Solutions to Exam 2 MATH 221 Spring 2010

#0: T , T , T , T , T

#1 $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{u} \underbrace{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}_{v^T}$; rank = 1 (all rows multiples of 1st).

#2 $\begin{bmatrix} a & a \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$; dim = # of el. in basis = 3

three elements constituting a basis

#3 rank = # of pivots = 2 ; dim $C(A) = 2$ with basis $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$; $\dim C(A^T) = 2$ with basis $(1\,0\,0\,2)$, $(0\,2\,1\,3)$

$\underbrace{x_3 = 1, x_4 = 0}_{\text{free var}}$: $x_1 = 0$, $2x_2 = -x_3 = -1$ so $\begin{bmatrix} 0 \\ -1/2 \\ 1 \\ 0 \end{bmatrix}$

$x_3 = 0, x_4 = 1$: $x_1 = -2x_4 = -2$, $2x_2 = -x_3 - 3x_4 = -3$ so $\begin{bmatrix} -2 \\ -3/2 \\ 0 \\ 1 \end{bmatrix}$

basis of $N(A)$;

$\dim N(A^T) = 3 - 2 = 1$, $(+1) r_1 + (3) r_2 = (+1, +2, +1, +5) + (3, -6, -3, -3) = (4, -4, -2, -2) = 2r_3$ these were given

so $1 \cdot r_1 + 3 \cdot r_2 - 2r_3 = 0$. Thus $(1, 3, -2)$ is basis of $N(A^T)$

Or, simpler: from $A^T \begin{bmatrix} 1 \\ 3 \\ a \end{bmatrix} = 0$ we get $1 + 3 + 2a = 0$ so $a = -2$.

f) Since $C(A) \perp N(A^T)$: $b \cdot (1, 3, -2) = b_1 + 3b_2 - 2b_3 = 0$

#4: a) $\alpha u + \beta v = (\alpha + \beta, \alpha + \beta, \beta) = (1, 0, 1)$ gives $\beta = 1$, $\alpha + \beta = 1$, $\alpha + \beta = 0$ can't be!

b) $q_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, q_2 = normalized v − proj. of v onto q_1 $= v - q_1^T v \cdot q_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

$\underbrace{\quad}_{2/3}$ normalized

so $q_2 = \underbrace{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}}_{\text{normalized}} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$. Got $Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix}$.

To find R: $u = \sqrt{3} q_1$

$v = q_1^T v \cdot q_1 + q_2^T v \cdot q_2 = \frac{2}{\sqrt{3}} q_1 + \frac{2}{\sqrt{6}} q_2$ so $R = \begin{bmatrix} \sqrt{3} & 2/\sqrt{3} \\ 0 & 2/\sqrt{6} \end{bmatrix}$.

c) $p = q_1^T b\, q_1 + q_2^T b\, q_2 = \frac{2}{\sqrt{3}} \cdot q_1 + \frac{-1}{\sqrt{6}} \cdot q_2 = \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$

d) Solving $Ax = p$

$x_1 + x_2 = 1/2$
$x_1 + x_2 = 1/2$
$x_1 = 1$

so $x_1 = 1$, $x_2 = 1/2 - 1 = -1/2$. Got $x = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$

Rk: Could also use $A^T A x = A^T b$ (but that's more work).

#5: basis of $C(A^T)$: $(1, -2)$ so $A \begin{bmatrix} t \\ -2t \end{bmatrix} = b$ gives $t + 4t = 9$ so $t = 9/5$ giving $x_r = \begin{bmatrix} 9/5 \\ -18/5 \end{bmatrix}$

basis of $N(A)$ is $(2, 1)$ obtained by solving $Ax = 0$. So $x = x_r + x_n = \begin{bmatrix} 9/5 \\ -18/5 \end{bmatrix} + a \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, a scalar.