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Monotonicity of rotation set for toroidal chaos of a resonantly kicked linear oscillator*

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Abstract. We investigate a model of a charged particle in a uniform magnetic field with resonant nonlinear forcing. The resonance is chosen so that the problem is invariant under a rank 2 lattice of translations in velocities and thus reduces to iteration of a torus map. The rotation set of this map, at least for well-pronounced nonlinearities, has a positive (and easy to estimate) area. This is an indication of 'toroidal chaos', and physically corresponds to stochastical heating of the plasma. Our main result is the monotonicity of the rotation set as a function of the amplitude of the wave in the case when the forcing is 'saw-tooth-like'.

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1. Introduction

This paper is concerned with a concrete family of maps of a two-dimensional torus that are isotopic to the identity and have the rotation set with nonempty interior—and thus they exhibit so-called 'toroidal chaos', see [11]. Its purpose is twofold. First, we wish to indicate the utilitarian role of the recently developed theory of rotation sets in the context of a classical example from physics. Secondly, we hope that this example is of theoretical interest to the coupled oscillators community as a relatively accessible testing ground for analysing the dependence on a parameter of a well-developed rotation set with nonempty interior. Unlike the mode-locked or partially mode-locked case [2, 6], this is a truly uncharted territory; particularly, due to scarcity of understood examples and inherent difficulties in numerical exploration of the boundary of the rotation set [1].

We commence with a brief introduction of the physical model. Consider a particle of mass M and charge e moving in $\mathbb{R}^3$ filled with a uniform vertical magnetic field of induction B_0. The force exerted by the field on the particle is proportional to the vector product of its velocity and the field. Thus, the trajectories are curled into spirals with circular projections on the horizontal plane. More precisely, if $x, y, z \in \mathbb{R}$ are the coordinates of the particle (z being the vertical one), then the Newtonian equations of motion are

$$M\ddot{x} = \frac{eB_0}{c}\dot{y},$$

* This paper is based on chapter 4 of my 1995 Stony Brook thesis [9].

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$$M\ddot{y} = -\frac{eB_0}{c}\dot{x},$$
$$M\ddot{z} = 0,$$

where c is the speed of light. The vector of horizontal velocity $(\dot{x}, \dot{y})$ rotates with the *Larmor frequency* $\omega = eB_0/cM$. The three momenta: $p_z := M\dot{z}$, $p_y := M\dot{y} + M\omega x$, and $p_x := M\dot{x} - M\omega y$, are conserved. This simple behaviour can change radically in the presence of an external electric field in the system. We will restrict our attention to the case when the electric field is aligned with the x-axis and does not depend on y, z coordinates [15]. These assumptions reduce the number of essential degrees of freedom to one since the momenta p_z, p_y continue to be integrals of the motion. Furthermore, we will require that the field is a 'tight' wavepacket with amplitude $E(x,t) = \sum_{n=-\infty}^{\infty} TE(x)\delta(t - Tn)$, where $T > 0$ is fixed, $E(x)$ is $2\pi/k$-periodic in x, and $\delta(\cdot)$ is the Dirac delta function. The meaning of the formula is that the packet interacts with the particle by boosting its x-velocity by $\frac{e}{M}TE(x)$ every T seconds. Thus, if $(\dot{x}_n, \dot{y}_n)$ is the horizontal velocity prior to the kick at the time nT $(n \in \mathbb{Z})$, then

$$(\dot{x}_{n+1}, \dot{y}_{n+1}) = R_\alpha \left(\dot{x}_n + \frac{eT}{M} E\left(\frac{p_y - M\dot{y}_n}{M\omega} \right), \dot{y}_n \right) \tag{1}$$

where R_α is the rotation by the angle $\alpha = \omega T$. In this way, the analysis of the physical model reduces to an investigation of dynamics on $\mathbb{R}^2$ generated by the mapping

$$F : (\dot{x}_n, \dot{y}_n) \mapsto (\dot{x}_{n+1}, \dot{y}_{n+1}).$$

This reduction was first carried out in [15].

What makes the map F interesting is that (even for small amplitudes of E) numerical experiments and physical heuristic suggest that it exhibits trajectories escaping towards infinity with a nonzero 'average acceleration'. A significant physical consequence is an unbounded diffusion of plasma particles called also *stochastical heating* [15]: the particles increase their kinetic energy (heat up) by exploiting the energy of the electric field.

