JOURNAL OF DIFFERENTIAL EQUATIONS **141**, 86–101 (1997)
ARTICLE NO. DE973319

Stability in a Semilinear Boundary Value Problem via Invariant Conefields

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Received October 31, 1996; revised May 1, 1997

We give a geometric proof of stability for spatially nonhomogeneous equilibria in the singular perturbation problem $u_t = \varepsilon^2 u_{xx} + f(x, u)$, $t \in \mathbf{R}^+$, $-1 \leqslant u \leqslant 1$, with the Neumann boundary conditions on $x \in [0, 1]$. The nonlinearity is of the form $f(x, u) := (1 - u^2)(u - c(x))$, where $c(x)$ is merely continuous with a finite number of zeros. The strength of the method is in dealing with non-transversal zeros of c, the case escaping the existing techniques of singular perturbations. The approach is also used for showing existence of unstable equilibria with one transition layer.

1. INTRODUCTION

The note concerns itself with the following much studied semilinear boundary value problem

$$\begin{cases} u_t = \varepsilon^2 u_{xx} + f(x, u), & x \in [0, 1], \quad t \in \mathbf{R}^+, \\ -1 \leqslant u \leqslant 1, & u_x(0) = u_x(1) = 0, \end{cases} \tag{1}$$

where $f(x, u) := (1 - u^2)(u - c(x))$ and $c(x)$ is an arbitrary continuous function c: $[0, 1] \to (-1, 1)$ with a finite number of zeros. Observe that $f(x, u)$ is the negative gradient with respect to u of a function with two wells, and the bottoms ± 1 are stable equilibria—in particular, $u = \pm 1$ satisfy

$$\varepsilon^2 u_{xx} = -f(x, u), \qquad u_x(0) = u_x(1) = 0, \qquad x \in [0, 1]. \tag{2}$$

The problem provides basic testing ground for phenomena occuring in a bistable spatially distributed system: think of $x \in [0, 1]$ worth of agents, each with a state variable $u(x)$, evolving under the gradient flow $u_t = f(x, u)$ while

[1] Partially supported by Grants ARO DAAH-0493G0199 and NIST G-06-605, and Polish KBN Grant 2 P301 01307 Iteracje i Fraktale, II.

0022-0396/97 $25.00

being subject to small diffusive coupling. Without diffusion ($\varepsilon = 0$), the states of agents tend to ± 1 independently, thus accounting for uncountably many stable equilibria. On the other extreme, if the preferences of all agents coincide (i.e. c is constant), even small diffusion makes all stable equilibria spatially homogeneous (equal to ± 1). Less apparent is the birth of non-homogeneous stable equilibria under even the slightest nonhomogeneity of the preferences. A number of papers put this phenomenon on a rigorous ground: [11, 12, 9, 2, 1, 14, 7], to name the most closely related; and Section 2.6 in [5] or 4.3 in [4] should be consulted for a broader introduction with an overview of the results and some proofs. In showing stability, all these works invoke a rather involved and delicate singular perturbation analysis. Our main goal is to achieve an elementary and geometrically clear treatment of this issue. While at present the scope of the approach is dwarfed by that of asymptotic techniques (c.f. [7]), certain advantages make it worthwhile. Most notably, it does not require any transversality conditions on c; and, due to its non-perturbative character, it easily reflects the effects vanishing to all orders in ε.

To proceed more systematically, we set off with a result from [1] (see also Fig. 1).

THEOREM 1 [1]. *For small $\varepsilon > 0$, if $\zeta_1 < \zeta_2 < \cdots < \zeta_k$ are zeros of c along which c changes sign in an alternating fashion* (i.e. $c(\zeta_i + 0^+) \cdot c(\zeta_{i+1} + 0^-) > 0$, $i = 1, ..., k-1$), *then* (1) *has an equilibrium $u(\cdot)$ with values $u(x)$ close to ± 1 except for x in the vicinity of the ζ_i's, where $u(\cdot)$ makes a transition between ± 1 in the direction going against the change of sign incurred by c (c.f.* $\mathbf{HU}_d$ *in Section* 2).

The proof amounts to constructing appropriate upper and lower solutions for (1); the equilibrium is trapped between the two (Fig. 2). Note that neither monotonicity of transition layers nor stability are asserted, as both are established in [1] only under the assumption of transversality of zeros of c. Also, the more general methods of [7], which yield existence, shape, and stability at the same time, require transversality.

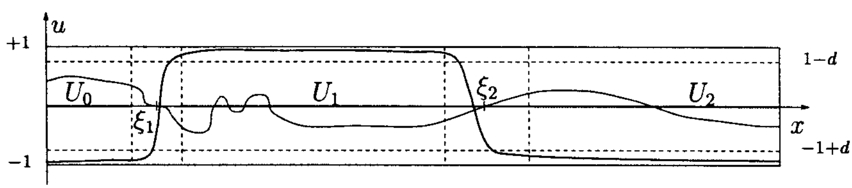


FIG. 1. Stable equilibrium with two transition layers, each associated with a zero of the preference c.

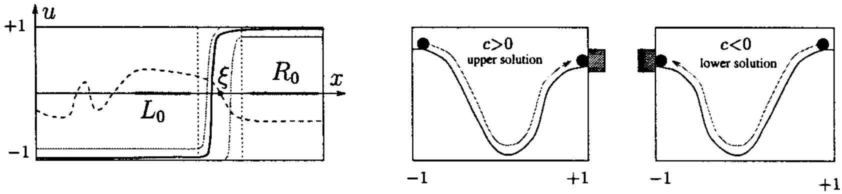


FIG. 2. Existence. Interpret the ODE (2) as describing a point mass in a potential well with walls of height difference $4c/3$. For the upper solution (the dotted graph on the left), hold the mass just right from the top of the left wall until x increases to near ζ and $c(x)>0$, let it swing to the opposite side and break at the right summit before c turns negative. For the lower solution, invert the orientation of u and the time x.

THEOREM 2. *The equilibria in Theorem* 1 *have monotone transition layers and are exponentially stable with respect to* (1).

