GRADING KEY

Max Score 160

Math 333 Final Exam (11 Dec 2012)

Name:

Show all work (unless instructed otherwise). Good Luck!

0.[10pts] Circle **True** or **False** without explanation. Below V is a finite dimensional vector space.

(**T** or **F**) Two vector spaces of the same (finite) dimension are isomorphic.

(**T** or **F**) Any independent subset can be extended to a basis.

(**T** or **F**) A linear $T : V \to V$ is one-to-one iff $\ker(T) = \{0\}$.

(**T** or **F**) The real eigenvalues of A are singular values of A.

(**T** or **F**) If $A \geq 0$ is irreducible then $A^N > 0$ for some N.

1.[10pts] Suppose that V is an inner product space and $\|x\| := \sqrt{\langle x|x\rangle}$. Demonstrate the Pythagorean Theorem:

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 \quad \text{for all } u, v \text{ in } V \text{ such that } u \perp v.$$

$$\|u+v\|^2 = \langle u+v|u+v\rangle = \langle u|u\rangle + \langle u|v\rangle + \langle v|u\rangle + \langle v|v\rangle^2$$

$$= \|u\|^2 + 0 + 0 + \|v\|^2$$

where we used $\langle u|v\rangle = \langle v|u\rangle = 0$ due to $u \perp v$.

Check if orthodox wording

2.[10pts] Find the operator norm of $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ with respect to the Euclidean vector norm.

5pts.

$\|A\| =$ the maximum singular value

$$AA^T = \left[\begin{array}{c|c} 2 & 1 \\ \hline 1 & 2 \end{array}\right] \quad T = 4, \; D = 3$$

$$\lambda^2 - 4\lambda + 3 = 0$$

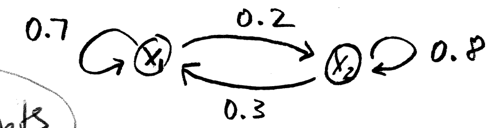
$$\lambda_{1,2} = \frac{4 \pm \sqrt{4^2 - 4\cdot 3}}{2} = \frac{4 \pm \sqrt{4}}{2} = 3, 1$$

Hence $\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$; $\sigma_2 = \sqrt{\lambda_2} = \sqrt{1}$

and $\|A\| = \sigma_1 = \sqrt{3}$

Forgot to take $\sqrt{\ }$ − 2pts.

3.[15pts] The following graph describes the flow of wealth between two partners.



7pts a) Compute the asymptotic growth rate.

Adjacency matrix is $A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$ 2pts

Evolution matrix is $P = A^T = \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}$ since $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto \begin{bmatrix} 0.7x_1 + 0.3x_2 \\ 0.2x_1 + 0.8x_2 \end{bmatrix}$

5pts
$T = 1.5$; $D = 0.56 - 0.06 = 0.5$
$\lambda^2 - 1.5\lambda + 0.5 = 0$
$(\lambda - 1)(\lambda - 0.5) = 0$ so $\lambda_1 = 1$, $\lambda_2 = 0.5$

The asymptotic growth rate is 0% (since $\lambda_1 = 1$)

8pts b) What is the asymptotic ratio of wealth between the two partners?

The initial vector $\vec{v}$ evolves according to

$$P^n \vec{v} \approx \lambda_1^n \langle \vec{v} | \vec{u}_1 \rangle v_1$$ where v_1 is the Perron eigenvector of P

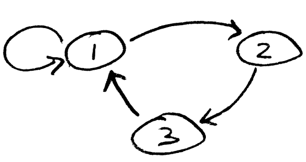
$$P - \lambda_1 I = \begin{bmatrix} -0.3 & 0.3 \\ 0.2 & -0.2 \end{bmatrix} \text{ so } v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The asymptotic ratio is $1/1 = 1$.

v ?

4.~~[15pts]~~ 10 pts Suppose $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $A^9 = \begin{bmatrix} 19 & 13 & 9 \\ 9 & 6 & 4 \\ 13 & 9 & 6 \end{bmatrix}$

4pts. a) Draw the graph for which A is the adjacency matrix.



3pts b) What is the number of paths of length 9 from the first vertex to itself?

It equals $a_{11}^{(9)} = 19$.

3pts c) Is A **primitive**? [*Justify.*]

$A^9 > 0$ so yes.

5.[10pts] A is 3×3 and $[1, 2, 3]^T$ is a basis of solutions to $A^T y = 0$.

