1 & 2 & 2\\0 & 1 & 1\end{bmatrix}$$

2.[20pts] For

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

10 a) Use Gauss-Jordan Elimination to find A^{-1}. (Indicate the row operations used.)

5 for Gauss
5 for Jordan

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[③-①]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[③-②]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{2}③]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right] \xrightarrow[②-③]{①-③} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{matrix}}_{A^{-1}}$$

5 b) Write out the LU-decomposition of A (based on a)).

$$LU = \cancel{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}}_{U}$$

5 c) Use A^{-1} to solve $Ax = b$ when $b = (1, 2, 3)$. (Use the suggested method.)

$$x = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/2 - 1 + 3/2 \\ -1/2 + 1 + 3/2 \\ 1/2 + 1 - 3/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

✓

3.[15pts] For $A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$

5 a) Use the Laplace expansion about the first row to compute det(A).

$$|A| = 2 \cdot (-1)^{1+1} \underbrace{\begin{vmatrix} 2 & -1 \\ 2 & 0 \end{vmatrix}}_{+2} + 0 \cdot \ldots + (-1)(-1)^{1+3} \underbrace{\begin{vmatrix} 0 & 2 \\ 2 & 2 \end{vmatrix}}_{-4} = 4 + 4 = 8$$

5 b) Test ~~A~~ the following matrix is positive definite: $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 4 \end{bmatrix}$

$$|2| = 2 > 0 \;, \quad \begin{vmatrix} 2 & 2 \\ 2 & 4 \end{vmatrix} = 4 > 0, \quad |A| = 8 > 0.$$

✓

5 c) $(-1, 1, 0)$ is an eigenvector of A. What is its eigenvalue?

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{so} \quad \lambda = 2$$

4.[15pts] The diagonalization of some B is given by

$$B = \begin{pmatrix} 0 & 0 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ -1/3 & -1/3 & 2/3 \\ -1 & 0 & 0 \end{pmatrix}$$

3 a) The eigenvalues of B are $2, -1, 1$ [← *Fill the blank.*]

More space ↕

4 b) The eigenvectors of B are $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ [← *Fill the blank.*]

4 c) Based on b), decide if B is symmetric (without computing B). Explain.

$v_1^T v_2 = 0 \cdot 0 + 2 \cdot (-1) + 1 \cdot 1 = -1 \neq 0$ so NO since eigenvectors of symmetric A are orthogonal.

4 d) Complete the three missing diagonal entries in

$$B^3 = \begin{pmatrix} 0 & 0 & -1 \\ 2 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ -1/3 & -1/3 & 2/3 \\ -1 & 0 & 0 \end{pmatrix}$$

5.[20pts] Consider the matrix

$$A = \begin{bmatrix} 3 & 0 & 4 \\ 0 & 2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 4 & 0 \end{bmatrix}$$

5pts a) Compute the singular values of A (by using AA^T).

$$AA^T = \begin{bmatrix} 25 & 0 \\ 0 & 4 \end{bmatrix} \quad \text{so} \quad \sigma_1^2 = 25, \quad \sigma_2^2 = 4$$

$$\boxed{\sigma_1 = 5, \quad \sigma_2 = 2}$$

3pts b) A maps the unit ball to an ellipse with semi-axes 5, 2 [← *Fill the blanks.*]

5pts c) Find the unique x_r in the row space solving $Ax_r = b$ when $b = (7, 2)$.

$$\begin{bmatrix} 3 & 0 & 4 \\ 0 & 2 & 0 \end{bmatrix} \underset{\substack{\uparrow \\ \text{an el. in} \\ \text{row space}}}{\begin{bmatrix} 3a \\ 2b \\ 4a \end{bmatrix}} = \begin{bmatrix} 9a + 16a \\ 4b \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix} \quad \text{so} \quad \begin{matrix} a = 7/25 \\ b = 1/2 \end{matrix}$$

$$x_r = \begin{bmatrix} 21/25 \\ 1 \\ 28/25 \end{bmatrix}$$

7pts. d) Give the complete solution to $Ax = b$ when $b = (7, 2)$.