By general results of Matano [10], the stability already implies uniqueness of an equilibrium satisfying the properties in Theorem 1. This enables one to enumerate stable equilibria of (1) for small ε by considering all suitable sets of zeros of c—see [1] for details.

Our argument is essentially a phase portrait analysis of the ODE for the equilibria, (2). The stability of $u(\cdot)$ is inferred from the rotation of the direction tangent to the initial condition manifold $\{u_x=0\}$ under the variational flow Ψ^x: $T_{(u(0),\, u_x(0))}\mathbf{R}^2 \to T_{(u(x),\, u_x(x))}\mathbf{R}^2$, $x\in[0,1]$. This method goes back to Prüfer in the beginning of this century and more recently was used in [3, 6, 13, 8]. Following Th. 4.3.13 in [4], one identifies all the tangent planes making up $T_{(u,\, u_x)}\mathbf{R}^2$ with a copy of $\mathbf{R}^2$ equipped with polar coordinates (θ, r) so that $\theta(\partial/\partial u)=0$ and $\theta(\partial/\partial u_x)=\pi/2$. The *total rotation* $\Delta\theta := \theta(\Psi^1(\partial/\partial u))$ is positive exactly when the equilibrium is exponentially stable. (Furthermore, hyperbolic equilibrium with d-dimensional unstable manifold is characterized by $\Delta\theta\in(-d\pi, (-d+1)\pi)$.)

Direct estimation of $\Delta\theta$ may be a daunting task and the novel part of our approach comes in tying it up with the natural geometry of the problem. In the autonomous (or piecewise autonomous [14]) case, there is the foliation into the phase curves of (2)—which is gone once we pass to the nonautonomous (and nonintegrable) setting. What persists though is preservation by the variational flow of a certain cone-field—just about enough to lock control over θ. This is elucidated by the change of coordinates described below.

Let $g(u) := u(1-u^2)$ and $\eta(u) := 1-u^2$ so that

$$f(x, u) = g(u) - c(x)\,\eta(u). \tag{3}$$

Map $u\in(-1, 1)$ to a new coordinate $v\in(-\infty, \infty)$ with the Jacobian $du/dv = 1-u^2 = \eta(u)$ and $u=0$ corresponding to $v=0$. (Here conveniently

$u = \tanh(v)$, but the method works with no modifications for any f admitting (3).) Then the ODE for equilibria (2) becomes

$$v_{yy} = u(2v_y^2 - 1) + c(x), \qquad v_y(0) = v_y(1/\varepsilon) = 0, \tag{4}$$

where $y = x/\varepsilon$, $y \in [0, 1/\varepsilon]$, is *the fast time* (as opposed to *the slow time* x).[2] The point is that, if q, p are the variations of v and v_y respectively, then they obey equations with no explicit dependence on time:

$$\frac{d}{dy}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (2v_y^2 - 1)\,\eta & 4v_y u \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}. \tag{5}$$

Moreover, the signs of the entries in the above matrix reveal that the positive cone $\{(p, q)\colon p, q > 0\}$ is preserved by the variational flow in the region where $|v_y| \geqslant \sqrt{1/2}$—which exactly corresponds to the outside of the heteroclinic loop for the autonomous flow with $c = 0$. (The rotation inside of the loop clearly precludes any cone preservation there.) The stability is established by showing that the trajectory of the equilibrium is mostly contained in this region. To illustrate the situation, the case of one transition layer is depicted in Fig. 3 in Section 3. The scenario that unfolds as x runs through $[0, 1]$ is as follows. Before x reaches L_0—where $c > 0$ only to vanish at ζ, Fig. 2—we have $v \leqslant -O(1/\varepsilon)$ so that $|\eta| \leqslant O(\exp(-1/\varepsilon))$ and $\partial/\partial u$ suffers virtually no rotation under (5). Over L_0, $c > 0$ makes v_y quickly raise above $\sqrt{1/2}$ and into the realms of the cone preservation, with the subsequent drop below $\sqrt{1/2}$ a priori prohibited as long as $c > 0$, that is for $x < \zeta$. For $x > \zeta$, all this applies with the time run backwards (and the complementary cones), which completes the description—c.f. Fig. 4 in Section 4.

Since one of the main strengths of our method is its elementary character, we assumed a rather detailed style of exposition with concrete inequalities preferred over compactness arguments. All the estimation is very robust and often much better kept track of by drawing the phase portraits. The multiple transition layer case reduces to that with a single layer by cutting u at critical points found between any two consecutive layers—see Section 5. Section 3 spells out the features of the shape of one such layer (a lap) in preparation for analysis of the variational flow carried out in Section 4. Finally, the geometry lying behind our arguments should help to deal with other aspects of PDE (1). To illustrate this point, in the last section, we find an unstable equilibrium of index one via a simple shooting procedure

[2] I apologize for this non-orthodox terminology but it seems in tune with the mechanistic intuitions behind many arguments, c.f. Fig. 2.

(Fig. 5). In this case the trajectory stays between $\pm\sqrt{1/2}$, where the clockwise rotation of the variational flow makes unstability totally apparent.

2. TECHNICAL FORMULATIONS

To formally describe the class of equilibria $u(\cdot)$ of interest here, along with the sequence $\zeta_1 < \zeta_2 < \cdots < \zeta_k$ of alternating zeros of c (see Theorem 1), we will fix $d > 0$, small, and a sequence of open intervals $U_i \subset [0, 1]$ such that $\sum_{i=1}^{k} |U_i| \geqslant 1 - d$ and $0 \in U_0 < \zeta_1 < U_1 < \zeta_2 < U_2 < \cdots < \zeta_k < U_k \ni 1$. (See Fig. 1.) We will use the following hypothesis, which describes $u(\cdot)$ from Theorem 1 somewhat more precisely.

$(\mathbf{HU})_d$. $u|_{U_i} \in B_d(\pm 1)$ and the sign of u over U_i is opposite to that of $c(\zeta_i + 0^+)$ and $c(\zeta_{i+1} + 0^-)$.

As explained in the introduction, the stability assertion in Theorem 2 follows via Prüfer's method from the following result.

