Solutions to Review Problems:

Sol 1:

a) $\begin{bmatrix}1&2&3\\2&3&2\\3&2&-9\end{bmatrix} \rightsquigarrow \begin{matrix}②-2\cdot①\\③-3\cdot①\end{matrix} \begin{bmatrix}1&2&3\\0&-1&-4\\0&-4&-18\end{bmatrix} \rightsquigarrow ③-4② \begin{bmatrix}1&2&3\\0&-1&-4\\0&0&-2\end{bmatrix} = U$

$L = \begin{bmatrix}1&&\\2&1&\\3&4&1\end{bmatrix}$ got LU factorization: $L\cdot U = \begin{bmatrix}1&&\\2&1&\\3&4&1\end{bmatrix}\begin{bmatrix}1&2&3\\&-1&-4\\&&-2\end{bmatrix} = \begin{bmatrix}1&2&3\\2&3&2\\3&2&-9\end{bmatrix} = A$

CHECKED BY MULTIPLYING

LDU factorization: $\begin{bmatrix}1&&\\2&1&\\3&4&1\end{bmatrix}\begin{bmatrix}1&&\\&-1&\\&&-2\end{bmatrix}\begin{bmatrix}1&2&3\\&1&4\\&&1\end{bmatrix}$

NOTE: transposes of each other since A is symmetric!

b) $Ax = b$ becomes $Lc = b$ and $Ux = c$, that is:

$$\begin{cases} c_1 = 5 \\ 2c_1 + c_2 = 14 \Rightarrow c_2 = 14 - 2\cdot 5 = 4 \\ 3c_1 + 4c_2 + c_3 = 21 \Rightarrow c_3 = 21 - 4\cdot 4 - 3\cdot 5 = -10 \end{cases}$$

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \Rightarrow x_1 = 5 - 2\cdot(-24) - 3\cdot 5 = 38 \\ -x_2 - 4x_3 = 4 \Rightarrow x_2 = -4 - 4\cdot 5 = -24 \\ -2x_3 = -10 \Rightarrow x_3 = 5 \end{cases}$$

c) Gauss El. $\rightarrow$ $\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\2&3&2&0&1&0\\3&2&-9&0&0&1\end{array}\right] \rightsquigarrow \begin{matrix}②-2\cdot①\\③-3\cdot①\end{matrix} \left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&-1&-4&-2&1&0\\0&-4&-18&-3&0&1\end{array}\right] \rightsquigarrow ③-4② \left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&-1&-4&-2&1&0\\0&0&-2&5&-4&1\end{array}\right]$

continues to Jordan El. $\rightarrow$ $\rightsquigarrow \begin{matrix}①+\frac{3}{2}\cdot③\\②-2\cdot③\end{matrix} \left[\begin{array}{ccc|ccc}1&2&0&17/2&-6&3/2\\0&-1&0&-12&9&-2\\0&0&-2&5&-4&1\end{array}\right] \rightsquigarrow ①+2② \left[\begin{array}{ccc|ccc}1&0&0&-31/2&12&-5/2\\0&-1&0&-12&9&-2\\0&0&-2&5&-4&1\end{array}\right]$

divide by pivots $\left[\begin{array}{ccc|ccc}1&0&0&-31/2&12&-5/2\\0&1&0&12&-9&2\\0&0&1&-5/2&2&1/2\end{array}\right]$ Got: A^{-1}

To solve $Ax = b$: $x = A^{-1}b = \begin{bmatrix}-31/2&12&-5/2\\12&-9&2\\-5/2&2&1/2\end{bmatrix}\begin{bmatrix}5\\14\\21\end{bmatrix}$

Got $x = \begin{bmatrix}-\frac{155}{2}+168-\frac{105}{2}\\60-126+42\\-\frac{25}{2}+28+\frac{21}{2}\end{bmatrix} = \begin{bmatrix}38\\-24\\5\end{bmatrix}$

Same as before!

SHOULD CHECK $\begin{bmatrix}-31/2&12&-5/2\\12&-9&2\\-5/2&2&1/2\end{bmatrix}\begin{bmatrix}1&2&3\\2&3&2\\3&2&-9\end{bmatrix} = I$

Sol (2):

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & a & 2 \\ 3 & 2 & -9 \end{bmatrix} \rightsquigarrow \begin{matrix} ② - 2① \\ ③ - 3① \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & a-4 & -4 \\ 0 & -4 & -18 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -18 \\ 0 & 0 & -4 \end{bmatrix} = U$$

row switch for $a=4$

b)

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{= P} A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & -4 & \\ & & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 9/2 \\ 0 & 0 & 1 \end{bmatrix}$$

note that 3 and 2 are switched because upon applying P to A first we get rows ② and ③ switched in PA.

In other words for

$$PA = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & -9 \\ 2 & 4 & 2 \end{bmatrix} \rightsquigarrow \begin{matrix} ② - 3① \\ ③ - 2① \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -18 \\ 0 & 0 & -4 \end{bmatrix}$$

c) $U \neq L^T$ because PA is not symmetric.

Sol (3):

$$P = \begin{bmatrix} & & & 1 \\ 1 & & & \\ & & 1 & \\ & 1 & & \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} & 1 & & \\ & & & 1 \\ & & 1 & \\ 1 & & & \end{bmatrix}$$

since it sends

$4 \longmapsto 1$
$1 \longmapsto 2$
$3 \longmapsto 3$
$2 \longmapsto 4$

(also $P^{-1} = P^T$ for permutation matrices.).