5pts

a) Find a value of the parameter λ so that $Ax = \begin{bmatrix} 1 \\ \lambda \\ 1 \end{bmatrix}$ has a solution x.

Need $\begin{bmatrix} 1 \\ \lambda \\ 1 \end{bmatrix} \perp \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ so $1 + 2\lambda + 3 = 0$

$2\lambda = -4$ so $\lambda = -2$

5pts

b) Is the solution x (of the system in a)) unique? *Explain.*

$\dim N(A^T) = 1$ since $N(A^T)$ has basis $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Hence the rank of A is $r = 3 - 1 = 2$ and

$\dim N(A) = 3 - 2 = 1$.

The general solution $x = x_p + x_n$ (where x_p is a particular solution and $x_n \in N(A)$).

takes then ∞-many values.

6.[15pts] The SVD for a 3×3 matrix A is (approximately) given by

$$A = \underbrace{\begin{bmatrix} 0.63 & -0.33 & -0.71 \\ 0.63 & -0.33 & 0.71 \\ 0.46 & 0.89 & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 2.2 & 0 & 0 \\ 0 & 1.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.21 & 0.58 & 0.79 \\ 0.79 & -0.58 & 0.21 \\ -0.58 & -0.58 & 0.58 \end{bmatrix}}_{V^T}$$

3pts a) The rank(A) is $\boxed{2}$, the # of (non-zero) singular values.

4pts b) The image (under T_A) of the unit sphere in R^3 is an ellipse with semi-axis lengths

$a = \boxed{2.2}$ $b = \boxed{1.1}$

4pts c) The semi-axes of the ellipse point in the direction of vectors

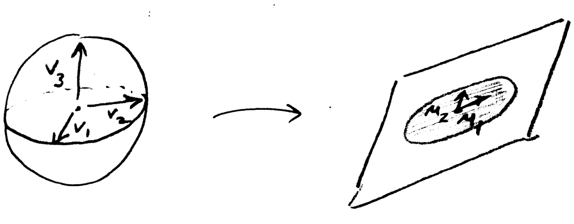
v lengthen boxes

Using V not U −3pts

$\underbrace{(\boxed{0.63}, \boxed{0.63}, \boxed{0.46})^T}_{u_1}$, $\underbrace{(\boxed{-0.33}, \boxed{-0.33}, \boxed{0.89})^T}_{u_2}$ (The remaining u_3 spans $N(A^T)$)

4pts d) Is the ellipse filled-in or not? [*Give an idea why.*]

Yes because T_A collapses one dimension. ($\ker(T_A) = \text{span}(v_3)$)



7.[10pts] Fix a non-zero $v \in \mathbf{R}^n$. Show that the following is a subspace of $M_{n \times n}$:

$$W_v := \{A \in M_{n \times n} : v \text{ is an eigenvector of } A\}.$$

Knowing what to do 5pts.

Suppose $A \in W_v$ and $B \in W_v$, i.e., $\exists\, \lambda, \mu \in \mathbb{R}$ $Av = \lambda v$ (1) and $Bv = \mu v$ (2).

Then $(A+B)v = Av + Bv \underset{\text{by (1) and (2)}}{=} \lambda v + \mu v = (\lambda+\mu)v$

so $A + B \in W_v$.

Also, for any scalar α, $(\alpha A)v = \alpha(Av) \overset{\text{by (1)}}{=} \alpha \lambda v$

so $\alpha A \in W_v$ as well.

Another solution to 8:

$a \cdot 1 + b \cdot x = x^2$

has to be solved for a, b in the sense of least squares

(we never did things in this way.)

$\sqrt{}$?

8.[15 pts] ~~[10pts]~~ Find the best linear approximation to the function x^2 in the inner product space of all continuous functions on $[0,1]$ with the inner product $\langle f|g\rangle := \int_0^1 f(x)g(x)dx$. [*Note: By "linear functions" we mean span*$\{1, x\}$.]

