0	1	2	3	4	5	6	7	8	9	

Math 333 First Exam (19 Oct 2010)

Name:

If you use a theorem indicate what it is. Show all work (unless instructed otherwise).
Good Luck!

0. Circle **True** or **False** without explanation:

(**T** or **F**) Every linearly independent set in V can be extended to a basis of V.

(**T** or **F**) $\mathbf{R}^9$ and $M_{3\times 3}$ are isomorphic.

(**T** or **F**) If $T \circ S$ is one-to-one then so is T.

(**T** or **F**) If U, W are subspaces then so is their union $U \cup W$.

(**T** or **F**) Any linear $T : \mathcal{P}_2 \to \mathcal{P}_2$ that is one-to-one is onto.

1. Find a basis and the dimension for the subspace (in $M_{3\times 3}$) given by
$W := \left\{ \begin{bmatrix} a & 0 & c \\ c & a & b \\ 0 & b & a \end{bmatrix} : a, b, c \in \mathbf{R} \right\}$. Do not prove anything.

2. Carefully verify that the set of all even functions on $\mathbf{R}$,

$$W := \{f \in \mathcal{F} : f(x) = f(-x) \text{ for all } x \in \mathbf{R}\},$$

is a subspace of $\mathcal{F}$.

3. Suppose that $T : V \to W$ is linear and $\{T(v_1), \ldots, T(v_n)\}$ is a linearly independent set. Prove that $\{v_1, \ldots, v_n\}$ is linearly independent but the opposite implication may fail.

4. Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be given by $T(p) := p''$ (the second derivative).
a) Describe kernel(T) and compute nullity(T).

nullity(T) = ...

b) Describe range(T) and compute rank(T).

rank(T) = ...

c) Write out the matrix of T in the standard basis $\{1, x, x^2\}$.

5. You learned that if $\dim V = \dim W$ (and finite) then V and W are isomorphic. Recall the construction of the isomorphism $T : V \to W$ (without proving its properties).

6. Suppose that $\{v_1, v_2, v_3\}$ is a basis (of some V).
a) Show that $\mathcal{B} = \{v_1 - v_2, v_2 - v_3, v_3 - v_1\}$ is not a basis.

b) Show that $\mathcal{B} = \{v_1 + v_2, v_2 + v_3, v_3 + v_1\}$ is a basis.

7. Suppose that the set $\mathcal{A} = \{v_1, \ldots, v_n\}$ spans all of V but fails to do so if one removes any of its vectors. Prove that $\mathcal{A}$ is independent.

8. Show that $\{e^x, e^{-x}\}$ is linearly independent (in $\mathcal{F}$).

9. A certain binary code sends a message $x = (x_1, x_2)^T$ to $b := Gx$ where the code generating matrix G is

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{so} \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = Gx = \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \\ x_1 \end{bmatrix}$$

a) Find the conditions ("check") for (b_1, b_2, b_3, b_4) to be an uncorrupted message (i.e. $b = Gx$ for some x).

b) How many bits in (b_1, b_2, b_3, b_4) can be corrupted and the original message (x_1, x_2) is still recoverable? (Give an answer and some indication of the reason.)

0	1	2	3	4	5	6	7	8	

Math 333 Second Exam (30 Nov 2010)

Name:

Show all work (unless instructed otherwise). Good Luck!

0. Circle **True** or **False** without explanation: Below vectors are in a finite dimensional vector space V, $\langle\cdot|\cdot\rangle$ denotes an inner product, and $\|\cdot\|$ its associated norm.

(**T** or **F**) $\langle u|v\rangle = \|u\|\|v\|$ only if one of u and v is a scalar multiple of another.

~~(**T** or **F**) Every norm on V comes from an inner product.~~

(**T** or **F**) $\overline{x} := A^+b$ is a **least squares solution** to $Ax = b$.

(**T** or **F**) For any square matrix A, the SVD gives a diagonalization of A.

1. Suppose that u, v, w are vectors in an inner product space and

$$\langle u|v\rangle = -3, \quad \langle u|w\rangle = -6, \quad \langle v|w\rangle = 6$$

$$\|u\| = 3, \quad \|v\| = \sqrt{5}, \quad \|w\| = \sqrt{8}.$$

a) Evaluate $\|u + w\| = \ldots$

b) Determine by computation if $u + v$ is perpendicular to w.

2. Derive the **Triangle Inequality** from the **Cauchy-Schwarz Inequality**. (State both!)

3. Show that $\langle p(x)|q(x)\rangle := p(0)q(0) + p(1)q(1)$ is not an inner product on $\mathcal{P}_2$.

4. Let $\|A\|$ be a matrix norm of a square matrix A compatible with a vector norm $\|x\|$. Show that if λ is an eigenvalue of A then

$$\|A\| \geq |\lambda|.$$

5. Prove (while justifying each step) that the Frobenius norm

$$\|A\|_F := \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}$$

is compatible with the Euclidean norm, i.e., $\|Ax\| \leq \|A\|_F \|x\|$ for all x in $\mathbf{R}^n$.