In studying the above behaviour, much attention has been paid to the resonant cases when α is rational [15, 3, 14]. Among those, the cases of choice are $\alpha = \pi/2$, $\alpha = \pi/3$, or $\alpha = 2\pi/3$—the only possibilities if we require that some positive iterate of F is a doubly periodic map of $\mathbb{R}^2$, i.e. it commutes with some faithful $\mathbb{Z}^2$-action on $\mathbb{R}^2$. We shall analyse in more detail the case of $\alpha = 2\pi/3$ as the other cases can be treated analogously. Thus, the third iterate F^3 is doubly periodic, and moded out by the $\mathbb{Z}^2$-action, it yields a torus diffeomorphism $f_3 : \mathbb{T}^2 \to \mathbb{T}^2$ which is isotopic to the identity.

In terms of the torus map, the presence of *stochastical heating* corresponds to the rotation set with interior. The rotation set, denoted $\rho(F^3)$, is by definition ([12], see also [13]) equal to the set of all limit points of the sequences of the form

$$\frac{F^{3n_i}(x_i) - x_i}{n_i}, \qquad i \in \mathbb{N}$$

where $n_i \to \infty$ and $x_i \in \mathbb{R}^2$ are arbitrarily chosen sequences. It is always a compact convex subset of the plane [12], and in view of (1), it can be interpreted as the totality of asymptotic average accelerations exhibited by the particles (with fixed momentum p_y). Clearly, the rotation set is one of the basic combinatorial characteristics of the diffusion in the stochastic web; however, it received little attention in this context. This is especially unjust now that a general theory linking the rotation set with other dynamical phenomena became sufficiently developed. For example, if we know just a single periodic point for f_3 with a nontrivial rotation vector $\rho = p/q$, i.e. we have $x \in \mathbb{R}^2$ with $F^{3q}(x) = x + p$ for

$q \in \mathbb{N}$ and $p \in \mathbb{Z}^2 \setminus \{(0,0)\}$, then we can conclude (see proposition 2.1) that $\rho(F^3)$ contains a whole triangle spanned by ρ and its rotations by $2\pi/3$ and $4\pi/3$. This already implies positive entropy [11], existence of periodic orbits with all rational rotation vectors in the triangle [4], and existence of compact invariant sets with their restricted rotation sets equal to an *a priori* chosen subcontinuum of the triangle [11]. On a more quantative note, the topological entropy of the map is explicitly estimated from below by a function proportional to the logarithm of the area of the triangle [8, 9]. Moreover, the entropy measured at any fixed rotation vector (i.e. roughly counting only trajectories with that rotation vector), is positive up to the boundary of the rotation set with an explicit lower bound proportional to the area and distance of the rotation vector from the boundary [9]. While we refer the reader to the cited works for a broader exposition and proofs, we hope that these highlights explain why the rotation set is well worth studying.

For our map, the basic properties of the rotation set are outlined in section 2. (We should however stress that, while we prove that the rotation set has nonempty interior and estimate its size for rather large values of the amplitude of E, we make no attempt to tackle the case of small E, for which we know no easy alternative to the complicated perturbation apparatus as in [5].)

Our main result is that of monotonicity of $\rho(F^3)$ as a function of the amplitude of E—also for rather large amplitudes, and for 'saw-tooth-like' E's only. The 'saw-tooth-like' nonlinearity is admittedly not the most natural; its choice is somewhat analogous to that of the tent map for the iteration of interval maps. Still the result comes amidst complete lack of any rigorous or numerical study of parameter dependence of a well-developed rotation set with nonempty interior. Numerics are difficult because of scarsity of trajectories that contribute to the boundary of the rotation set. Theoretical considerations are usually stuck on the reef of computing the rotation set. Unless one knows a Markov partition for the map, this is difficult and has only been accomplished in a few specially doctored cases [7, 10].

The proof hinges on a formulation of the problem in terms of a second-order recurrence equation and on a shadowing-type result (lemma 3.2) that assures that periodic orbits do not go away as E increases, which implies the monotonicity of the rotation set via the result by Franks [4].