THEOREM 3 (Main result). *There are d, $\varepsilon_0 > 0$ with the following property. If u satisfies $(\mathbf{HU})_d$ and solves (2) with $\varepsilon < \varepsilon_0$, then the time one map Ψ^1 of the variational flow along u maps the positive cone $\{t(\partial/\partial u) + s(\partial/\partial u_x)$: $t, s > 0\}$ at $(u(0), 0)$ to the corresponding cone at $(u(1), 0)$ with zero total rotation.*

As mentioned before, the discussion of the equilibria $u(\cdot)$ satisfying $(\mathbf{HU})_d$ is simplified by considering the consecutive "transitions" between ± 1 separately. Specifically, $u(\cdot)$ must have a critical point a_i between any two zeros, and, for small d, one can require that $a_i \in U_i$, $i = 0, ..., k$. (Actually, a_i sits well inside U_i—see Section 5 for technicalities.) The restrictions of u to $[a_i, a_{i+1}]$ will be referred to as *laps*[3] of u. Upon rescaling of its domain back to $[0, 1]$, a lap can be thought of as equilibrium given by Theorem 1 for a single zero of the appropriate restriction of c (see Fig. 2). *From now on, we assume that u is such an equilibrium corresponding to a zero ζ of c* (see Fig. 2). Also, without loss of generality we can take $c(\zeta + 0^-) > 0$. Note that the time one map Ψ^1 of the variational flow for any multi-transition equilibrium is a composition of the corresponding maps for the laps. If those maps preserve the cone with zero total rotation so does Ψ^1. Hence, to establish Theorem 3, it indeed suffices to argue for one lap only.

We will extract the basic characteristics of a lap $u(\cdot)$ in explicit dependence on some quantitative features of c. Particularly useful will be Δ, $\omega \in (0, 1)$ for which the following hypothesis holds.

[3] We will prove later in the paper that they are indeed monotone, thus justifying the terminology.

$(\mathbf{HC})_{\Delta,\omega}$. The function c: $[0, 1] \to (-1, 1)$ is continuous with a finite number of zeros, and

(i) $1 - |c(x)| \geqslant \Delta$ for $x \in [0, 1]$ and

(ii) $c|_{L_0} \geqslant \omega$ and $c|_{R_0} \leqslant -\omega$ for certain intervals L_0, R_0 in $[0, 1]$ such that $L_0 \subset U_0$, $R_0 \subset U_1$, $L_0 < \zeta < R_0$, and $|L_0|$, $|R_0| \geqslant \Delta$.

Different laps correspond to different c's, but they all will share uniform Δ and ω.

3. SHAPE OF A LAP

The flow generated by (4) in the (v, v_y)-plane is easy to grasp, and we use it here to verify the form of a lap suggested by Fig. 2 and Fig. 3. In particular, we establish the monotonicity of the transition layer asserted in Theorem 2. Recall that $x = \varepsilon y$ and $v_x = v_y/\varepsilon = u_x \cdot \eta$, where $\eta(u) = 1 - u^2 = 1/\cosh^2(v)$.

LEMMA 1 (Shape of a lap, see Fig. 3). *For any* $0 < \Delta, \omega < 1$, *there are* ε_0, $d > 0$ *such that if* $c(\cdot)$ *satisfies* $(\mathbf{HC})_{\Delta,\omega}$, $u(\cdot)$ *satisfies* $(\mathbf{HU})_d$ *and solves* (2) *with* $\varepsilon < \varepsilon_0$, *then* $u_x(x) > 0$ *for* $0 < x < 1$ *and* u *is monotonically increasing.*

Moreover, there are intervals L, R, $I \subset [0, 1]$, $L < \zeta < R$, $L \cup R \subset I$, *that depend only on* Δ *and* ω, *such that*

(i) $v_y \geqslant \sqrt{1/2}$ *for* $x \in I$;

(ii) $v_y \geqslant \sqrt{1/2 + \omega/4}$ *for* $x \in L \cup R$;

(iii) $v|_L \leqslant -|L|/(100\varepsilon)$ *and* $v|_R \geqslant |R|/(100\varepsilon)$.

Remark 1 (Symmetry). Note that our assumptions and conclusions are unchanged if we replace u, c, x by $-u$, $-c$, $-x$. Thus we will provide only arguments for $c > 0$ most of the time.

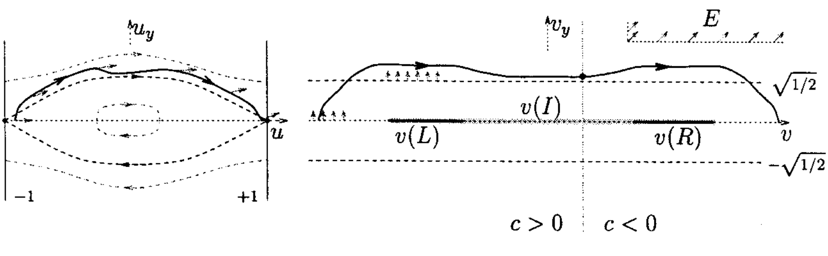


FIG. 3. On the left, the stable one transition layer equilibrium (a lap) from Fig. 2 with the rotation of the initial vector $\partial/\partial u$. On the right, the same lap in the preferred coordinate; some of the x-independent features of the vectorfield are indicated.

Proof of Lemma 1. (ii). On the line $v_y = 0$ we have $v_{yy} = -u + c$. For small d, $d < \Delta/2$; and, by $(\mathbf{HU})_d$, $u(x) < -1 + d$ for $x \in U_0$, hence $v_{yy}(x) \geqslant -(-1+d) - (1-\Delta) \geqslant \Delta/2 > 0$ there. Since, $v_y(0) = 0$ this shows that $v_y > 0$ over U_0. Moreover, for $x \in L_0$, we additionally have $c(x) \geqslant \omega$ by (ii) of $(\mathbf{HC})_{\Delta,\omega}$, so v_y grows there with a definite[4] y-speed $v_{yy} \geqslant (2v_y^2 - 1)\,u + \omega \geqslant \omega/2$, if only $2v_y^2 - 1 \leqslant \omega/2$, i.e. $v_y \leqslant \sqrt{1/2 + \omega/4}$. Thus, after fixing any subinterval $L \subset \operatorname{int}(L_0)$, we may find ε_0 so that $v_y(x) \geqslant \sqrt{1/2 + \omega/4}$, $x \in L$, as there is $O(1/\varepsilon)$ worth of y-time in $[\inf L_0, \inf L]$ for v_y to grow. Analogous arguments, with the time run backwards, give $v_y(x) > 0$, for $x \in U_1$, and $v_y(x) \geqslant \sqrt{1/2 + \omega/4}$, for $x \in R$, where $R \subset \operatorname{int}(R_0)$.

