MATH 221 Final Review Part I

(Some "heavier" problems)

① For $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 1 \end{bmatrix}$, find the singular decomposition.

What is the effect of applying A to the unit ball in $\mathbb{R}^3$?

(ball of radius = 1)

Note: This is cooked to be easy. For a more realistic example, tweak one entry and try again.

② What symmetric matrix A gives

$$x^T A x = x_1^2 + 2x_2^2 + 4x_3^2 - 4x_1x_2 + 2x_2x_3$$

a) Perform the LDL^T decomposition of A and use it to complete square. Is A positive definite? (Check in several different ways.)

b) Compute the eigenvalues and eigenvectors and diagonalize A.

c) Modify the coefficient at x_2^2 to make A positive definite. Attempt to find the eigenvalues, if only approximately using a calc.!

! **Extra** (for those who took multivar. calc) What figure is given by $x^T A x = 1$? both in a) and c)

Fin. Dec 2002 ③ $P = \frac{1}{21}\begin{bmatrix} 1 & 2 & -4 \\ 2 & 4 & -8 \\ -4 & -8 & 16 \end{bmatrix}$ is a projection matrix (it satisfies $P^2 = P$) and is symmetric.

a) Find a basis of the space P projects onto; call it V.

b) What is the distance from $b = (1,1,1)$ to this subspace V.

c) What is the projection matrix Q onto $V^\perp$ (Hint: The projection of x onto $V^\perp$ is $x - Px$. Draw a picture.)

d) Factor P into $u \cdot v^T$ where u, v are vectors.

e) What are the eigenvalues of P? Can it be diagonalized?

④ On planet Markovia the weather follows the following rule: rain today implies 50/50 percent chance of rain or shine tomorrow, sunshine today implies 10/90 chance of rain or shine tomorrow.

a) Write a matrix A so that if today's chances of rain or shine are given by $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then tomorrow's are $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. (remember the example from class?)

b) You are going to arrive on Markovia some time in 2013. What is your best bet guess for the weather on that day

(Hint: Compute eigenvectors of A.).

⑤ A sequence a_1, a_2, a_3 is given by $a_1 = 4$, $a_2 = 1$ and then by the recursive formula $a_{n+2} = \frac{3}{2}a_{n+1} - \frac{1}{2}a_n$. (for $n = 0, 1, 2, \ldots$)
Find the formula for a_n and the $\lim_{n\to\infty} a_n$

⑥ a) Use the "big formula for det" to find det of $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$
(If you like, you can draw the associated graph.)

b) Explain why the above det equals det $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$. (Hint: row operations

c) Use Gauss El. to find det A, then do this again with Laplace expansion. (about a row or column of your choosing).

d) Find A^{-1} in two different ways; by Gauss-Jordan, by cofactors.

⑦ a) Find QR decomp of $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

b) Find orthonormal basis of the subspace spanned by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$. (use a)

c) Find the least squares solution to $\begin{cases} x_1 = 2 \\ x_2 = 0 \\ x_1 + x_2 = 1 \end{cases}$

Solutions (to Final Review PART I (more coming))

① $A^TA = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ $AA^T = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Char Poly = ... unplesant computation... Char Poly $(\lambda-2)(\lambda-3)$ since diagonal.

COPY

OR cleverly $= -\lambda(\lambda-2)(\lambda-3)$

(because the non zero eigenvalues of AA^T and A^TA are the same!) And, of course, char poly AA^T has to be of degree three.

We got singular values $\sigma_2 = \sqrt{2}$, $\sigma_1 = \sqrt{3}$. Now we need eigenvect. of A^TA:

$\lambda_1 = 3$: $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{matrix} \rightarrow x_1 = x_2 \\ \\ \rightarrow x_2 + x_3 = 0 \end{matrix}$ so $v_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ (normalized eigenvector)

$\lambda_2 = 2$: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{matrix} \rightarrow x_2 = 0 \\ \rightarrow x_1 = x_3 \end{matrix}$ so $V_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_3 = 0$: $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{matrix} \rightarrow x_1 = -\frac{1}{2}x_2 \\ \\ \rightarrow x_2 = 2x_3 \end{matrix}$ so $V_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

Thus $V = [v_1, v_2, v_3]$ and $U = [u_1, u_2]$ where

$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ -3 \end{bmatrix}$ $u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Answer: Singular Decomp of A:

$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

3×3 3×2

Solutions:

②
$$A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix} \underset{②-(-2)①}{\rightsquigarrow} \begin{bmatrix} 1 & -2 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & 4 \end{bmatrix} \underset{③-(\frac{-1}{2})②}{\rightsquigarrow} \underbrace{\begin{bmatrix} 1 & -2 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 9/2 \end{bmatrix}}_{U}$$

a)
$$A = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 1 & & \\ & -2 & \\ & & \frac{9}{2} \end{bmatrix}} \underset{L^T}{\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}}$$

pivots: $1, \underset{\uparrow <0 \text{ so not pos. def}}{-2}, 9/2$

Thus $x^TAx = 1\cdot(x_1 - 2x_2)^2 - 2\cdot(x_2 - \frac{1}{2}x_3)^2 + \frac{9}{2}x_3^2$

not pos. definite because:
- pivot $-2 < 0$
- $\begin{vmatrix} 1 & -2 \\ -2 & 2 \end{vmatrix} = 2 - 4 < 0$
- $x^TAx = -2$ if $\begin{cases} x_1 - 2x_2 = 0 \\ x_2 - \frac{1}{2}x_3 = 1 \\ x_3 = 0 \end{cases}$

i.e. $x_3 = 0, x_2 = 1, x_1 = 2$.

b) see next page

c) Take $x_1^2 + 5x_2^2 + 4x_3^2 - 4x_1x_2 + 2x_2x_3$ for which

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 1 \\ 0 & 1 & 4 \end{bmatrix} \underset{②-(-2)①}{\rightsquigarrow} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 4 \end{bmatrix} \underset{③-②}{\rightsquigarrow} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$ pivots: $1, 1, 3 > 0$

Char poly:
$$\begin{vmatrix} 1-\lambda & -2 & 0 \\ -2 & 5-\lambda & 1 \\ 0 & 1 & 4-\lambda \end{vmatrix} = (1-\lambda)(5-\lambda)(4-\lambda) - 1\cdot 1\cdot(1-\lambda) - (4-\lambda)(-2)(-2)$$
$$= -\lambda^3 + \lambda^2(4+1+5) - \lambda(20+4+5) + 20 - 1 + \lambda - 16 + 4\lambda$$
$$= -\lambda^3 + 10\lambda^2 - 24\lambda + 3 = 0$$ with approx sol:

$\lambda_1 \approx 6.218,\ \lambda_2 \approx 3.650,\ \lambda_3 \approx 0.132$

Sol to extra:

in a) $x^TAx = 1$ represents a hyperboloid of one sheet

in c) $x^TAx = 1$ represents an ellipsoid with semiaxes $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}$.

Solutions:

② b) Char Poly is $-\lambda^3 + 7\lambda^2 - 9\lambda - 9$, which (luckily) has one easy solution $\lambda = 3$ so it factors via long division

$$\begin{array}{l} \quad\ \lambda^2 - 4\lambda - 3 \\ \hline \lambda^3 - 7\lambda^2 + 9\lambda + 9 \div (\lambda - 3) \\ -\ \lambda^3 + 3\lambda^2 \\ \hline \qquad -4\lambda^2 + 9\lambda \\ \qquad +4\lambda^2 - 12\lambda \\ \hline \qquad\qquad -3\lambda + 9 \\ \qquad\qquad\ \ 3\lambda - 9 \\ \hline \qquad\qquad\quad 0 \end{array}$$

so $\ldots = -(\lambda - 3)(\lambda^2 - 4\lambda - 3) = 0$; $\lambda_{2,3} = \frac{4 \pm \sqrt{28}}{2}$

Got eigenvalues: $\lambda_1 = 3$, $\lambda_2 = 2 + \sqrt{7}$, $\lambda_3 = 2 - \sqrt{7}$

Eigenvectors of $\lambda_1 = 3$:

$$\begin{bmatrix} -2 & -2 & 0 \\ -2 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \begin{array}{l} \longleftarrow 1^{st} : x_1 + x_2 = 0 \\ 2^{nd} : \text{redundent} \\ \longleftarrow 3^{rd} \quad x_2 + x_3 = 0 \end{array} \quad \text{so } v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

is OK!