2. The mapping and its rotation set

The rotation $R_{2\pi/3}$ is linearly conjugated to the map

$$L : (u, v) \mapsto (v, -u - v), \qquad u, v \in \mathbb{R}.$$

By choosing the conjugacy that sends the $\dot{x}$-axis to the u-axis and by rescaling the variables, one conjugates F (as prescribed by (1)) to G given by

$$G : (u, v) \mapsto L(u + \phi(v), v) = (v, -u - v - \phi(v)), \tag{2}$$

where $\phi(v) = eT/M\omega k \cdot E(-kv + \frac{p_y}{M\omega})$ is 2π-periodic. The third iterate of G commutes with the translations of $\mathbb{R}^2$ by vectors in $(2\pi\mathbb{Z})^2$, and so it factors to a torus map

$$g_3 : \mathbb{T}^2 \to \mathbb{T}^2.$$

Even though G and g_3 depend on the choice of ϕ, we suppress this dependence in the notation since it will always be clear which ϕ we are talking about. Hereafter, if we do not state otherwise, ϕ is an arbitrary fixed continuous function of period 2π. However, we will remember that ϕ naturally comes embedded in the two-parameter family

$$\phi_{K,\eta}(x) := K\phi(x + \eta), \qquad x \in \mathbb{R}, \tag{3}$$

where $\eta \in \mathbb{R}$ and $K > 0$ correspond to the momentum p_y and the strength of the forcing respectively.

Fact 2.1. *For any $\eta \in \mathbb{R}$, replacing $\phi(\cdot)$ with a shifted function $\phi(\cdot + \eta) + 3\eta$ is equivalent to conjugating G by the translation $(u, v) \mapsto (u - \eta, v - \eta)$. In particular, the rotation set $\rho(G^3)$ of G^3 is unaltered by the shift.*

For $r \in \mathbb{R}$, we will denote by $\mathrm{Hex}(r)$ the hexagon with vertices $\pm(r, 0)$, $\pm L(r, 0) = \pm(0, -r)$, $\pm L^2(r, 0) = \pm(-r, r)$. The following are the most basic properties of the rotation set $\rho(G^3)$.

Proposition 2.1. *If G is defined by (2), then:*
(i) $\rho(G^3)$ has a threefold symmetry, namely $\rho(G^3) = L(\rho(G^3)) = L^2(\rho(G^3))$;
(ii) if ϕ is odd, then $\rho(G^3) = -\rho(G^3)$;
(iii) $\rho(G^3)$ is contained in $\mathrm{Hex}(\sup\phi - \inf\phi)$.

Proof of proposition 2.1(i). Let $Q : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$(u, v) \mapsto (u + \phi(v), v),$$

so that $G = L \circ Q$. The displacement of a point $p \in \mathbb{R}^2$ under $n + 1$ iterates of G is

$$G^{n+1}(p) - p = L \circ Q \circ G^n(p) - L \circ G^n(p) + L(G^n(p) - p) + Lp - p.$$

The first and last differences on the right-hand side are bounded, so $(1/(n+1))(G^{n+1}(p) - p)$ is asymptotically equal to $L((1/n)(G^n(p) - p))$, as $n \to \infty$; and thus $\rho(G^3) = L(\rho(G^3))$. □

Proof of proposition 2.1(ii). Just note that if ϕ is odd, then Q commutes with the central symmetry $p \mapsto -p$, and so does the whole map G. □

Proof of proposition 2.1(iii). Write the displacement of $p \in \mathbb{R}^2$ under G^3 as follows

$$G^3(p) - p = L(Q \circ G^2(p) - G^2(p)) + L^2(Q \circ G(p) - G(p)) + L^3(Q(p) - p).$$

The arguments of $L, L^2, L^3 = \mathrm{id}$, above, belong to a segment I with endpoints $(\inf\phi, 0)$ and $(\sup\phi, 0)$, so $G^3(p) - p$ sits inside the algebraic (Minkowski) sum $L(I) + L^2(I) + I$, which equals $\mathrm{Hex}(\sup\phi - \inf\phi)$. □

Proposition 2.1(iii) can be sharp, as it is shown by the example below. For convenience, we note the following first.

