Exam 1 Math 333 (11 Oct 2012)

Name:

<u>**Show all work**</u> **(unless instructed otherwise). If you use a theorem indicate what it is.** Good Luck!

0.[10pts] Circle **True** or **False** without explanation:
Below, V and W are vector spaces of finite dimension, U and U' are subspaces of V, and $T : V \to W$ is linear.

(**T** or **F**) $\dim(U + U') = \dim(U) + \dim(U') - \dim(U \cap U')$.

(**T** or **F**) $U \subset V$ and $\dim(U) = \dim(V) \implies U = V$.

(**T** or **F**) V is isomorphic to $\mathbf{R}^n$ for some n.

(**T** or **F**) T is one-to-one iff $\ker(T) = \{0\}$.

(**T** or **F**) If $\{v_1, \dots, v_n\}$ spans V then $\{Tv_1, \dots, Tv_n\}$ spans range(T).

1.[10pts] In $\mathcal{M}_{2\times 2}$, the subspace W has basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right\}$.

a) Find the A with $[A]_{\mathcal{B}} = [1, 2, 3]$. *(Indicate your method.)*

$A = \dots$

b) Show that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ does not belong to W.

2.[10pts] Give a bullet-proof argument for independence of $\{1, \sin x, \cos x\}$ in $\mathcal{F}$.

3.[10pts] Let B be some fixed $n \times n$ matrix. Verify that

$$V = \{ \ A \in \mathcal{M}_{n\times n} : \ AB = BA \ \}$$

is a subspace of $\mathcal{M}_{n\times n}$.

4.[15pts] Consider linear $T : \mathcal{P}_2 \to \mathcal{P}_2$ given by $(Tp)(x) = xp'(x)$.

a) Find $[T]_{\mathcal{B}\leftarrow\mathcal{B}}$ when $\mathcal{B} = \{1,\ x+1,\ x^2\}$. *(Note that $\mathcal{B}$ is non-standard!)*

b) Describe $\ker(T)$ and find its basis.

c) Use the Rank Theorem to compute $\dim(\text{range}(T))$. *(Indicate the meaning of the numbers in your computation.)*

5.[15pts] Consider $T : \mathcal{M}_{n\times n} \to \mathcal{M}_{n\times n}$ given by $T(A) := A - A^T$.

a) Carefully verify that T is linear.

b) Compute and simplify the formula for $T \circ T$:

$T \circ T(A) = \ldots$

6.[10pts] Let $\mathcal{B}_{st} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

Find the change of basis matrix

$P_{\mathcal{C} \leftarrow \mathcal{B}_{st}} = \ldots$

7.[10pts] A binary code encodes $x = (x_1, x_2)^T$ to $(b_1, b_2, b_3, b_4)^T := Gx$ where G is 4×2 matrix of rank **two** and with rows satisfying

$$r_1 + r_2 + r_3 = 0 \quad \text{and} \quad r_1 = r_4.$$

Find the conditions for $(b_1, b_2, b_3, b_4)^T$ to be an uncorrupted message, that is $b = Gx$ for some x. *Explain your reasoning!*

8.[10pts] Let $T : V \to W$ be a linear transformation. Argue that if $\{Tv_1, \ldots, Tv_n\}$ is independent then $\{v_1, \ldots, v_n\}$ is independent. *Move by clear steps and justify each.*