Math 333 First Exam (9 Oct 2014)

Name:

If you use a theorem indicate what it is. Show all work (unless instructed otherwise).
Good Luck!

0. Circle **True** or **False** without explanation:

(**T** or **F**) Every non-zero vector v in V belongs to some basis of V.

(**T** or **F**) The spaces $\mathcal{P}_3$ and $\mathcal{M}_{2\times 2}$ are isomorphic (to each other).

(**T** or **F**) If $\{v_1, \ldots, v_n\}$ is independent then so is $\{Tv_1, \ldots, Tv_n\}$ (for a linear T).

(**T** or **F**) If A is a square matrix and $A^2 = 0$ then $A = 0$.

(**T** or **F**) For any b in $\mathbb{Z}_8$, the equation $6x = b$ has a solution x in $\mathbb{Z}_8$.

1. Find a basis and the dimension for the space $\mathcal{A}_{3\times 3}$ of all 3×3 **antisymmetric** matrices. Start by **filling-in the six blanks** in the general from of such a matrix.

$$\mathcal{A}_{3\times 3} := \left\{ \begin{bmatrix} & a & b \\ & & c \\ & & \end{bmatrix} : a, b, c \in \mathbf{R} \right\}.$$ Do not prove anything.

2. Quickly and decisively show that the set $\mathcal{I}_{n\times n}$ of all invertible $n \times n$ matrices **is not** a linear subspace of $\mathcal{M}_{n\times n}$ (for any natural n).

3. Carefully prove that the following subset W of $\mathcal{F}$ is a subspace:

$$W := \{f \in \mathcal{F} : f(2x) = f(x) \text{ for all } x \in \mathbf{R}\},$$

(Plainly, W consists of all functions on $\mathbf{R}$ unchanged by the substitution $x \mapsto 2x$.)

4. Give a bullet-proof verification of independence of $\{1, e^x, e^{-x}\}$ in $\mathcal{F}$.
(If too hard, do $\{e^x, e^{-x}\}$ for partial credit.)

5. Consider two transformations T and S on $\mathcal{M}_{n\times n}$ given by

$$S(A) := \frac{1}{2}(A + A^T) \quad \text{and} \quad T(A) := \frac{1}{2}(A - A^T).$$

a) Carefully verify that S is linear.

b) Compute and simplify the formula for the composition $S \circ T$

$S \circ T(A) = \ldots$

6. Consider $T : \mathcal{P}_2 \to \mathcal{P}_2$ given by $T(p(x)) := p(x) + p'(x)$.

a) Compute the matrix $[T]_{\mathcal{B} \leftarrow \mathcal{B}}$ when $\mathcal{B} = \{1, x+1, x^2\}$.

b) Determine the kernel of T.

c) Based on b), can you tell readily if T is onto. Explain.

7. Prove that if $T : V \to W$ is linear and $\ker(T) = \{0\}$ then T is one-to-one.

8. A binary code encodes $x = (x_1, x_2, x_3)^T$ to $(b_1, b_2, b_3, b_4, b_5)^T := Gx$ where G is 5×3 matrix of rank **three** and with rows satisfying

$$r_1 + r_2 + r_3 = r_4 + r_5 \quad \text{and} \quad r_2 = r_4 + r_3.$$

Find the conditions for $(b_1, b_2, b_3, b_4, b_5)^T$ to be an uncorrupted message (i.e., $b = Gx$ for some x). *Explain your reasoning!*