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A toral flow with a pointwise rotation set that is not closed

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Abstract
We give an example of a C^1 flow on the two-dimensional torus for which the pointwise rotation set is not closed.

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1. Introduction

The classical idea of the Poincaré rotation number of an orientation preserving circle homeomorphism found many fruitful adaptations in the context of other dynamical systems for which one can sensibly talk about limiting average displacement (or winding) of orbits in the phase space. (See [2, 17] for surveys.) In particular, in the theory of nonlinear oscillations, a system of d coupled periodically forced oscillators often exhibits an invariant d-dimensional torus and therefore induces a homeomorphism $F : \mathbb{T}^d \to \mathbb{T}^d$, $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, that is homotopic to the identity. (See, e.g. [1] and references therein.) For such F, upon fixing a lift $\tilde{F} : \mathbb{R}^d \to \mathbb{R}^d$, one can speak of the average winding along the fundamental cycles of $\mathbb{T}^d$ for an orbit segment of a point $p \in \mathbb{T}^d$ as measured by the average displacement

$$\frac{\tilde{F}^n(\tilde{p}) - \tilde{p}}{n},$$

where $\tilde{p} \in \mathbb{R}^d$ is a lift of p. In the uncoupled linear case, the above quotient is just the vector of frequencies of the individual oscillators, $(k_1, \dots, k_d)$. In general, the rotation set is meant to collect all asymptotic 'frequency' vectors exhibited by the system and thus is formed by taking a limit of the average displacements as $n \to \infty$. However, how the limit is formalized is a matter of choice and may lead to different sets. Perhaps the most natural impulse is to take only the well defined pointwise limits and form the following *pointwise rotation set*:

$$\rho_{\mathrm{pp}}(\tilde{F}) := \left\{ \lim_{n\to\infty} \frac{1}{n}(\tilde{F}^n(\tilde{p}) - \tilde{p}) : \ \tilde{p} \in \mathbb{R}^d, \text{ the limit exists} \right\}. \qquad (1.1)$$

This set is of obvious interest as it accounts for asymptotic behaviour of actual orbits, but it is awkward to study because the limit often fails to exist for many orbits and is hard to compute for those orbits for which it does exist. That difficulty can be elegantly minimized (see [14] and also [8, 16]) by either considering the average displacements of invariant measures or, more simply, by taking the very liberal approach of admitting into the rotation set all accumulation points of the average displacements for long orbit segments:

$$\rho(\tilde{F}) := \left\{ \lim_{i\to\infty} \frac{1}{n_i}(\tilde{F}^{n_i}(\tilde{p}_i) - \tilde{p}_i) : \ \tilde{p}_i \in \mathbb{R}^d, \ n_i \in \mathbb{N}, \ n_i \to \infty \right\}. \tag{1.2}$$

An immediate advantage of this approach is that $\rho(\tilde{F})$ is compact by design; and it is convex for $d \leqslant 2$ by a result in [14]. This offers a welcome *a priori* simplification in the still very incomplete taxonomy of possible rotation sets [5, 11, 12]. On the other hand, simple examples (e.g. example 1 in [13]) show that $\rho_{\mathrm{pp}}(\tilde{F})$ may be a non-convex proper subset of $\rho(\tilde{F})$. It is the purpose of this note to reinforce the sentiment that $\rho_{\mathrm{pp}}(\tilde{F})$ is more unwieldy than $\rho(\tilde{F})$ and show that it can fail to be closed even for dynamics of as low complexity as those associated with a flow on $\mathbb{T}^2$.

Theorem 1.1. *There exists a C^1-diffeomorphism $F : \mathbb{T}^2 \to \mathbb{T}^2$ that is the time-one-map of a C^1-flow for which the pointwise rotation set $\rho_{\mathrm{pp}}(\tilde{F})$ is not closed. Specifically, $\rho(\tilde{F})$ is an irrationally sloped segment containing $(0, 0)$ and $\rho_{\mathrm{pp}}(\tilde{F})$ contains a sequence of points converging to a limit point strictly inside the segment $\rho(\tilde{F})$ but fails to contain that limit point.*

Note that any flow $(F^t)_{t\in\mathbb{R}}$ on $\mathbb{T}^2$ uniquely lifts to a flow $(\tilde{F}^t)_{t\in\mathbb{R}}$ on $\mathbb{R}^2$, so the time-one-map F^1 has a preferred lift $\tilde{F} := \tilde{F}^1$. By a result in [5], the rotation set $\rho(\tilde{F})$ of that lift is either a point or a line segment that has to contain 0 as an endpoint if its slope is irrational.

Theorem 1.1 should be contrasted with the situation for degree one circle maps and orientation preserving homeomorphisms of the annulus for which the analogues of $\rho_{\mathrm{pp}}(\tilde{F})$ and $\rho(\tilde{F})$ coincide (and thus $\rho_{\mathrm{pp}}(\tilde{F})$ is closed) as shown in [9] and [7], respectively. On the other hand, in higher dimensions, $d \geqslant 3$, the absence of the restrictions imposed by the planar topology facilitates a number of counterexamples (cf [13, 14]) including a construction in [16] of a real analytic diffeomorphism of $\mathbb{T}^3$ for which both $\rho_{\mathrm{pp}}(\tilde{F})$ and the *subsequential pointwise rotation set*,

$$\rho_{\mathrm{p}}(\tilde{F}) := \left\{ \lim_{i\to\infty} \frac{1}{n_i}(\tilde{F}^{n_i}(\tilde{p}) - \tilde{p}) : \ \tilde{p} \in \mathbb{R}^d, \ n_i \in \mathbb{N} \text{ with } n_i \to \infty \right\}, \tag{1.3}$$

fail to be closed.