(i). Set $I = [\inf L, \sup R]$. For $x \in L \cup R$, we already have $v_y(x) > \sqrt{1/2}$ by (ii). As long as $x \in [\sup L, \zeta)$, $c(x) > 0$, so $v_y(x)$ can not cross below $v_y = \sqrt{1/2}$ where $v_{yy}(x) = c(x) > 0$. Analogously, as long as $x \in (\zeta, \inf R]$, $c < 0$, so v_y can not cross below $\sqrt{1/2}$ with the time run backwards—(i) is proved.

Monotonicity of u. In the proof of (ii) we saw that $v_y > 0$ on U_0 and U_1, which put together with (i) implies that $v_y > 0$ for all $x \in (0, 1)$.

(iii). We will adjust L, R to get (iii) with (i) and (ii) left intact. First check that $E := \{(v, v_y) : u \geqslant 1/2,\ v_y \geqslant 2\}$ is invariant under the flow so that $(v(x), v_y(x)) \notin E$ because clearly $(v(1), v_y(1)) \notin E$. As a consequence any interval $J \subset L$ such that $u|_J > 0$ is short, i.e. of order $O(\varepsilon)$. Indeed, if $x \in J$, then $v_y(x) \geqslant \sqrt{1/2}$ and $v_{yy}(x) \geqslant (2v_y^2 - 1)\,u + c \geqslant c \geqslant \omega$ by (ii) of $(\mathbf{HC})_{\Delta,\omega}$. Thus, for $x_0 = \sup J$, we have $v(x_0) \geqslant \sqrt{1/2}\,|J|/\varepsilon$ and $v_y(x_0) \geqslant \omega\,|J|/\varepsilon$. This implies $|J| = O(\varepsilon)$ for otherwise we hit E. In this way, by shrinking L and R a bit (of order ε), we can get $v|_L \leqslant 0$ and $v|_R \geqslant 0$. Because v moves fast over L, R, namely $v_y \geqslant \sqrt{1/2}$, further shrinking of L and R by 1/50 of their length yields (iii). ∎

The complement of $L \cup R$ in $[0, 1]$ consists of three intervals: the *central piece* $C := I \setminus (L \cup R)$ and the two *fringes* $F_- := [0, \inf L]$ and $F_+ := [\sup R, 1]$. By monotonicity, the partition of the x-time interval $[0, 1]$ into $F_- \cup L \cup C \cup R \cup F_+$ maps via $x \mapsto v(x)$ to a partition of $\mathbf{R}$ into $v(F_-) \cup v(L) \cup v(C) \cup v(R) \cup v(F_+)$. We need some rough understanding of v over the fringes $F_\pm$—c.f. Fig. 3.

Lemma 2 (Fringe addendum). *The following assertions can be added to Lemma* 1:

(i) $v_y(x) \geqslant \min\{\Delta y/2, \sqrt{\Delta}/4\}$ *for* $x \in F_-$,

[4] We use the word *definite* in reference to quantities of order $O(1)$.

(ii) $v_y(x) \geqslant \min\{\Delta(\varepsilon^{-1} - y)/2, \sqrt{\Delta}/4\}$ *for* $x \in F_+$,

(iii) $|v_y(x)| \leqslant 10$ *for* $x \notin C$, *i.e.* $x \in F_- \cup L \cup R \cup F_+$.

Proof of Lemma 2. We show (i); use the mirror argument for (ii). Since $\rho := -(1-\Delta/2)/(1-\Delta/8) > -1$, from (iii) of Lemma 1, we see that $-1 < u(x) \leqslant \min\{\rho, -1/2\}$, for $x \in L$, ε_0 small; and this is still true for $x \in F_- \cup L$ by monotonicity of u. As long as $v_y(x) \leqslant \sqrt{\Delta}/4$ and $x \in F_- \cup L$, we have

$$v_{yy}(x) = (2v_y^2 - 1)\,u + c \geqslant (\Delta/8 - 1)(-(1-\Delta/2)/(1-\Delta/8)) - (1-\Delta) = \Delta/2.$$

It follows that $v_y(x) \geqslant \Delta/2 \cdot y$ before it reaches the cutoff $\sqrt{\Delta}/4$.

Part (iii) is more crude. As noted above, $u(x) < -1/2$ for $x \in F_- \cup L$. So, along the line $v_y = 10$, we have $v_{yy} = (2v_y^2 - 1)\,u + c \leqslant (200-1)\cdot(-1/2) \leqslant -99$. This makes it impossible for $v_y(x)$ to climb over $v_y = 10$ while $x \in F_- \cup L$. The argument for $R \cup F_+$ is analogous. ∎

4. THE PROJECTIVE ACTION

As in the introduction, identify the tangent bundle to the (v, v_y) plane with $\mathbf{R}^2 \times \mathbf{R}^2$ via coordinate (v, v_y, p, q), where q, p are the variations of v and v_y correspondingly. The variational flow is given by (5), and the signs of the entries on the right side immediately reveal the following properties of the fundamental solution $\Phi(y, y_0) \in \mathrm{Gl}(\mathbf{R}^2)$, $y, y_0 \in [0, 1/\varepsilon]$:

(P1) if $|v_y|_{[y_0, y]}| > \sqrt{1/2}$, then $\Phi(y, y_0)$ transforms the standard cone $\Gamma = \{(q, p) : pq > 0\}$ strictly into itself;

(P2) if $|v_y(y_0)| < \sqrt{1/2}$, then $\Phi(y, y_0)$ moves the vector $(1, 0)$ clockwise outside of Γ for small $y - y_0 > 0$;

(P3) $\Phi(y, y_0)$ moves $(0, 1)$ clockwise inside Γ for small $y - y_0 > 0$.