Fact 2.2. *If $(u, v) \in \mathbb{R}^2$ satisfies $\phi(-v - u - \phi(v)) = \phi(v)$ and $\gamma := \phi(v) - \phi(u) \in 2\pi\mathbb{Z}$, then (u, v) is a lift of a fixed point of g_3 and $G^3(u, v) = (u, v) + (0, \gamma)$, so that $(0, \gamma) \in \rho(G^3)$.*

Proof. We apply G to (u, v) thrice

$$\begin{aligned}(u, v) &\mapsto (v, -u - v - \phi(v))\\ &\mapsto (-u - v - \phi(v), u + \phi(v) - \phi(-v - u - \phi(v)))\\ &= (-u - v - \phi(v), u) \mapsto (u, v + \phi(v) - \phi(u)).\end{aligned}$$

□

Actually, from the proof, one can see that the inverse of the fact is also true.

Example. Consider any ϕ normalized so that $\sup\phi = K$ and $\inf\phi = -K$ (see fact 2.1). By fact 2.2, if there are u_1, u_2, v_1, v_2 such that $\phi(v_1) = \phi(v_2) = \phi(-v_1 - u_1) = K$ and $\phi(u_1) = \phi(u_2) = \phi(-v_2 - u_2) = -K$, we conclude that both $\pm(0, 2K)$ are in the rotation set provided also $K = 0 \pmod{2\pi}$. By (i) of proposition 2.1, the rotation set is then actually equal to $\mathrm{Hex}(2K)$. It is not difficult to draw a graph of ϕ with the required placement of the maxima and minima. (Use for example $u_1 = -\pi/2$, $u_2 = -\pi/6$, $v_1 = \pi/6$, $v_2 = \pi/2$.) (One can see that any suitable ϕ has to attain one of its global extrema at least three times per period.)

Another consequence of fact 2.2 is the following very crude lower bound on the rotation set.

Proposition 2.2. *If $l := \sup\phi - \inf\phi \geqslant 24\pi$, then the triangle with vertices $L^i(0, l/2)$, $i = 1, 2, 3$, is contained in the rotation set $\rho(G^3)$.*

Proof. Using fact 2.1, we can shift the graph of ϕ so that there are v_0, u_0 with $\phi(v_0) = \sup\phi \geqslant 12\pi$ and $\phi(u_0) = \inf\phi \leqslant -12\pi$. Now, as we push v continuously away from v_0 (in either direction), before $\phi(v)$ drops below $\sup\phi/2$, the expression $-2v - \phi(v)$ changes by at least $-4\pi + 6\pi = 2\pi$, thus sweeping the whole period. In particular, $\phi(-2v - \phi(v))$ sweeps, in the process, an interval containing $[\inf\phi, \inf\phi/2]$; and so, by the intermediate value theorem, there is v such that $l/2 = \sup\phi/2 - \inf\phi/2 \leqslant \phi(v) - \phi(-2v - \phi(v)) \in 2\pi\mathbb{Z}$. Set $u = -2v - \phi(v)$, and see that (u, v) satisfies the assumption of fact 2.2, thus placing $(0, \phi(v) - \phi(u))$ in the rotation set. By (i) of proposition 2.1, $L^i(0, \phi(v) - \phi(u))$, $i = 1, 2$, are in the rotation set as well. □

3. The monotonicity theorem

We will call ϕ *unimodal*, if it has only two intervals of monotonicity over its minimal period. We will refer to the endpoints of the maximal intervals of monotonicity as *critical points of* ϕ. The following theorem is our main result.

Theorem 3.1. *For a piecewise-smooth unimodal and odd ϕ with*

$$D := \inf_x |\phi'(x)| \geqslant 6,$$

the rotation set of G^3 corresponding to $K\phi$ is nondecreasing in K for $K \in [1, +\infty)$, i.e. $K_1 \leqslant K_2$ implies $\rho(G^3_{K_1}) \subseteq \rho(G^3_{K_2})$. Actually, all periodic orbits of the torus map can be continued as K increases.

Note that the rotation set is not constant in K by proposition 2.2.

