#0. T, T, T, F, T

#1 $x = s(1,0,-1)^T + t(1,-1,0)^T = (s+t, -t, -s)^T$, which we plug into $Ax = b$ to get

a) $\begin{cases} s+t-(-s) = 5 \\ s+t-(-t) = 4 \end{cases} \equiv \begin{cases} 2s+t = 5 \Rightarrow 2(4-2t)+t = 5 \\ s+2t = 4 \Rightarrow s = 4-2t \end{cases}$ ↖ PLUG

so $-3t = -3$
so $t = 1$
so $s = 4-2 = 2$

Got $x_r = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$

b) Need $N(A)$: $Ax = 0$ gives $\begin{matrix} x_1 = x_3 \\ x_1 - x_2 = 0 \end{matrix}$ with x_3 free set to 1 yielding $x_1 = 1$, $x_2 = 1$

Complete solution $x = x_r + x_n = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} + a \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

#2

a) $q_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$; $q_2 = \text{norm}\left(v_2 - v_2^T q_1 \cdot q_1\right) = \text{norm}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\cdot 2 \cdot \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right) = \text{norm}\begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$

$q_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ so $Q = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$. Also $v_1 = 2q_1$

$v_2 = v_2^T q_1 \cdot q_1 + v_2^T q_2 \cdot q_2 = q_1 + q_2$

so $R = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

b) The projection is $q_1^T b \cdot q_1 + q_2^T b \cdot q_2 = \frac{1}{2} \cdot 4 q_1 + \frac{1}{2} 4 q_2 = 2q_1 + 2q_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$

c) $x = R^{-1} Q^T b = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

#3

a) $\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{① - 2③} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$

b) $|A| = (-1)\cdot 1\cdot 1\cdot 1$ with the (-1) due to the row switch

c) $A - \lambda I = \begin{bmatrix} -1-\sqrt{2} & 0 & 1 \\ 0 & -\sqrt{2} & 0 \\ 1 & 0 & 1-\sqrt{2} \end{bmatrix}$ Can ignore the third equation since it must be dependent so $x_3 = (1+\sqrt{2})x_1$, $x_2 = 0$ so $V = \begin{bmatrix} 1 \\ 0 \\ 1+\sqrt{2} \end{bmatrix}$

d) Use that the outside matrices are transposes of each other to fill the blanks: $\boxed{\begin{matrix} \frac{1}{4+2\sqrt{2}} \\ 0 \\ \frac{1+\sqrt{2}}{4+2\sqrt{2}} \end{matrix}}$ and $\boxed{1 \mid 0 \mid 1}$

e) $|A| < 0$ so no.

#4:

a) $AA^T = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}$ $T = 7$, $D = 2\cdot 5 - 2\cdot 2 = 6$ $\lambda^2 - 7\lambda + 6 = 0$, $(\lambda-6)(\lambda-1) = 0$ so $\lambda_1 = 6$, $\lambda_2 = 1$.

Singular values: $\sigma_1 = \sqrt{6}$, $\sigma_2 = \sqrt{1}$

b) The semi-axes are $\sqrt{6}$ and $\sqrt{1}$

c) u_1 from $AA^Tu_1 = 6u_1$ so $AA^T - 6I = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$ so $x_1 = 2x_2$

$u_1 =$ normalized $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

d) v_1 from $A^TAv_1 = 6v_1$ which is $\begin{bmatrix} 5 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} v_1 = 6v_1$

#5. From $(0,2,0,-1)^T \in N(A^T)$ $2r_2 - r_4 = 0$ so $r_4 = 2r_2 = (2,0)$

b) $b^T(0,2,0,-1) = 0$ and $b^T(1,0,-1,0) = 0$ so $2b_2 = b_4$ and $b_1 = b_3$

#6. a) $\det(A) = 2\cdot 1\cdot(-1) = -2$ b) 2^2+1, 1^2+1, $(-1)^2+1$

c) It must since the eigenvalues are distinct. d) It must since A is.

#7 $\begin{cases} x = 1 \\ x = 2 \end{cases}$ $\quad \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{A} [x] = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{b}$ so $A^T A = [2]$, $A^T b = 3$

so <u>normal equations</u>: $A^T A x = A^T b$ are $2x = 3$ so $x = 3/2$

#8 a) $\det(A) = 0$, $\text{rank}(A) = \#$ of pivots $= 3$, $\dim(N(A)) = 4-3$, $\dim(N(A^T)) = 4-3$

b) $C(A)$: $\begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ OR $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, c) $C(A^T)$: $(2,1,1,1)$, $(0,1,0,1)$, $(0,0,0,1)$.

d) This is $N(A)$: $2x_1 + x_2 + x_4 = -x_3$ ~ free setting $x_3 = 1$

$x_2 + x_4 = 0 \Rightarrow x_2 = 0$ we get

$x_4 = 0$ $\quad x_1 = -\frac{1}{2}$

<u>Answer</u>: $\text{lin}\begin{bmatrix} -1/2 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \text{lin}\begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ <u>NOTE</u>: Could have guessed this one.

#9 $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} a_{n+1} + \frac{1}{2} a_n \\ a_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$ with $T = \frac{1}{2}$; $D = -\frac{1}{2}$

$\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = (\lambda - 1)(\lambda + \frac{1}{2}) = 0$ so $\lambda_1 = 1$, $\lambda_2 = -\frac{1}{2}$

$A - \lambda_1 I = \begin{bmatrix} -1/2 & 1/2 \\ 1 & -1 \end{bmatrix}$ so $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $A - \lambda_2 I = \begin{bmatrix} 1 & 1/2 \\ 1 & 1/2 \end{bmatrix}$ so $v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

so $S = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$, $S^{-1} = \frac{1}{-3}\begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

$\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = A^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & (-1/2)^n \end{bmatrix} \cdot \frac{1}{3} \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

$= \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -(-\frac{1}{2})^n \end{bmatrix} = \frac{1}{3}\begin{bmatrix} \cdots \\ 1 + 2(-\frac{1}{2})^n \end{bmatrix}$ so $a_n = \frac{1}{3} + \frac{2}{3}(-\frac{1}{2})^n$

$\lim_{n\to\infty} a_n = \frac{1}{3} + \lim_{n\to\infty} \frac{2}{3}(-\frac{1}{2})^n = \frac{1}{3} + 0 = 0.$