In fact, (P1) is the prevailing mechanism (along a lap) as expressed by the following proposition.

PROPOSITION 1. *For any* $0 < \Delta$, $\omega < 1$ *and* $\delta > 0$, *there are* d, $\varepsilon_0 > 0$ *such that, if* $c(\cdot)$ *satisfies* $(\mathbf{HC})_{\Delta,\omega}$ *and* $u(\cdot)$ *satisfying* $(\mathbf{HU})_d$ *solves* (2) *with* $\varepsilon < \varepsilon_0$, *then* $\Phi(y, 0)\,\Gamma$ *is contained in the* δ (*projective*) *neighborhood of* Γ *for* $y \in [0, \zeta/\varepsilon]$. *Moreover*, $\Phi(\zeta/\varepsilon, 0)$ *maps* Γ *strictly into* Γ *with zero total rotation* (*in particular, any vector is rotated by less than* $\pi/2$).

Proof of Theorem 3 *from Proposition* 1. From Proposition 1, $\Gamma^+ := \Phi(\zeta/\varepsilon, 0)\,\Gamma \subset \Gamma$ with zero rotation. The mirror version of the proposition

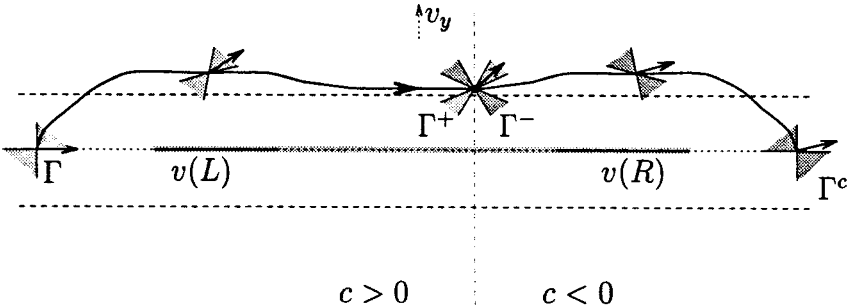


FIG. 4. The cones along a lap. The cone Γ, while not preserved by the variational flow at all times, is preserved by the (slow) time-one-map. Indeed, we show that Γ and Γ^c (its complement and dual under time inversion) have non-overlapping images at $c = 0$.

gives[5] $\Gamma^- := (\Phi(1/\varepsilon, \zeta/\varepsilon))^{-1}\,\Gamma^c \subset \Gamma^c$ with zero rotation, and so $\Phi(1/\varepsilon, \zeta/\varepsilon)$ $(\Gamma^-)^c = \Gamma$ with zero rotation. Since $\Gamma^+ \subset \Gamma \subset (\Gamma^-)^c$, we can *pipeline* as follows, see Fig. 4,

$$\begin{aligned}\Phi(1/\varepsilon, 0)\,\Gamma &= \Phi(1/\varepsilon, \zeta/\varepsilon)(\Phi(\zeta/\varepsilon, 0)\,\Gamma)\\ &= \Phi(1/\varepsilon, \zeta/\varepsilon)(\Gamma^+) \subset \Phi(1/\varepsilon, \zeta/\varepsilon)(\Gamma^-)^c = \Gamma.\end{aligned}$$

The inclusion is strict and the total rotation is zero. ∎

Proof of Proposition 1. One has to look at the boundary of the cone. It suffices to prove that $\Phi(y, 0)(1, 0)$ stays in the δ-neighborhood of Γ and that $\Phi(\zeta/\varepsilon, 0)(1, 0)$ sits strictly inside Γ. The analogous conclusions for $(0, 1)$ are then immediate from (P3). Clearly only the projective action of Φ is relevant so we consider the slope $s := p/q$, $q > 0$, for which (5) means

$$s_y = V(s, y) := -s^2 + 4v_y u \cdot s + (2v_y^2 - 1)\,\eta, \tag{6}$$

with the initial condition $s(0) = 0$, which is the slope of $(1, 0)$.[6]

Before we go on let us outline the argument. Observe that $\eta(u(x)) = 1/\cosh^2(v(x))$, although increasing along $L \cup F_-$, is extremely small there (of order $1/\cosh^2(\varepsilon^{-1})$) by (iii) of Lemma 1, and so is $b := (2v_y^2 - 1)\,\eta$ because of Lemma 2, (iii). On the other hand $a := -4v_y u$ is a definite positive quantity over L, by Lemma 1, (i) and (iii). The vectorfield in (6) has a stable nearly stationary point near b/a closely followed by s, which must then stay extremely close to 0 over F_- and eventually get positive over L, where $b > 0$. The following claim formalizes this description.

[5] The superscript c indicates the complementary cone.

[6] Note that $s(\cdot)$ can not blow up to $+\infty$ by (P3). By Claim 1, it also does not blow up to $-\infty$.

Claim 1. We have, for small enough d, $\varepsilon_0 > 0$, that

(i) $s(x) \geqslant -1000 \cdot \eta(u(\inf L))/\sqrt{\Delta}$, $x \in F_-$;

(ii) the above bound extends to $x \in (\sup F_-, \zeta)$ and $s(\zeta) > 0$.

Moreover, the right side in (i) tends to 0 as ε shrinks to 0.

The following technical lemma is proved in the end of this section.