Before we embark on the proof we need some preliminaries. A convenient way of viewing a trajectory of a point $(u, v) \in \mathbb{R}^2$ under G is to label it as a sequence $(x_k)_{k\in\mathbb{Z}}$ so that $x_0 = u$, $x_1 = v$ and $(x_k, x_{k+1}) = G(x_{k-1}, x_k)$, for $k \in \mathbb{Z}$. It is then characterized by the following recurrence equation

$$x_{k-1} + x_k + \phi(x_k) + x_{k+1} = 0, \qquad k \in \mathbb{Z}. \tag{4}$$

For $n \in 3\mathbb{N}$, a periodic orbit of G of period n thus corresponds to $x = (x_k)_{k=0}^{n-1} \in \mathbb{R}^n$ satisfying (4) with a cyclic indexing modulo n. For $\phi = 0$, this is when every point is fixed by G^3, the two-dimensional space of solutions to (4) is conveniently spanned by

$$e = (1, -1, 0, 1, -1, 0, \ldots, 1, -1, 0),$$
$$Se = (0, 1, -1, 0, 1, -1, \ldots, 0, 1, -1),$$
$$S^2e = (-1, 0, 1, -1, 0, 1, \ldots, -1, 0, 1),$$

where $S : \mathbb{R}^n \to \mathbb{R}^n$ shifts the coordinates cyclically to the right, and $S^3e = e$.

The orbits of G that correspond to periodic orbits of g_3 on the torus with nonzero rotation vector come from $(x_k)_{k\in\mathbb{Z}}$ that satisfy (4) together with the periodicity condition $x_{k+n} - x_k = v_{k \pmod 3}$, where $v_0, v_1, v_2 \in 2\pi\mathbb{Z}$, $n \in 3\mathbb{N}$. To search for them, we may use the following scheme. (Below $\langle x, y\rangle := \sum_{i=0}^{n-1} x_i y_i$, for $x, y \in \mathbb{R}^n$.)

Lemma 3.1. *For $n \in 3\mathbb{N}$, if $z = (z_k)_{k=0}^{n-1} \in \mathbb{R}^n$ satisfies*

$$z_{k-1} + z_k + \phi(z_k) + z_{k+1} = \xi_k, \qquad 0 \leqslant k \leqslant n-1, \tag{5}$$

where $\xi \in (2\pi\mathbb{Z})^n$ and the indexing is cyclic, then there is $\rho \in (2\pi\mathbb{Z})^{\mathbb{Z}}$ satisfying

$$\rho_{k-1} + \rho_k + \rho_{k+1} = -\xi_{k \pmod n}, \qquad k \in \mathbb{Z}, \tag{6}$$

and the sequence $x_k := z_{k \pmod n} + \rho_k$ is a solution to (4). The displacement over a period is given by

$$x_{k+n} - x_k = \langle \xi, S^k e\rangle, \qquad k \in \mathbb{Z}, \tag{7}$$

so the corresponding rotation vector for G^3 is $(1/n)(\langle\xi, e\rangle, \langle\xi, Se\rangle)$. Moreover, one can obtain in this way all of $(x_k) \in \mathbb{R}^{\mathbb{Z}}$ corresponding to orbits of G that cover periodic orbits of g_3 on the torus.

Proof. Solving (6) is straightforward: fix $\rho_0, \rho_1 \in 2\pi\mathbb{Z}$ and calculate the rest of ρ_i's successively. That $x_k = z_{k \pmod n} + \rho_k$ satisfy (4) follows by combining (5) and (6). To calculate the displacement, replace in (4) k with $k+1$ and subtract unaltered (4) to obtain

$$x_{k+3} - x_k = \phi(x_{k+1}) - \phi(x_{k+2}) = \phi(z_{k+1 \pmod n}) - \phi(z_{k+2 \pmod n}). \tag{8}$$

Sum over the range $k, k+3, \ldots, k+n-3$ to obtain

$$x_{k+n} - x_k = \langle (\phi(z_{j \pmod n}))_{j=k+1}^{k+n}, S^k e\rangle = \langle \xi, S^k e\rangle,$$

where we used the fact that

$$\langle (z_{j-1 \pmod n} + z_{j \pmod n} + z_{j+1 \pmod n})_{j=k+1}^{k+n}, S^k e\rangle = 0.$$