Using $\perp$ projection formula without orthogonalizing -7 pts

We are seeking $P := \text{proj}_{\text{span}\{1,x\}} x^2$

Since $\langle 1|x\rangle = \int_0^1 x\,dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2} \neq 0$ we apply Gram Schmidt

to get $q_0(x) = 1$ (which is normalized $\|1\| = \int_0^1 1\,dx = 1$)

and $q_1(x) = \text{normalized}(x - \langle x|1\rangle 1)$

$= \text{normalized}\left(x - \frac{1}{2}\right) = \sqrt{12}\left(x - \frac{1}{2}\right)$

where we used $\|x - \frac{1}{2}\| = \sqrt{\int_0^1 (x-\frac{1}{2})^2 dx} = \sqrt{\int_0^1 x^2 - x + \frac{1}{4}\,dx} = \sqrt{\frac{1}{3} - \frac{1}{2} + \frac{1}{4}}$

$= \sqrt{\frac{4-6+3}{12}} = \sqrt{\frac{1}{12}}$

Now, $\text{span}\{1, x\} = \text{span}\{q_0, q_1\}$

and $\{q_0, q_1\}$ is orthonormal so

$$P = \text{proj}_{\{q_0,q_1\}} x^2 = \langle x^2|q_0\rangle q_0 + \langle x^2|q_1\rangle q_1$$

$$= \int_0^1 x^2 dx \cdot 1 + \int_0^1 x^2\left(x - \frac{1}{2}\right)dx \cdot 12 \cdot \left(x - \frac{1}{2}\right)$$

$$= \frac{1}{3} + \left(\frac{1}{4} - \frac{1}{2}\cdot\frac{1}{3}\right) \cdot 12\left(x - \frac{1}{2}\right)$$

$$= \frac{1}{3} + \frac{1}{12} \cdot 12\left(x - \frac{1}{2}\right) = x - \frac{1}{2} + \frac{1}{3} = x - \frac{1}{6}$$

9.[10pts] Show that if T is a linear transformation and $\{Tv_1, Tv_2, \ldots, Tv_n\}$ are linearly independent then $\{v_1, v_2, \ldots, v_n\}$ are linearly independent as well.

Suppose $\alpha_1 v_1 + \ldots + \alpha_n v_n \overset{(*)}{=} 0$. for some scalars $\alpha_1, \ldots \alpha_n$.

Our goal is to show that $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$.

Apply T to both sides of $(*)$:

$$T(\alpha_1 v_1 + \ldots + \alpha_n v_n) = T(0)$$

By linearity this is: $T(\alpha_1 v_1) + \ldots + T(\alpha_n v_n) = 0$

and further $\alpha_1 Tv_1 + \ldots + \alpha_n Tv_n = 0$.

Since $\{Tv_1, \ldots, Tv_n\}$ are lin. indep., we have $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$; as promised!

10.[20pts] Consider $T : P_2 \to P_2$ given by $T(p(x)) = p'(0)x$. (be careful) add space or dot

Describe ker(T) a) Find a basis of kernel(T).

(6pts)

Consider $p(x) = a + bx + cx^2$

$$(Tp)(x) = (b + 2cx \text{ evaluated at } 0) \cdot x = bx$$

+ 1cm

$p \in \ker(T)$ iff $bx \equiv 0$ i.e. iff $b = 0$

So $\boxed{\ker(T) = \{a + cx^2 : a, c \in \mathbb{R}\} = \text{Span}\{1, x^2\} \quad \text{Basis is } \{1, x^2\}}$

(5pts) b) Find the matrix $[T]$ of T with respect to the standard basis $\{1, x, x^2\}$.

Since
$$\begin{aligned} T(1) &= 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2 \\ T(x) &= 1\cdot x = 0\cdot 1 + 1\cdot x + 0\cdot x^2 \\ T(x^2) &= 0\cdot x = 0\cdot 1 + 0\cdot x + 0\cdot x^2 \end{aligned}$$
we have $[T] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(4pts) c) What is the rank(T)? $\text{rank}(T) = \dim P_2 - \dim \ker(T) = 3 - 2 = 1$

(Equally well, $\text{rank}(T) = \text{rank}([T]) = 1$ since)

(5pts) d) Show that $S = I + T$ is invertible. (Here I is the identity.)

$$[S] = [I] + [T] = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is an invertible matrix so S is invertible as well

11.[15pts] Find a basis of the space of all 3×3 marices A that are anti-symmetric, i.e., $A^T = -A$. [*Explain why your basis is independent and spanning.*]

☆ Anti-symmetric matrices have the form $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$
(since the entries in the symmetric places have to be opposites.)

Hence $A = a \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

This shows that these three matrices span the space.

They are also independent since if their linear combination is zero: $a \begin{bmatrix} & 1 & \\ -1 & & \\ & & \end{bmatrix} + b \begin{bmatrix} & & 1 \\ & & \\ -1 & & \end{bmatrix} + c \begin{bmatrix} & & \\ & & 1 \\ & -1 & \end{bmatrix} = 0$

then $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so $a = b = c = 0$.