LEMMA 3 (Comparison). *Suppose* $z(0) = 0$ *and* $dz/dy = -az + z^2 - b$, $y \geqslant 0$. *If* $a(y) \geqslant 0$ *and* $|b(y)| \leqslant \beta$ *with* β *so small that*

$$2\sqrt{\beta} \cdot \min\{y, (\inf_{t \geqslant y} a(t))^{-1}\} < 1, \tag{7}$$

then

$$z(y) \leqslant 2\beta \min\{y, (\inf_{t \geqslant y} a(t))^{-1}\}, \qquad y \geqslant 0. \tag{8}$$

Proof of the claim, (i). Set $z := -s$, $a = -4v_y u$ and $b := (2v_y^2 - 1)\eta$. For sufficiently small ε_0 and $x \in F_-$, we have $u(x) < -\frac{1}{2}$ by (iii) of Lemma 1, and $a(x) \geqslant 4v_y(x) \cdot \frac{1}{2} \geqslant 2\min\{\Delta y/2, \sqrt{\Delta}/4\}$ by (i) of Lemma 2. Part (iii) of Lemma 2, yields $|b(x)| \leqslant |(2v_y(x)^2 - 1)\,\eta(u(x))| \leqslant 200 \cdot \eta(u(\inf L)) = 200/\cosh^2(v(\inf L)) =: \beta$, also for $x \in F_-$.

In this way, for $x = y\varepsilon \in F_-$, we have

$$\min\{y, (\inf_{t \geqslant y} a(t))^{-1}\} \leqslant \min\{y, \min\{\Delta y, \sqrt{\Delta}/2\}^{-1}\} \leqslant 2/\sqrt{\Delta},$$

where we verify the second inequality by inspecting the cases $y \geqslant 1/(2\sqrt{\Delta})$ and $y \leqslant 1/(2\sqrt{\Delta})$. To satisfy the conditions of Lemma 3 we confirm that $2\sqrt{\beta} \cdot 2/\sqrt{\Delta} < 1$. Indeed, $v(\inf L) \to -\infty$ as $\varepsilon_0 \to 0$ by (iii) of Lemma 1, so also $\beta \to 0$. Now, (i) is a consequence of (8) in Lemma 3:

$$-s(x) \leqslant 2\beta \min\{y, (\inf_{t \geqslant y} a(t))^{-1}\} \leqslant 2\beta \cdot 2/\sqrt{\Delta}, \qquad x \in F_-. \quad \blacksquare$$

Proof of the claim, (ii). Factor $V(s, y) = (s - s_-)(s_+ - s)$ where $-a = s_+ + s_- = 4uv_y$ and $-b = s_+ s_- = -(2v_y^2 - 1)\eta$. As in (i), for sufficiently small ε_0 and all $x \in L$, one has

$$u(x) < -1/2, \qquad \eta(u(x)) \leqslant 0.0001\sqrt{\Delta} \leqslant 0.0001$$

by (iii) of Lemma 1, and

$$\sqrt{1/2 + \omega/4} \leqslant v_y \leqslant 10$$

by (ii) of Lemma 1 and (iii) of Lemma 2. Hence,

$$\eta\omega/2 \leqslant b \leqslant 200\eta \text{and} 2\sqrt{1/2} = -4(-1/2)\sqrt{1/2} \leqslant a \leqslant -4\cdot(-1)\,10 = 40.$$

Because $\eta < 0.0001$ we see that $b/a \leqslant a$ and $s_- < 0 < s_+$, so:

$$s_- = -a - s_+ \leqslant -a \leqslant -2\sqrt{1/2}, \qquad s_+ = -b/s_- \leqslant b/a \leqslant a \leqslant 40,$$

$$s_- = -a - s_+ \geqslant -2a \geqslant -80, \qquad s_+ = -b/s_- \geqslant b/2a \geqslant \frac{\eta\omega/2}{80} = \eta\omega/160.$$

Thus, if $x \in L$ and $s(x) \in \Omega := [-1000\cdot\eta(u(\inf L))/\sqrt{\Delta},\ 0]$, then the first and the last inequality yield

$$s_y(x) \geqslant (-1000\cdot\eta(u(\inf L))/\sqrt{\Delta} + 2\sqrt{1/2})(s_+ - 0) \geqslant \sqrt{1/2}\cdot\eta\omega/160.$$

Since $\sup F_- = \inf L$, $s(\inf L) \in \Omega$ by the already proved (i). From the above estimate, $s(x)$ increases and leaves Ω by becoming positive and the amount of y-time it needs for that is at most

$$\frac{1000\cdot\eta(u(\inf L))/\sqrt{\Delta}}{\sqrt{1/2}\cdot\eta(u(\inf L))\,\omega/160} \leqslant 160000/(\sqrt{1/2}\,\omega\Delta).$$

In this way, if only ε_0 is small enough to make the right side above dominated by $|L|/\varepsilon$, s exits Ω through 0 and $s(x_0) > 0$ for some $x_0 \in L$. Because $v_y(x) > \sqrt{1/2}$, for $x \in [x_0, \zeta]$, the property (P1) implies that $s(x)$ stays positive for those x; in particular, $s(\zeta) > 0$. ∎

Proof of Lemma 3. We compare $z(\cdot)$ to $\gamma(\cdot)$ that solves $\gamma_y = -a\gamma + 2\beta$, $\gamma(0) = 0$ and begin with showing that

$$\gamma(y) \leqslant 2\beta \min\{y, (\inf\{a(t): y \leqslant t\})^{-1}\}, \qquad y \geqslant 0. \tag{9}$$

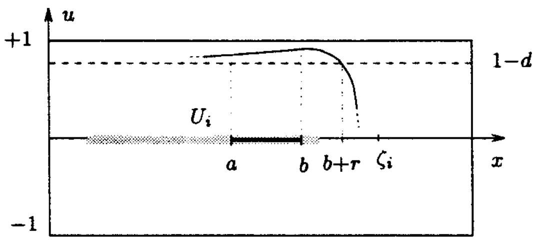
First, $\gamma(\cdot) \geqslant 0$ because $\gamma_y = 2\beta > 0$ when $\gamma = 0$. Hence $\gamma_y \leqslant 2\beta$ and $\gamma(y) \leqslant 2\beta y$, $y \geqslant 0$. Still, (9) could fail on some interval (y_0, y_1) in that $\gamma|_{(y_0, y_1)} > 2\beta/\inf\{a(t): y_0 \leqslant t \leqslant y_1\}$. Consider such an interval maximal with respect to inclusion. From the differential equation, $\gamma_y(y) < 0$ for all $y \in (y_0, y_1)$, and so $\gamma(y) \leqslant \gamma(y_0)$ there. But then (9) holds at $y \in (y_0, y_1)$ because the right side of (9) is non-decreasing in y and (9) holds at y_0 by the maximality of (y_0, y_1). This is a contradiction.

