0	1	2	3	4	5	6	7	
10	10	25	10	8	6	8	8	85

Math 221 First Exam (12 Feb 2010)

Name:

No calculators. Circle the answers. Show all work (unless instructed otherwise).
Good Luck!

0.[10pts] Circle **True** or **False** without explanation:

(**T** or **F**) The row picture for a system of two equations in three unknowns consists of two planes in the 3D space.

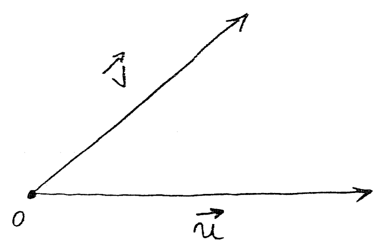
(**T** or **F**) For any two square matrices, $(A+B)^2 = A^2 + 2AB + B^2$.

(**T** or **F**) For any vector, $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$.

(**T** or **F**) Knowing the solutions to $A\mathbf{x} = \mathbf{b}$ for two right sides $\mathbf{b} = (1,0)^T$ and $\mathbf{b} = (0,1)^T$, is sufficient to determine the inverse A^{-1} (for a 2×2 matrix A).

(**T** or **F**) If $A\mathbf{x} = \mathbf{b}$ has a solution for some $\mathbf{b}$ then A is non-singular.

1.[10pts] Consider the vectors $\mathbf{u}$ and $\mathbf{v}$ depicted below:



a) Draw the vector $\mathbf{u} - \mathbf{v}$.

b) Lightly shade the region filled by all the linear combinations $c\mathbf{v} + d\mathbf{u}$ where $0 \le c \le 1$ and $0 \le d \le 1$.

c) Put a visible dot at the end of the vector $\frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{v}$.

2.[25pts] Perform the Gauss-Elimination on

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 4 \end{bmatrix}$$

a) Write out the two elimination matrices E (one for each step).

b) The LU factorization is $A =$

c) If you were to use the LU factorization to solve $A\mathbf{x} = \mathbf{b}$, what would be the two systems that you would solve? (Do not solve them!)

d) Use Gauss-Jordan elimination to find A^{-1}, the inverse of

$$\left[\begin{array}{ccc|} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 4 \end{array}\right.$$

e) Use A^{-1} to solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = (1, 1, 1)^T$.

$\mathbf{x} =$

3.[10pts] Suppose that A and B are $n \times n$ matrices. Simplify (if you can) the following matrix products:

a) $(AB)^{-1}B =$

b) $(AB)^{-1}A =$

c) $(A^{-1}B)^T A^T =$

4.[8pts] Write the following linear combinations as a product of a suitable matrix by the vector (c, d, e) treated as a 3×1 or 1×3 matrix, accordingly.

$$c \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + d \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + e \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} =$$

$$c \begin{bmatrix} 1, & 2, & 3 \end{bmatrix} + d \begin{bmatrix} 4, & 5, & 6 \end{bmatrix} + e \begin{bmatrix} 7, & 8, & 9 \end{bmatrix} =$$

5.[6pts] Write out the definition of the inverse matrix. Referring to this definition, explain why $(A^T)^{-1} = (A^{-1})^T$.

6.[8pts] For the system

$$\begin{cases} x_3 = 1 \\ x_2 + 2x_3 = 3 \\ x_1 + x_2 + 3x_3 = 4 \end{cases}$$

a) Write the augmented matrix $[A|\mathbf{b}]$.

b) Write the permutation matrix P such that PA is upper triangular.

7.[8pts] Consider vectors $\mathbf{u} = (0, 1, 1)$, $\mathbf{v} = (1, 0, 2)$, and $\mathbf{w} = (2, 1, a)$.

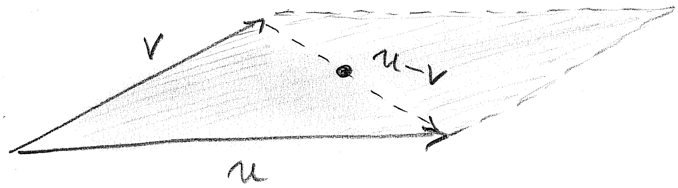
a) Pick the parameter a so that the three vectors are coplanar.

b) Pick the parameter a so that two of the vectors are perpendicular.

Condensed solutions to Exam I.

#0 : T, F, T, T, F

#1



#2 $A=\begin{bmatrix}1&0&0\\0&1&-1\\2&1&4\end{bmatrix} \xrightarrow{③-2\cdot①} \begin{bmatrix}1&0&0\\0&1&-1\\0&1&4\end{bmatrix} \xrightarrow{③-1\cdot②} \begin{bmatrix}1&0&0\\0&1&-1\\0&0&5\end{bmatrix} = U$

a) $E=\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}$ $\quad E=\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}$

b) $A=\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\2&1&1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}1&0&0\\0&1&-1\\0&0&5\end{bmatrix}}_{U}$

c) $Lc=b$ and $Ux=c$

d) $\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&-1&0&1&0\\2&1&4&0&0&1\end{array}\right]$ two steps compressed $\underset{\substack{③-2\cdot①\\③-1\cdot②}}{\rightsquigarrow}$ $\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&-1&0&1&0\\0&0&5&-2&-1&1\end{array}\right] \rightsquigarrow ②+\frac{1}{5}③ \left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&-2/5&4/5&\\0&0&5&-2&-1&\end{array}\right]$

$A^{-1}=\begin{bmatrix}1&0&0\\-2/5&4/5&1/5\\-2/5&-1/5&1/5\end{bmatrix}$ $\quad$ e) $A^{-1}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\3/5\\-2/5\end{bmatrix}$

#3 $(AB)^{-1}B =$ nothing simpler $\quad (AB)^{-1}A = B^{-1}A^{-1}A = B^{-1} \quad (A^{-1}B)^TA^{-T} = B^T(A^{-1})^TA^{-T} = B^T(A^T)^{-1}A^{-T} = B$

#4 $\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}\begin{bmatrix}c\\d\\e\end{bmatrix}$ $\quad \begin{bmatrix}c&d&e\end{bmatrix}\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$

#5 see the book for def.
Justification:
$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$
$(A^{-1})^TA^T = (AA^{-1})^T = I^T = I$

#6 $\left[\begin{array}{ccc|c}0&0&1&1\\0&1&2&3\\1&1&3&4\end{array}\right]$, $P=\begin{bmatrix}&&1\\&1&\\1&&\end{bmatrix}$

#7 a) $u+2v=\begin{pmatrix}2\\1\\5\end{pmatrix}$ so $a=5$ will do. b) $u\cdot w = 1+a$ so $a=-1$ will make $u \perp w$