Being spanning and linearly independent, the three matrices form a basis.

5pts

extra explanation of ☆: $A^T = -A$ is $\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = - \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

So $a_{11} = -a_{11}$, $a_{22} = -a_{22}$, $a_{33} = -a_{33}$ giving $a_{11} = a_{22} = a_{33} = 0$;

and $a_{21} = -a_{12}$, $a_{31} = -a_{13}$, $a_{32} = -a_{23}$. By renaming $a := a_{12}$, $b = a_{13}$, $c = a_{23}$

we see that $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ as claimed.

✓✓✓ 12.[10pts] Recall that the Frobenius norm of A is given by $\|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2}$. Show that $\|AB\|_F \leq \|A\|_F \|B\|_F$. (*Justify key steps.*)

Brutal Method:

$$\|AB\|^2 = \sum_{i,j} \Big(\sum_k a_{ik} b_{kj}\Big)^2 \underset{\text{Cauchy Schwarz}}{\leq} \sum_{i,j} \sum_k a_{ik}^2 \sum_k b_{kj}^2 = \sum_i \sum_k a_{ik}^2 \sum_j \sum_k b_{kj}^2$$
$$= \|A\|^2 \|B\|^2$$

Taking $\sqrt{\ }$, gives $\|AB\| \leq \|A\| \cdot \|B\|$.

Slick Method: $\|AB\|_F^2 = \|Ab_1\|^2 + \dots + \|Ab_n\|^2$ where b_i are the columns of B

Because Frobenius norm is adapted to Euclidean norm $\|Ab_i\| \leq \|A\|_F \|b_i\|$ so

$$\|AB\|_F^2 \leq \|A\|_F^2 \|b_1\|^2 + \dots + \|A_F\|^2 \|b_n\|^2 = \|A\|_F^2 (\|b_1\|^2 + \dots + \|b_n\|^2)$$
$$= \|A\|_F^2 \cdot \|B\|_F^2$$

Extra: Proof of "adaptation": $\|Ax\|^2 = \sum_i (a_i \cdot x)^2 \leq \sum_i \|a_i\|^2 \|x\|^2 = \sum_i \|a_i\|^2 \cdot \|x\|$

↑ dot product of i^{th} row of A and x $= \|A\|_F^2 \|x\|$

Taking $\sqrt{\ }$ gives $\|Ax\| \leq \|A\|$ ...

EXTRA PROBLEMS

13.[10pts] Suppose $n \times n$ matrix A has a nonzero vector in the intersection of $N(A)$ and $C(A)$. Show that the characteristic polynomial of A has $\lambda = 0$ as a double root. (*Hint: Let that vector be $Av \neq 0$ and let $\mathcal{B}$ be a basis obtained by extending $\{v, Av\}$. Consider $[T_A]_{\mathcal{B}}$.*)

14.[10pts] Suppose that λ and μ are two different real eigenvalues of A. Show that any eigenvector of λ is contained in $C(A - \mu I)$.

15.[10pts] Suppose $n \times n$ matrix A has n distinct real eigenvalues $\lambda_1, \ldots, \lambda_n$ with the corresponding (real) eigenvectors for A and A^T denoted as follows

$$A: \quad v_1, \ldots, v_n$$

$$A^T: \quad u_1, \ldots, u_n.$$

This means that $N(A - \lambda_i I) = \text{span}(v_i)$ and $N(A^T - \lambda_i I) = \text{span}(v_i)$.

a) Show that $u_i \perp v_j = 0$ for $i \neq j$.
(*Hint: Use that* $Av_j = \lambda_j v_j$ *and* $\lambda_j \neq \lambda_i$ *to first show that* $v_j \in C(A - \lambda_i I)$.)

b) Show that $v = \alpha_1 v_1 + \ldots + \alpha_n v_n$ where $\alpha_i = u_i^T v$ under **cross-normalization** $u_i^T v_i = 1$ for all i. (*Assume a) if you cannot show it.*)

c*) Show that $u_i^T v_i \neq 0$.
(*Hint: Otherwise,* $v_i \in N(A^T - \lambda_i)^\perp = C(A - \lambda_i)$ so $N(A - \lambda_i) \cap C(A - \lambda_i) \neq \{0\}$.)