2pts special sol.
3pts nullspace.
2 pts put together.

$$\begin{bmatrix} 3 & 0 & 4 \\ 0 & 2 & 0 \end{bmatrix}$$

$$\underbrace{\uparrow \quad \uparrow}_{\text{pivot}} \quad \underset{\text{free}}{\uparrow}$$

$$N(A) = \left\{ c \cdot \begin{bmatrix} -4/3 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \text{special solution} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{\begin{cases} \text{set } x_3 = 1 \\ 3x_1 = -4 \\ 2x_2 = 0 \end{cases}}$$

$$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} -4/3 \\ 0 \\ 1 \end{bmatrix}$$

6.[15pts] Consider the system $\begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

6pts

a) What equation on b_1, b_2, b_3 assures solvability of the system?

$b \perp N(A^T)$ $\quad A^T = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix}$ $\quad \dim N(A^T) = 3 - 2 = 1$

rank $= 2$

basis of $N(A^T)$: $c_1 + c_2 = c_3$

so we get $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

so $b \perp \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ which is $\boxed{b_1 + b_2 - b_3 = 0}$

3pts

b) Is there b for which the system has infinitely many solutions? Explain.

For b for which the sol. exists they have $\dim N(A) = 2 - 2 = 0$ free parameters. So NO

6pts

c) Give the least squares solution when $b = (0, 6, 6)$.

$A^T A x = A^T b$ is $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \\ 6 \end{bmatrix}$

$\begin{bmatrix} 6 & 6 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 18 \\ 12 \end{bmatrix}$ so $- \begin{cases} 3x_1 + 3x_2 = 9 \\ 3x_1 + 4x_2 = 6 \end{cases}$

$X_2 = -3$

$X_1 = \frac{9+9}{3} = 6.$

7.[15pts] Find the formula for a_n given by $a_0 = 0$, $a_1 = 1$ and

$$a_{n+2} = 3a_{n+1} + 4a_n \quad \text{for } n = 0, 1, 2, 3, \ldots.$$

3 pts

$$\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$$

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pts,

trace = 3 $\quad \lambda^2 - 3\lambda - 4 = 0$

det = −4 $\quad (\lambda - 4)(\lambda + 1) = 0$ so $\lambda_1 = -1$, $\lambda_2 = +4$

Eigenvectors: @ $\lambda_1 = -1$: $\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ so $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

@ $\lambda_2 = 4$: $\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$ so $x_1 = 4x_2$ so $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Thus $A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ +4 & 1 \end{bmatrix}$.

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pts Finally $a_n = 2^{nd}$ comp of $\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = 2^{nd}$ comp. $A^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$

$$a_n = 2^{nd} \text{comp } \frac{1}{5} \cdot \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^n & \\ & 4^n \end{bmatrix} \underbrace{\begin{bmatrix} 1 & -4 \\ +4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

$$a_n = 2^{nd} \text{comp of } \frac{1}{5} \begin{bmatrix} (-1)^n & 4^{n+1} \\ (-1)^{n+1} & 4^n \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$a_n = \frac{1}{5} \left((-1)^{n+1} + 4^n \right)$$

8.[15pts] Consider

$$A = \cancel{\begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

5pts a) find an orthonormal basis of the column space $C(A)$ of A.

$$q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \underbrace{\frac{1}{2} \cdot 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\text{normalized}} = \underbrace{\begin{bmatrix} -1/2 \\ 1 \\ 1/2 \end{bmatrix}}_{\text{norm.}} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

5pts b) Use a) to find the projection of $(1, 1, 1)$ onto A.

$$q_1^T b \cdot q_1 + q_2^T b \, q_2 = \frac{1}{2} \cdot 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{2}{6} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - 1/3 \\ 0 + 2/3 \\ 1 + 1/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \\ 4/3 \end{bmatrix}$$

5pts. c) Find the QR decomposition of A.

$$v_1 = \sqrt{2} \, q_1$$

$$v_2 = \frac{1}{\sqrt{2}} [0, 1, 1] \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot q_1 + \frac{1}{\sqrt{6}} [0, 1, 1] \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} q_2$$

$$= \frac{1}{\sqrt{2}} q_1 + \frac{3}{\sqrt{6}} q_2$$

so $R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} \end{bmatrix} = \sqrt{3}/\sqrt{2} = \sqrt{6}/2$