We finish by proving that $z \leqslant \gamma$. If $E := \{y: z(y) > \gamma(y)\}$ is nonempty we take $y_* := \inf E$. Clearly $z(y_*) = \gamma(y_*)$ and, also at y_*, $0 \geqslant \gamma_y - z_y \geqslant \beta - \gamma^2$, i.e. $\gamma \geqslant \sqrt{\beta}$. In view of (9), this contradicts the assumption on β. ∎

5. THE LAP DECOMPOSITION

In the introduction we promised to show that any equilibrium $u(\cdot)$ satisfying hypothesis $(\mathbf{HU})_d$ can be decomposed into laps. For the decomposition we need to know that $u(\cdot)$ has a critical point inside each U_i between two zeros ζ_i and ζ_{i+1}. To be specific assume that $u|_{U_i} \geqslant 1-d$. We actually need to know that the critical point is in a definite distance from the endpoints of U_i because we want the resulting laps to satisfy the $(\mathbf{HC})_{\Delta,\omega}$ assumption on c with uniform Δ and ω. Suppose that a suitable critical point does not exist. Then u would have to be monotone, say increasing, on most of (ζ_i, ζ_{i+1}) only to drop very sharply in the vicinity of ζ_{i+1}. The following lemma shows that this is impossible: it takes $O(1)$ stretch of x before u drops back to mere $1-d$.

LEMMA 4. *For any* $0<\Delta<1$, *there are* r, $C>0$ *with the following property for all sufficiently small* d, $\varepsilon>0$ *and* c *satisfying* (i) *of* $(\mathbf{HC})_{\Delta,\omega}$. *If* u *solves* (2) *and* $u(x) \geqslant 1-d$ *with* $u_x(x)>0$ *for all* x *in some interval* $[a, b]$, *then* $u|_{[b, b+r]} \geqslant 1-d$ *or* $b-a \leqslant C\varepsilon$.



Proof of Lemma 4. For small enough d, $u \approx 1$ over $[a, b]$, so one can find $\kappa>0$ such that $(2\kappa^2-1)u+(1-\Delta) \leqslant -\Delta/2$ over $[a, b]$. Consider $\tilde{J} := [a, b-(b-a)/10]$. There are two cases.

Case 1. $v_y(x_0)<\kappa$ *for some* $x_0 \in \tilde{J}$. For x with $0 \leqslant v_y < \kappa$, we have $v_{yy} \leqslant (2\kappa^2-1)u+c \leqslant -\Delta/2$, so v_y decreases from $v_y(x_0)$ to reach $v_y(x_1)=0$ for some $x_1>x_0$ with $x_1-x_0 \leqslant \varepsilon\kappa/(\Delta/2)$. Since $x_1 \notin [a, b]$, it follows that $(b-a)/10 \leqslant 2\kappa\varepsilon/\Delta$, that is $b-a \leqslant C\varepsilon$ for $C=20\kappa/\Delta$. We may assume that $(b-a) \geqslant C\varepsilon$ from now on.

Case 2. $v_y|_{\tilde{J}} \geqslant \kappa$. Then $v(b) \geqslant v(a)+\kappa 0.9(b-a)/\varepsilon$. Let $x_1>b$ be maximal such that $u|_{[b, x_1]} \geqslant 1-d$ so that $r=x_1-b$. Clearly $u(b)>u(a) \geqslant 1-d=u(x_1)$, so $v(b)-v(x_1) \geqslant v(b)-v(a)$ and consequently $(x_1-b)\nu \geqslant \kappa 0.9(b-a)/\varepsilon$, with $\nu := \sup\{|v_y(x)|: b<x<x_1\}$. Hence, $\varepsilon\nu r \geqslant (b-a)\kappa 0.9$. The lemma follows once we observe that $\nu \leqslant 2$. For $x \in [b, x_1]$, we have $u \geqslant 1-d \geqslant 1/2$ and, as in the proof of (iii) of Lemma 1, $v_y \leqslant 2$ must hold or otherwise $v_y \geqslant 2$ forever—which is a contradiction. ∎

6. SHOOTING FOR UNSTABLE LAPS

To give another application of our approach to a problem beyond the grasp of present singular perturbation methods, we will exhibit an unstable equilibrium to (1) that changes sign in the same direction as c. Existence of such equilibria under transversality conditions has been established in [7]. We assume a rather informal style and restrict to the case of c with one zero. More detailed arguments, similar to those in the previous sections, would be needed for extension to the many lap case.

PROPOSITION 2. *Suppose that* $c(\cdot)$ *has only one zero* ζ, $c|_{[0,\zeta)}<0$, *and* $c|_{(\zeta,1]}>0$. *For sufficiently small* $\varepsilon>0$, *the problem* (1) *has an increasing exponentially unstable equilibrium* $u(\cdot)$.

A key fact is that, for u of interest, the velocity u_x has to change sign at its zeros.

LEMMA 5 (No libration). *Suppose that* $c(\cdot)$ *has only one zero* ζ *and* $c|_{[0,\zeta)}<0$ *and* $c|_{(\zeta,1]}>0$. *For sufficiently small* d, $\varepsilon>0$, *if* $u(\cdot)$ *solves* (2) *with* $u(0)<-1+d$ *and* $u_x(x)\geqslant 0$ *for all* $x\in[0,1]$, *then* u_x *does not vanish except possibly at* $x=0, 1$.

