Phantom Quiz on 3.5 and 3.6 MATH 221

(Time: 30 min)

① Consider

$$A = \begin{bmatrix} 2 & 4 & 2 & 6 \\ 2 & 4 & 2 & 7 \\ 3 & 6 & 3 & 13 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for each of $C(A)$, $C(A^T)$, $N(A)$, $N(A^T)$

$C(A)$'s basis: $\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 13 \end{bmatrix}$. ; $C(A^T)$'s basis: $[1,2,1,3]^T, [0,0,0,1]^T$.

$\dim N(A) = 4-2 = 2$. Since $col_2 = 2\,col_1$ and $col_3 = col_1$, $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ are in $N(A)$.
These being independent, they form a basis.

$\dim N(A^T) = 3-2 = 1$; We are seeking a lin. comb of row_1 and row_2 that gives row_3:
$a \cdot row_1 + b \cdot row_2 = row_3$ gives $2a+2b = 3$ and $6a+7b = 13$,
which solves to $b = 4$ and $a = -5/2$.
Hence $(-5/2, -4, 1)$ is in $N(A^T)$. Because $\dim N(A^T) = 1$, this one vector is a basis of $N(A^T)$.

② Find a basis and the dimension of the space of all 3×3 antisymmetric matrices, that is A such that $A^T = -A$.

$$-\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$ gives $a_{11} = a_{22} = a_{33} = 0$, $a_{21} = -a_{12}$; $a_{31} = -a_{13}$, $a_{32} = -a$

That is our subspace consists of all 3×3 matrices of the form

$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$, which can be written as lin. combination

$$a\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & +1 \\ -1 & 0 & 0 \end{bmatrix}.$$

Thus the three matrices $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$ span the subspace.

Since they are also independent (why?), they are a basis.