Now, suppose that $(x_k)_{k\in\mathbb{Z}}$ covers a periodic orbit of period $n \in 3\mathbb{N}$. Then $x_{k+n} - x_k =: v_{k \pmod 3} \in 2\pi\mathbb{Z}$, and, because both x_k's and the shifted sequence x_{k+n}'s satisfy (4), the sequence $(v_{k \pmod 3})_{k\in\mathbb{Z}}$ obeys the linear equation $v_{k-1 \pmod 3} + v_{k \pmod 3} + v_{k+1 \pmod 3} = 0$. This makes $\xi_{k \pmod n} := x_{k-1 \pmod n} + x_{k \pmod n} + \phi(x_{k \pmod n}) + x_{k+1 \pmod n}$, $k \in \mathbb{Z}$, well defined. To satisfy (5), set $z_k := x_k$, $k = 0, \ldots, n-1$. □

Solving (5) for large n is a daunting task regardless of our choice of ϕ. There is then little hope that the rotation set can be calculated exactly for a continuum of values of the parameters K and η in (3). Our theorem comes, here, as a little consolation and is based on the following 'closing lemma' holding for solutions of (5).

Lemma 3.2. *Fix $\epsilon > 0$ and suppose that $z = (z_k)_{k=0}^{n-1} \in \mathbb{R}^n$ satisfies*

$$z_{k-1} + z_k + K\phi(z_k) + z_{k+1} = \xi_k + \epsilon_k,$$

where $\epsilon_k \in (-\epsilon, \epsilon)$. If, for some open intervals I_k that are free of the critical points of ϕ and contain z_k, $k = 0, \ldots, n-1$, we have

$$d := \inf\{|1 + K\phi'(t)| : t \in I_k, 0 \leqslant k \leqslant n-1\} > 2, \tag{9}$$

and

$$B_{\frac{d}{d-2}\epsilon}(z_k + K\phi(z_k)) \subset \{z + K\phi(z) : z \in I_k\}, \tag{10}$$

then there is $z^* = (z_k^*) \in \mathbb{R}^n$ such that $z_k^* \in I_k$ and

$$z_{k-1}^* + z_k^* + K\phi(z_k^*) + z_{k+1}^* = \xi_k, \qquad 0 \leqslant k \leqslant n-1. \tag{11}$$

Proof. We obtain z^* as the limit of a sequence of corrections of z, each diminishing the supremum of ϵ_k's by a definite multiplicative factor $\lambda < 1$. Hypothesis (9) guarantees that $z_k + K\phi(z_k)$ changes faster (in z_k) than the 'coupling term' $z_{k-1} + z_{k+1}$, thus ϵ_k can be effectively decreased by manipulating z_k only. Hypothesis (10) secures enough 'maneuvering space' to perform all the infinitely many corrections. With this in mind, it will be convenient to think of $\frac{d}{d-2}\epsilon$ as follows. Fix $\kappa \in (0, 1)$ and let $\lambda = 1-\kappa+2\kappa/d < 1$, then

$$\frac{d}{d-2}\epsilon = \frac{\kappa}{\kappa - 2\kappa/d}\epsilon = \kappa\frac{1}{1-\lambda}\epsilon = \kappa\epsilon + \kappa\lambda\epsilon + \kappa\lambda^2\epsilon + \cdots. \tag{12}$$

We describe the first correction. From (12), $\kappa\epsilon \leqslant \frac{d}{d-2}\epsilon$, so, by (10),

$$B_{\kappa\epsilon}(z_k + K\phi(z_k)) \subset \{z + K\phi(z) : z \in I_k\}. \tag{13}$$

It follows that we can find $z_k' \in I_k$, $k = 0, \ldots, n-1$, so that

$$|z_{k-1} + z_k' + K\phi(z_k') + z_{k+1} - \xi_k| = (1-\kappa)|\epsilon_k|$$

and

$$|z_k' + K\phi(z_k') - z_k - K\phi(z_k)| = \kappa|\epsilon_k| \leqslant \kappa\epsilon. \tag{14}$$

Hypothesis (9) then implies that

$$|z_k' - z_k| \leqslant \frac{\kappa}{d}|\epsilon_k|.$$

By combining the two inequalities we obtain

$$z_{k-1}' + z_k' + K\phi(z_k') + z_{k+1}' = \xi_k + \epsilon_k',$$

for some ϵ_k' such that

$$|\epsilon_k'| \leqslant (1-\kappa)|\epsilon_k| + \frac{\kappa}{d}|\epsilon_{k-1}| + \frac{\kappa}{d}|\epsilon_{k+1}| \leqslant \left(1 - \kappa + 2\frac{\kappa}{d}\right)\epsilon = \lambda\epsilon.$$