Consider for a moment $c(\cdot)$ constant and equal to $c\in[-1+\Delta, 1-\Delta]$. Then the vector-field $X_c=(v_y, u(2v_y^2-1)+c)$, $u=\tanh(v)$, whose flow ϕ^y in the (v, v_y) plane generates solutions to (4) and (2), has only one rest point p_c at $v_y=0$, $u=c$. This is an elliptic point and the variational equation (5) yields the fundamental solution of the linearized flow:

$$\exp\left(y\cdot\begin{pmatrix}0 & 1\\ -\eta & 0\end{pmatrix}\right), \qquad \eta=1-c^2, \quad y\in[0, 1/\varepsilon].$$

The vectorfield X_c is integrable so one has a neighborhood V_c of p_c that is an *elliptic island*—meaning that:

(i) V_c is a union of closed trajectories of X_c;

(ii) for any $p\in V_c$, the angle of the ray from p_c to $\phi^y(p)$ increases, and its y-derivative is greater than $\sqrt{\eta}/C$ where $C>1$ is a constant accounting for the eccentricity of the orbits.

Now, obtain a *fast annulus* A_c from the elliptic island V_c by cutting out its center, i.e. remove from V_c one of the orbits and take for A_c the component that does not contain the center p_c (see Fig. 5). It is clear that one can do this construction continuously in c; in particular, the size and the rotation speed of A_c are uniform in c as long as $c\in[-1+\Delta, 1-\Delta]$.

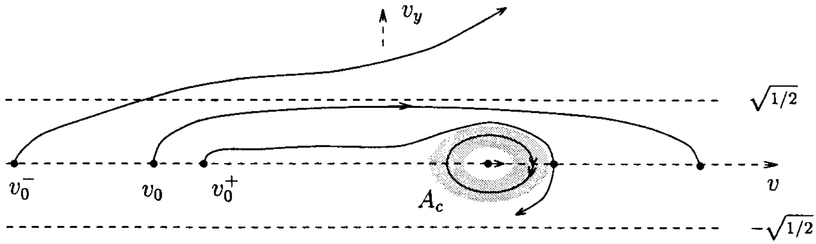


FIG. 5. Our trajectories start far from the fast annulus A_c, which spins rapidly while drifting slowly to the right. Only inside A_c could $v_y = 0$ without changing sign; and, on traversing A_c radially, v_y would have to change sign many times.

Sketch of proof of Lemma 5. For small d, $v(0)$ is so close to $-\infty$ that $(v(0), v_y(0)) \notin V_c$ for any $c \in [-1+\Delta, 1-\Delta]$. On the other hand, since all the time $v_y \geqslant 0$, if $v_y(x_0) = 0$ then $v_{yy}(x_0) = 0$, and $(v(x_0), v_y(x_0)) = p_{c(x_0)} \in U_{c(x_0)}$ from the equation. In this way, continuity forces existence of $x_* \in (0, x_0)$ such that $(v(x_*), v_y(x_*)) \in A_{c(x_*)}$. If c were fixed and equal to $c(x_*)$, this would lead to a contradiction because the motion within A_c is a fast rotation leading to $v_y < 0$ in O(1) of y-time. Actually c drifts very slowly in y-time, $|dc/dy| = O(\varepsilon)$, and we still get a contradiction for small enough ε. How small ε must be taken does not depend on $u(\cdot)$ but only on the properties of our fast annuli—these are uniform. ▮

Sketch of proof of Proposition 2. Let $v(\cdot)$ be a solution of (4) with initial data $v(0) = v_0$ and $v_y(0) = 0$. We will use shooting to find v_0 such that $v_y(1) = 0$ and $v_y(x) > 0$ for $x \in (0, 1)$. Observe that, as long as $c < 0$, v_y can not climb over $\sqrt{1/2}$. Similarly, once $v_y > \sqrt{1/2}$ and $c > 0$, it stays that way.

Claim 1. *For any fixed small* $\varepsilon > 0$, v_y *does not vanish except at* $x = 0$ *for all initial conditions* v_0 *sufficiently close to* $-\infty$. Let x_1 be maximal such that $v_y|_{[0, x_1)} \leqslant \sqrt{1/2}$. As observed, vanishing of v_y can happen only on $[0, x_1]$. For $v_0 \leqslant -\sqrt{1/2} \cdot O(1/\varepsilon^2)$, we clearly have $v(x_1) \leqslant -O(1/\varepsilon)$ so that $u(x_1) \leqslant -1 + \Delta/2$ and consequently $u|_{[0, x_1]} \leqslant -1 + \Delta/2$. Therefore $v_y|_{(0, x_1]}$ can not touch $v_y = 0$ because $v_{yy} = -u + c \geqslant \Delta/2 > 0$ there.

Claim 2. *For any fixed* $v_0 \approx -\infty$, $v_y|_{(0, 1]}$ *has a zero for all small enough* $\varepsilon > 0$. Pick $b \in (0, \zeta)$ and look at v_y over $L := [0, b]$ where $v_y \leqslant \sqrt{1/2}$ by the earlier observation. At first $v_{yy} > \Delta/2$ and v_y becomes definite positive; however, we will argue that $c < -\omega := \sup c|_L < 0$ over L will soon force v_y to get back to 0. One of two things can happen a priori. Either the trajectory of (v, v_y) enters the fast annulus A_c, and then it gets rotated into $v_y = 0$, or (v, v_y) stays away from A_c and thus has definite $v_y > O(1) > 0$. This however gets (v, v_y) into $u > 0$ where $v_{yy} < c \leqslant -\omega$ quickly pushes v_y down to $v_y = 0$ and beyond —all that happening in O(1) of the y-time, that is still $x \in L$, if only the ε was small.

Let v_0^- and v_0^+ be the initial conditions provided by Claim 1 and Claim 2 respectively—see Fig. 5. Between v_0^- and v_0^+ there must be the supremum v_* of those v_0 for which v_y is positive for all $x > 0$. For the corresponding $u(\cdot)$, $v_y(\cdot)$, and thus also $u_x(\cdot)$, has a zero —and it must be at $x = 1$ by the lemma. Hence, $u(\cdot)$ is an equilibrium of (1).

From our first observation, $0 \leqslant v_y < \sqrt{1/2}$ for all times. Using (P2) of Section 4, one immediately concludes that $u(\cdot)$ is unstable via Prüfer method, as explained in the introduction. In fact, the unstable manifold of u is one-dimensional. ▮

ACKNOWLEDGMENTS

The author is grateful to Jack Hale for introduction to the problem and encouragement. Special thanks go to Thomas Wanner for his helpful remarks on the manuscript.

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