Eig. of $\lambda_3 = 2 - \sqrt{7}$:

$$\begin{bmatrix} -1+\sqrt{7} & -2 & 0 \\ -2 & \sqrt{7} & 1 \\ 0 & 1 & 2+\sqrt{7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \begin{array}{l} \longleftarrow 1^{st} \quad x_1 = \frac{2x_2}{\sqrt{7}-1} \\ \longleftarrow 2^{nd} \text{ redundent} \\ \longleftarrow 3^{rd} \quad x_2 = -(2+\sqrt{7})x_3 \end{array} \quad \text{so } v_3 = \begin{bmatrix} -3-\sqrt{7} \\ -2-\sqrt{7} \\ 1 \end{bmatrix}$$

Eig. of $\lambda_2 = 2 + \sqrt{7}$: ... similar as above (with $-\sqrt{7} \Leftrightarrow +\sqrt{7}$) $\quad v_2 = \begin{bmatrix} -3+\sqrt{7} \\ -2+\sqrt{7} \\ 1 \end{bmatrix}$

Normalized eigenvectors form the diagonalizing matrix

$$Q = \begin{bmatrix} 1/\sqrt{3} & \frac{-3+\sqrt{7}}{2\sqrt{7-\sqrt{7}}} & \frac{-3-\sqrt{7}}{2\sqrt{7+\sqrt{7}}} \\ -1/\sqrt{3} & \frac{-2+\sqrt{7}}{2\sqrt{7-\sqrt{7}}} & \frac{-2-\sqrt{7}}{2\sqrt{7+\sqrt{7}}} \\ 1/\sqrt{3} & 1/2\sqrt{7-\sqrt{7}} & 1/2\sqrt{7+\sqrt{7}} \end{bmatrix} \quad \text{so that } Q^T A Q = \begin{bmatrix} 3 & & \\ & 2+\sqrt{7} & \\ & & 2-\sqrt{7} \end{bmatrix}$$

Solutions :

③ a) three colinear columns so $C(P)$ 1-dim with basis $\begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$

b) $Pb = \frac{1}{21}\begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}$ is the projection so the distance is $\| b - Pb \|$ equal to

$$\sqrt{\left(1-\frac{-1}{21}\right)^2+\left(1-\frac{-2}{21}\right)^2+\left(4-\frac{4}{21}\right)^2} = \frac{\sqrt{(22)^2+(23)^2+(80)^2}}{21}$$

c) $x - Px = (I-P)x$ so it is $I - P = \ldots$

d) $P = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} \begin{bmatrix} 1/21 & 2/21 & -4/21 \end{bmatrix}$

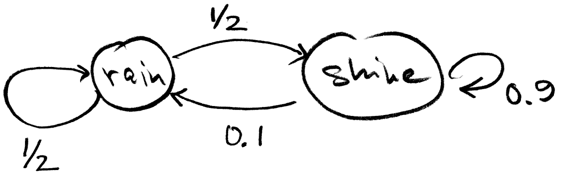
e) As a projection P has eigenvalues that are either 0 or 1.
(If you forgot, this is because $P^2 = P$ so $\lambda^2 = \lambda$ for any eigenvalue of P)
Since rank$(P) = 1$, dim $N(P) = 3-2$ so $\lambda = 0$ is a double eigenvalue
The third eigenvalue must be 1.
(So the characteristic poly of P is $-\lambda^2(\lambda-1)$.)

LOOK : We never computed any determinants!

Finally, P is diagonalizable because it is symmetric.

Solutions:

(4)



$$A = \begin{bmatrix} .5 & .1 \\ .5 & .9 \end{bmatrix}$$

so $A\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix} .5\cdot x_1 + 0.1\cdot x_2 \\ .5\cdot x_1 + 0.9\cdot x_2 \end{bmatrix}$ ok!

Eigenvalues of A: $\lambda^2 - 1.4\lambda + (0.45 - 0.5) = 0$

$$\lambda_{1,2} = \frac{1.4 \pm \sqrt{(1.4)^2 - 4\cdot 0.4}}{2} = \frac{1.4 \pm 0.6}{2} = 0.4 \text{ and } 1$$

$\lambda_1 = 1$ has eigenvector $\begin{bmatrix} -0.5 & .1 \\ .5 & -.1 \end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = 0$ so $.5x_1 = .1x_2$ so $v_1 = \begin{bmatrix}2\\10\end{bmatrix}$

After n-days the chances of rain or shine are

$A^n\begin{bmatrix}x_1\\x_2\end{bmatrix}$ which gets closer and closer aligned with the line of the "dominant" eigenvector v_1 i.e with constant. $\begin{bmatrix}2\\10\end{bmatrix}$

↑ initial chances

Since probabilitiess add up to 1, the limiting probability (when $n \to \infty$) is $\begin{bmatrix}\frac{2}{12}\\[4pt] \frac{10}{12}\end{bmatrix}$.

<u>Answer</u> Sunshine with probability $10/12$.