6. Find all least squares solutions to the system

$$\begin{cases} x_1 + x_2 = 1 \\ x_1 + x_2 = 2 \end{cases}$$

7. The normal equation $A^T A x = A^T b$ is an expression of perpendicularity of a **certain** vector to a certain subspace. What vector and subspace are involved? Draw a figure and explain how $A^T A x = A^T b$ follows.

8. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

a) Compute the semi-axis of the ellipse that is the image of the unit circle under the linear transformation T_A of the plane $\mathbf{R}^2$ induced by A.

b) What is the value of the operator norm $||A|| := \max_{||x||=1} ||Ax||$? (Here $||x||$ is the ordinary Euclidean norm.)

c)* Sketch the image of the square $\{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$ under T_A and guess-sketch the ellipse (from a)).

0	1	2	3	4	5	6	7	8	9	10	

Math 333 Final Exam (15 Dec 2010)

Name:

Show all work (unless instructed otherwise). Good Luck!

0. Circle **True** or **False** without explanation. Below V is a finite dimensional vector space.

(**T** or **F**) V is isomorphic to $\mathbf{R}^n$ provided $n = \dim(V)$.

(**T** or **F**) Any two bases of V have the same number of elements.

(**T** or **F**) If $T : V \to V$ is linear and onto then it is one-to-one.

(**T** or **F**) For orthogonally diagonalizable A, the singular values are eigenvalues.

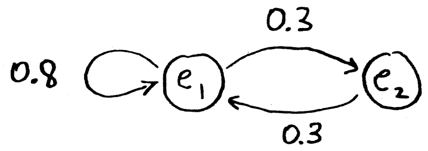
(**T** or **F**) For $A \geq 0$, $A^3 > 0$ implies that A is irreducible.

1. V is an inner product space and $\|x\| := \sqrt{\langle x|x\rangle}$ is the associated norm.

a) Demonstrate the Parallelogram Law:

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2 \quad \text{for all } u, v \text{ in } V.$$

2. The following graph describes the flow of wealth in a partnership of two agents e_1 and e_2.



a) What is the asymptotic ratio of the wealth of e_1 to the wealth of e_2?

b) Compute the growth rate λ_1. Is this partnership going to prosper or fizzle out?

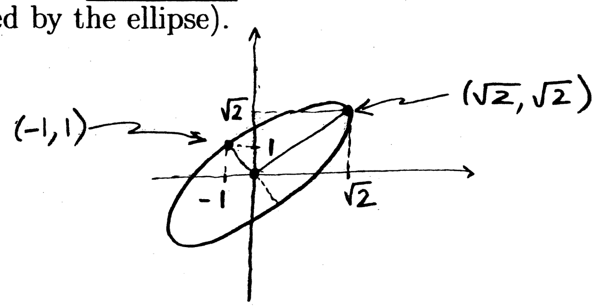
3. Let $\|A\|$ be a matrix norm of a square matrix A compatible with a vector norm $\|x\|$. Suppose there is x with $\|x\| = 1$ such that

$$\|Ax\| = 2 \quad \text{and} \quad \|A^2x\| = 9.$$

a) Prove that $\|A\| \geq 2$.

b)* Prove that $\|A\| \geq 3$.

4. For a 2×3 matrix A, the image of the unit sphere under T_A is the depicted solid ellipse (i.e., the oval area bounded by the ellipse).



b) What is the rank(A)?

a) What are the singular values of A?

c) What is the U is in the SVD $A = U\Sigma V^T$?

5. Suppose that B is a fixed 2×2 matrix.

a) Check that the set $W_B := \{A \in M_{2\times 2} : AB = BA\}$ is a subspace (of $M_{2\times 2}$).

b) Find a basis of W_B for the specific B given by $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

6. Let A be $n \times m$ with independent columns and QR-decomposition $A = QR$. (Here Q is $n \times m$ and R is $m \times m$.)

a) Give an idea why $C(A) = C(Q)$.

b) Write the formula for the projection of $b \in \mathbf{R}^n$ onto $C(A)$ in terms of Q and b.

c) Express the least squares solution to $Ax = b$ by using Q, R, and b. Justify.

7. Assuming $\{v_1, v_2, v_3\}$ is linearly independent, show that $\{v_1, v_1 + v_2, v_1 + v_2 + v_3\}$ is linearly independent as well.

8. Consider $T : \mathcal{P}_2 \to \mathcal{P}_2$ given by $T(p(x)) = xp'(x)$.

a) What polynomials $p(x)$ make up the kernel(T)?

b) Find the matrix $[T]$ of T with respect to the standard basis $\{1, x, x^2\}$.

c) What is the rank(T)? (Use a).)

d) Show that $S = I + T$ is invertible. (Here I is the identity so S is explicitly given by $S(p(x)) = p(x) + xp'(x)$; although, you may just use $[S]$ in your solution.)

9. Construct an orthogonal basis of $\mathcal{P}_2$ with the inner product $\langle p|q\rangle := \int_0^1 p(x)q(x)dx$.

10. In Crystallographic Restriction Theorem, a rotation matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ preserved some lattice Γ in $\mathbf{R}^2$, $R_\theta\Gamma = \Gamma$. Explain why $\text{trace}(R_\theta) = 2\cos\theta$ had to be an integer.