(1) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$ $\quad \begin{aligned} x_1 &= -x_3 \\ x_2 &= x_3 - x_4 \end{aligned}$ basis: $\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

free $\qquad$ free

(2) $q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$; $q_2 = \text{norm.}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right) = \text{norm} \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \\ 1 \end{bmatrix} = \text{norm} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \sqrt{2}\, q_1$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} q_1 + \frac{5}{\sqrt{10}} q_2$ so $Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{10} \\ 0 & 2/\sqrt{10} \\ 1/\sqrt{2} & -1/\sqrt{10} \\ 0 & 2/\sqrt{10} \end{bmatrix}$, $R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & 5/\sqrt{10} \end{bmatrix}$.

(3) $P = QQ^T = \begin{bmatrix} \frac{1}{2} + \frac{1}{10} & \frac{2}{10} & \frac{1}{2} - \frac{1}{10} & \frac{2}{10} \\ \frac{2}{10} & \frac{4}{10} & -\frac{2}{10} & \frac{4}{10} \\ \frac{1}{2} - \frac{1}{10} & -\frac{2}{10} & \frac{1}{2} + \frac{1}{10} & -\frac{2}{10} \\ \frac{2}{10} & \frac{4}{10} & -\frac{2}{10} & \frac{2}{10} \end{bmatrix}$

proj. onto V

proj. onto $V^{\perp}$ matrix is $I - P$ where P

(4) $b \perp N(A^T)$ so need basis of $N(A^T)$: $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ from $r_1 - r_2 + r_4 = 0$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ from $r_1 + r_3 - r_4 = 0$

Answ: $b_1 - b_2 + b_4 = 0$ and $b_1 + b_3 - b_4 = 0$.

(5) $x \in C(A^T)$ is of the form $x = s[1,1,1,1]^T + t[1,0,1,0]^T = [s+t, s, s+t, s]^T$

which plugged into $Ax = b$ gives $\begin{cases} s+t+s+s+t+s = 6 \\ s+t+s+t = 4 \end{cases}$ i.e. $\begin{cases} 4s+2t = 6 \\ 2s+2t = 4 \end{cases}$

$2s = 2$ so $s = 1, t = 1$

Answer: $x = [2,1,2,1]^T$. NO since $\dim N(A) = 4-2 > 0$.

(6) Yes, with eigenvalue $\lambda^2, \lambda^{-1}, \lambda^2 + \lambda^{-1}$ (This is assuming that A^{-1} exists. Then $\lambda \neq 0$.)

(7) The eigenvectors are independent, even orthogonal if A is symmetric.

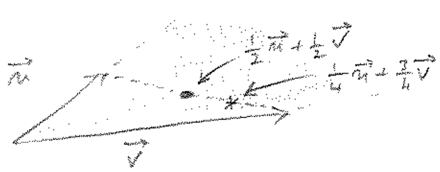
(9) SVD: $A = U \Sigma V^T$ is diagonalization iff $V = U$ and Σ is a square diagonal matrix, say $n \times n$

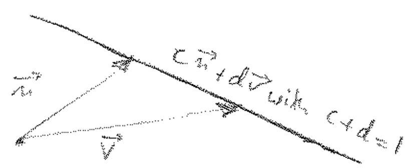
That is A is of the form $A = U \Sigma U^T$ making it symmetric with non-negative eigenvalues.

← Answ

(10) $A = \boxed{}$ 5, 3 so $\dim N(A) = 3 - r$, $\dim N(A^T) = 5 - r$, $\dim C(A) = r = \dim C(A^T)$ where the rank r is one of $0, 1, 2, 3$.

(11)





(12) A is square, say $n \times n$, and so is A^TA. Moreover, A^TA is also invertible because $N(A^TA) = \{0\}$

Indeed, suppose $A^TAx = 0$. Then $x^TA^TAx = 0$, i.e., $(Ax)^TAx = 0$, i.e., $\|Ax\|^2 = 0$ so $Ax = 0$, and application of A^{-1} gives $x = 0$.