(<u>NOTE</u>: I did not have to compute v_2 for this. Knowing that $\lambda_2^n = 0.4^n \xrightarrow[n\to\infty]{} 0$ was enough to draw the conclusion.

If you do not understand this, draw a complete figure of the action of A! As done in class.)

Solutions: (to Final Review PART II)

⑤ So $$\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 3/2 & -1/2 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$$ with $u_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$$u_{n+1} = A \cdot u_n$$

eigenval of A: $\lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = 0$ so $\lambda_{1,2} = \dfrac{\frac{3}{2} \pm \sqrt{\frac{9}{4} - 2}}{2}$

$\lambda_1 = \dfrac{3/2 + 1/2}{2} = 1$: $\begin{bmatrix} 1/2 & -1/2 \\ 1 & -1 \end{bmatrix}$ so $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = \dfrac{3/2 - 1/2}{2} = \frac{1}{2}$ $\begin{bmatrix} 1 & -1/2 \\ 1 & -1/2 \end{bmatrix}$ so $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ so $S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$$u_n = A^{n-1} u_1 = S \Lambda^{n-1} S^{-1} u_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1^{n-1} & 0 \\ 0 & \left(\frac{1}{2}\right)^{n-1} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & (1/2)^{n-1} \\ 1 & (1/2)^{n-2} \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 + \left(\frac{1}{2}\right)^{n-1} \cdot 3 \\ -2 + (1/2)^{n-2} \cdot 3 \end{bmatrix}$$

so $a_n = 2^{nd}$ component of $u_n = -2 + \dfrac{3}{2^{n-2}}$ so $\lim_{n \to \infty} a_n = -2$

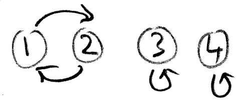
⑥ a) To use big formula we list all possible ways of picking an entry from each row so that all columns are represented.

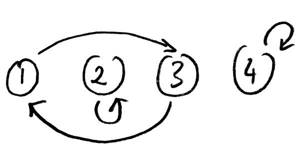
These are

$$\begin{array}{|c|c|c|c|}\hline & 1 & & \\ \hline 1 & & & \\ \hline & & -1 & \\ \hline & & & -1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|}\hline & & 1 & \\ \hline & -1 & & \\ \hline 1 & & & \\ \hline & & & -1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|}\hline & & & 1 \\ \hline & -1 & & \\ \hline & & -1 & \\ \hline 1 & & & \\ \hline \end{array}$$

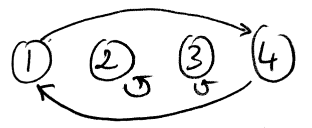
$$= \underbrace{(-1)^1}_{} \cdot 1 \cdot 1 \cdot (-1) \cdot (-1) + (-1)^1 \cdot 1 \cdot (-1) \cdot 1 \cdot (-1) + (-1)^1 \cdot 1 \cdot (-1)(-1) \cdot 1$$

because there is only one transposition:





one transposition

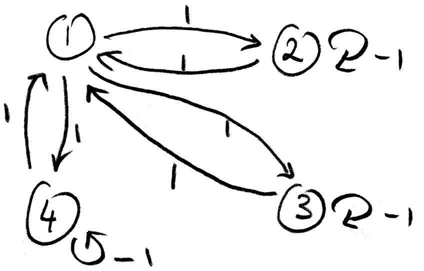


one transposition

$$= -3$$

⑥ cont BTW, the associated graph is

(optional →)



b) Just add row 1 to all other rows. Doing so does not affect det.

c, d) Gauss – Jordan el.:

ivot →

$$\left[\begin{array}{cccc|cccc} & 1 & 1 & 1 & 1 & & & \\ 1 & -1 & & & & 1 & & \\ 1 & & -1 & & & & 1 & \\ 1 & & & -1 & & & & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccccccc} & 1 & 1 & 1 & 1 & & & \\ 1 & -1 & & & & 1 & & \\ 0 & 1 & -1 & & & -1 & 1 & \\ 0 & 1 & & -1 & & -1 & & 1 \end{array}\right] \xrightarrow[\text{swith}]{\text{row}} \left[\begin{array}{cccccccc} 1 & -1 & & & & 1 & & \\ & 1 & -1 & & & -1 & 1 & \\ & 1 & & -1 & & -1 & & 1 \\ & 1 & 1 & 1 & 1 & & & \end{array}\right]$$