In this way

$$\epsilon' = \max_{k=0}^{n-1}|\epsilon_k'| \leqslant \lambda\epsilon. \tag{15}$$

To repeat the procedure leading from $z = (z_k) \in \mathbb{R}^n$ to $z' = (z_k') \in \mathbb{R}^n$ with z' as our new z, we need to verify (13) with z_k and ϵ replaced by z_k' and ϵ' respectively. In view of (10) and (14), it suffices to check that

$$\kappa\epsilon + \kappa\epsilon' \leqslant \frac{d}{d-2}\epsilon.$$

This is, however, a consequence of (15) and (12). It should now be clear how the series in (12) unfolds as we need to guarantee the possibility of carrying out the infinitely many consecutive steps. □

Proof of theorem 3.1. Fix $K > 1$. We want to prove that the rotation set of G^3 corresponding to $K\phi$ contains that corresponding to ϕ. Note that, by (i) of proposition 2.1, the rotation set is either equal to $\{(0,0)\}$ or it has nonempty interior. In the latter case, by Franks' theorem [4], the rotation set is the closure of the totality of the rotation vectors of periodic orbits. Consequently, it is enough to show the following. For each periodic orbit of the torus map generated by ϕ, there is one with the same rotation vector for $K\phi$. In view of lemma 3.1, it suffices to find, for $z \in \mathbb{R}^n$ satisfying (5), a point $z^* \in \mathbb{R}^n$ satisfying (11).

Let $M := \sup\phi = -\inf\phi$. Clearly, ϕ and $K\phi$ have the same intervals of monotonicity, $K\phi$ has a larger image, and

$$|K\phi(x) - \phi(x)| \leqslant (K-1)M, \qquad \text{for all } x \in \mathbb{R}. \tag{16}$$

Thus, if $z \in \mathbb{R}^n$ satisfies (5) for some $\xi \in (2\pi\mathbb{Z})^n$, we can find for each z_k a point y_k in the same interval of monotonicity such that $K\phi(y_k) = \phi(z_k)$. By (16) and the assumption on the derivative

$$|z_k - y_k| \leqslant (K-1)M/(KD) \leqslant (K-1)M/(6K).$$

In this way

$$y_{k-1} + y_k + K\phi(y_k) + y_{k+1} = \xi_k + \epsilon_k,$$

with $|\epsilon_k| \leqslant \epsilon := 3(K-1)M/(6K) = (K-1)M/(2K)$.

We will use lemma 3.2 to find $z^* \in \mathbb{R}^n$ which obeys (11). Hypothesis (9) is satisfied because $d \geqslant D - 1 \geqslant 5$. Let I_k be the monotonicity interval containing y_k, that is $I_k = [c_k, c_k']$ where c_k and c_k' are the critical points of ϕ closest to y_k. To verify hypothesis (10), observe that image under $K\phi$ of any maximal interval of monotonicity extends exactly $(K-1)M$ beyond the endpoints of the corresponding image for ϕ. This and the fact that $|K\phi'| \geqslant KD$ yield

$$\begin{aligned}|K\phi(c_k) + c_k - K\phi(y_k) - y_k| &\geqslant |K\phi(c_k) - K\phi(y_k)| - |c_k - y_k| \\ &\geqslant |K\phi(c_k) - K\phi(y_k)|\left(1 - \frac{1}{KD}\right) \geqslant (K-1)M\left(1 - \frac{1}{KD}\right),\end{aligned}$$

and we have the analogous inequality for c_k'. Since $\frac{d}{d-2}\epsilon \leqslant \frac{D-1}{D-3}\frac{(K-1)M}{2K}$, to satisfy (10), we need $\frac{D-1}{D-3}\frac{(K-1)M}{2K} \leqslant (K-1)M(1-1/(KD))$. Dividing both sides by $(K-1)M$ and setting $D = 6$, we obtain a stronger inequality $\frac{5}{3\cdot 2K} \leqslant 1 - \frac{1}{6K}$, which is true because $K \geqslant 1$. □

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