Sol (4):

$$E^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \Big(\underbrace{(2I)\cdot I - 0 \cdot E}_{\text{inverse determinant}}\Big)^{-1} \cdot \left[\begin{array}{c|c} I & -0 \\ \hline -E & 2I \end{array}\right]$$

so

$$A^{-1} = \frac{1}{2} \left[\begin{array}{c|c} I & 0 \\ \hline -E & 2I \end{array}\right] = \left[\begin{array}{cc|cc} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ \hline -1/2 & 0 & 1 & 0 \\ -1 & -1/2 & 0 & 1 \end{array}\right]$$

Sol (5):

$$B(B^{-1}A^{-1}A + I) - B = BB^{-1} + B - B = I$$

$$A(B^{-1})^T(A^T + B^T)(A^{-1})^T = A(B^T)^{-1}A^T(A^T)^{-1} + A(B^T)^{-1}B^T(A^T)^{-1}$$
$$= A(B^T)^{-1} + A(A^T)^{-1}$$

Sol ⑥: a) col. 2 of AB $= A \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

col 2 of B

b) row 2 of AB $= \underbrace{\begin{bmatrix} 3 & -2 \end{bmatrix}}_{\text{2 row of } A} \begin{bmatrix} 1 & 0 & 4 \\ 1 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

c) row 2 of $A^2 = \underbrace{\begin{bmatrix} 3 & -2 \end{bmatrix}}_{\text{row 2 of } A} \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$

row 2 of $A^3 = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{\text{row 2 of } A^2} \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \end{bmatrix}$

Sol ⑦: $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 8 \\ 3 & 0 & 12 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 18 \\ -2 & 0 & -12 \end{bmatrix}$

$= \begin{bmatrix} 5 & 0 & 26 \\ 1 & 0 & 0 \end{bmatrix}$

Sol ⑧:

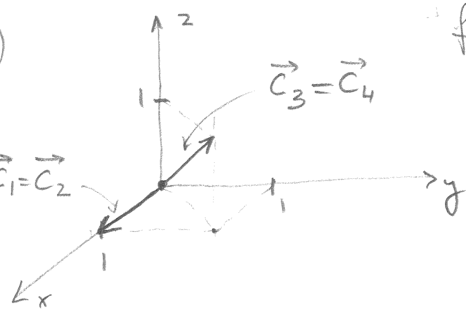
a) pivots

$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 0 & 1 & 1 \\ & & 1 & 1 \end{bmatrix}$

b) $(0, 0, 1)$ (since eq. 2 and 3 clash!)

c) $(0, 0, 0)$ or $(2, 2, 2)$ or ... do the job.

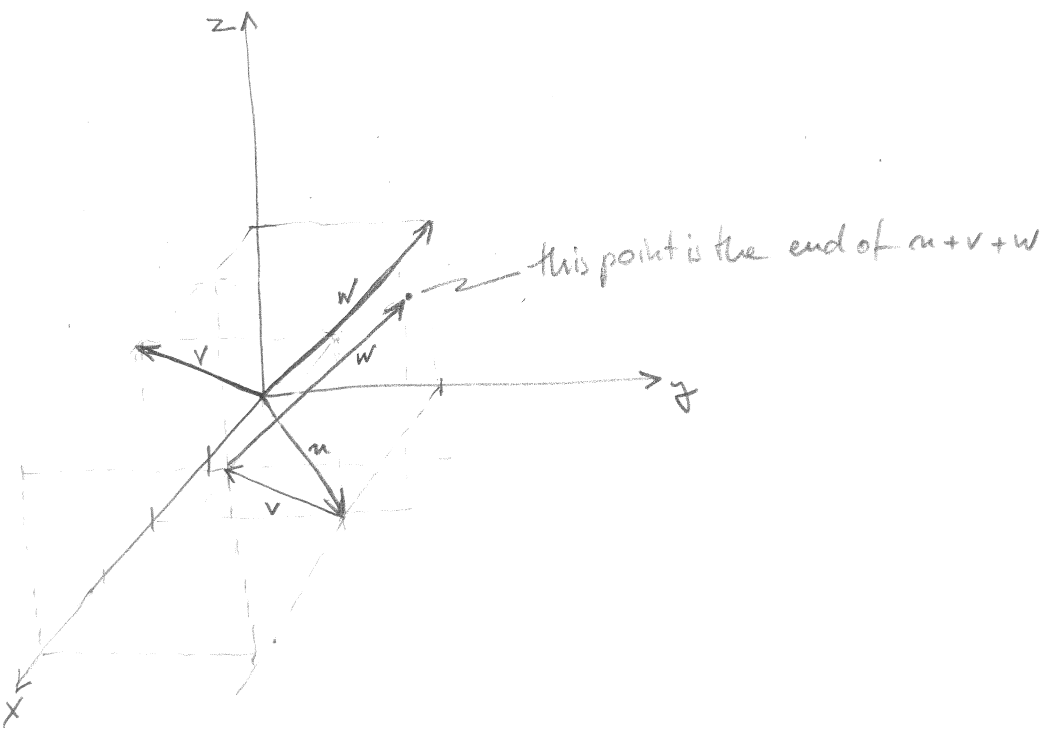
d)



four coplanar vectors: $\vec{c}_1, \vec{c}_2, \vec{c}_3, \vec{c}_4$ of which first two and last two coincide.

e) three 3D subspaces in 4D.

sol (9)



sol (10): Checking for zero dot product: $\mu^2 + 2\mu - 6 = 0$ so $\mu = \dfrac{-2 \pm \sqrt{28}}{2}$

sol (11): a point: all zero; a line: all equal; a plane: third a comb. of two

say $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\1\\2\end{pmatrix}$

(but make sure not collinear)

all of 3D: $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$

or any "random" choice.

$2\vec{u} - \vec{v} = \vec{w}$ so not all 3D. Since not collinear, must form a plane.

sol (12):