1st pivot

2nd pivot

$$\rightsquigarrow \left[\begin{array}{cccccccc} 1 & -1 & & & & 1 & & \\ & 1 & -1 & & & -1 & 1 & \\ & & 1 & -1 & & 0 & -1 & 1 \\ & & 2 & 1 & 1 & 1 & -1 & \end{array}\right] \rightsquigarrow \left[\begin{array}{cccccccc} 1 & -1 & & & & 1 & & \\ & 1 & -1 & & & -1 & 1 & \\ & & 1 & -1 & & & -1 & 1 \\ & & & 3 & 1 & 1 & 1 & -2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & & & & 1/3 & 1/3 & 1/3 & 1/3 \\ & 1 & & & 1/3 & -2/3 & 1/3 & 1/3 \\ & & 1 & & 1/3 & 1/3 & -2/3 & 1/3 \\ & & & 1 & 1/3 & 1/3 & 1/3 & -2/3 \end{array}\right]$$

3rd pivot

4th pivot

← This is A^{-1}

Also $\det A = \underbrace{1 \cdot 1 \cdot 1 \cdot 3}_{\text{product of pivots}} \cdot (-1)^3 = -3$ OK!

↖ sign change due to row ~~switch~~ permutation (actually 3 switches)

Inverse by cofactors:

$$A^{-1} = \frac{1}{-3} \left[\begin{array}{c|c|c|c} + & - & + & - \\ \hline - & + & - & + \\ \hline + & - & + & - \\ \hline - & + & - & + \end{array}\right] \leftarrow \text{sixteen cofactors}$$

For instance this is $\left|\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{array}\right| = 1$ etc.

↖ I just indicated signs before 3×3 dets like ↰.

Laplace expansion:

$$|A| = (-1)^{1+2} \begin{vmatrix} 1 & & \\ 1 & -1 & \\ 1 & & -1 \end{vmatrix} + (-1)^{1+3} \begin{vmatrix} 1 & -1 & \\ 1 & & \\ 1 & & -1 \end{vmatrix} + (-1)^{1+4} \begin{vmatrix} 1 & -1 & \\ 1 & & -1 \\ 1 & & \end{vmatrix}$$

$$= -1 + (-1) + (-1) = -3$$

⑦ a) $q_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}$; $q_2 =$ normalized $\begin{bmatrix}2\\0\\1\end{bmatrix} - \frac{1}{\sqrt{2}}\cdot 3\,\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}$

so $q_2 =$ normalized $\begin{bmatrix}2\\0\\1\end{bmatrix} - \begin{bmatrix}3/2\\0\\3/2\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}$

$q_3 = v_3 - (q_1^T v_3)\cdot q_1 - (q_2^T v_3) q_2$ normalized

$= \;.\,.\,.\,.\,.\; = \begin{bmatrix}0\\1\\0\end{bmatrix}$ Got $Q = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\ 0 & 0 & 1\\ 1/\sqrt{2} & -1/\sqrt{2} & 0\end{bmatrix}$

To find R: $v_1 = \sqrt{2}\, q_1$

$v_2 = q_1^T\cdot v_2 \cdot q_1 + q_2^T v_2 \cdot q_2 = \frac{3}{\sqrt{2}} q_1 + \frac{1}{\sqrt{2}} q_2$

$v_3 = q_1^T v_3 q_1 + q_2^T v_3 q_2 + q_3^T v_3 q_3 = \frac{1}{\sqrt{2}} q_1 - \frac{1}{\sqrt{2}} q_2 + q_3$

so $R = \begin{bmatrix}\sqrt{2} & & \\ 3/\sqrt{2} & 1/\sqrt{2} & \\ 1/\sqrt{2} & -1/\sqrt{2} & 1\end{bmatrix}^T$ Should be upper triangular

b) Just $q_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}$ and normalized $\begin{bmatrix}0\\1\\1\end{bmatrix} - \frac{1}{\sqrt{2}}\cdot 1\cdot q_1 = \begin{bmatrix}0\\1\\1\end{bmatrix} - \begin{bmatrix}1/2\\0\\1/2\end{bmatrix}$

$= \begin{bmatrix}-1/2\\1\\1/2\end{bmatrix}$ normalized $= \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\2\\1\end{bmatrix}$

Answ: These two vectors

c) This is $Ax = b$ for $b = \begin{bmatrix}2\\0\\1\end{bmatrix}$ and $A = \begin{bmatrix}1 & 0\\0 & 1\\1 & 1\end{bmatrix}$

Least squares equation: $A^T A x = A^T b$ is: $\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$

so $\begin{bmatrix}x_1\\x_2\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}5/3\\-1/3\end{bmatrix}$

answer: $x_1 = 5/3$, $x_2 = -1/3$