One can show that $\rho_{\mathrm{p}}(\tilde{F})$ coincides with $\rho(\tilde{F})$ in our example so it is still an open problem if $\rho_{\mathrm{p}}(\tilde{F})$ can fail to be closed for a homeomorphism of $\mathbb{T}^2$. Also, our reliance on the Denjoy counterexample leaves open a perhaps more important question whether $\rho_{\mathrm{pp}}(\tilde{F})$ can fail to be closed for a diffeomorphism of $\mathbb{T}^2$ that is C^2 smooth.

The rest of this introduction is devoted to an outline of the construction. Our example belongs to the class of flows on $\mathbb{T}^2$ that have a single stationary point of the saddle-node type. Such flows can be constructed (section 2) by taking an orientation preserving C^1-diffeomorphism of the circle, $f : \mathbb{T} \to \mathbb{T}$, and moving along the orbits of the suspension flow of f with variable speed prescribed by a non-negative C^1 function V that is positive except for a single zero at some point $p_0 = (s_0, y_0)$. This construction has been used to illustrate various dynamical pathologies for quite some time: it can be found in [15], and it was injected into the theory of rotation sets by an example (attributed in [6] to Katok) of a flow for which $\lim_{t\to\infty}(\tilde{F}^t(p) - p)/t$ fails to exist for some p. In [5], it was also suggested for a more detailed study of the rotation set (with f taken to be an irrational rotation).

We restrict attention to the interesting case when the rotation number α of f is irrational and the resulting flow is transitive. The diffeomorphism f has a unique invariant measure λ, and the flow has an invariant measure μ equivalent to the suspension of λ as soon as the function $1/V$ is integrable with respect to μ, which is exactly when the return time $\phi(x)$ of a point $(x, 0)$ back to the horizontal cross-section $\mathbb{T} \times \{0\}$ is integrable with respect to λ. The generic points of the measure μ contribute to $\rho_{\mathrm{pp}}(\tilde{F})$ a non-zero vector ρ of magnitude inversely proportional to the average return time $\bar{\psi} := \int \phi \, \mathrm{d}\lambda$ (see fact 2.2). That is in addition to $(0, 0) \in \rho_{\mathrm{pp}}(\tilde{F})$ contributed by the stationary point p_0. Crucially, it can be arranged (section 5) that no other vectors beside ρ and 0 are present in $\rho_{\mathrm{pp}}(\tilde{F})$. This requires sufficiently benign behaviour of ϕ near the singularity s_0 where $\phi(s_0) = +\infty$ and hinges on the orbits of f avoiding repeated extremely close passes near s_0. Specifically, we show that $\rho_{\mathrm{pp}}(\tilde{F}) = \{0, \rho\}$ provided $V(x, y)$ behaves like $|x - s_0|^b + |y - y_0|^c$ near $p_0 = (s_0, y_0)$ with $b, c > 1$ satisfying $1/b + 1/c > 1$ (to make $1/V$ integrable) and the rotation number α is the *golden mean*. Those hypotheses are far from being optimal and rather aim at simplifying the exposition. In particular, we benefit from the particularly simple *self-inducing* scheme for the golden rotation (section 3).

Our ultimate example utilizes f that is a *Denjoy example* (section 4) exhibiting a minimal Cantor set Ω whose complement is made of infinitely many orbits of wandering components: $\dots, I_{-1}^{(k)}, I_0^{(k)}, I_1^{(k)}, \dots,$ $k \in \mathbb{N}$. The measure μ is supported on a *Denjoy continuum* (the suspension of Ω) the complement of which is made of infinitely many immersed discs $D^{(k)}$, $k \in \mathbb{N}$, *winding* on $\mathbb{T}^2$. This allows carrying out a sequence of perturbations (with the C^1-norms tending to zero) each judiciously slowing the flow on $D^{(k)}$ with a goal of placing in the pointwise rotation set $\rho_{\mathrm{pp}}(\tilde{F})$ a vector ρ_k, $k \in \mathbb{N}$, so that $\lim_{k\to\infty} \rho_k = C\rho$ with $C \in (0, 1)$ and $C\rho \notin \rho_{\mathrm{pp}}(\tilde{F})$ (section 6).

In order to preserve the C^1 smoothness, slowing within each $D^{(k)}$ is effected on an infinite sequence of squares that converge to the stationary point p_0. This step depends on the squares being sufficiently big to allow for *gentle* (C^1-small) slowing and therefore—in the simplest instance when $c = b$ with $b \in (1, 2)$—requires that, on passing near p_0, the *width* of $D^{(k)}$ is bounded from below by the distance to p_0 raised to a power $b'' < b$ (see (6.9)). Because the squares have to be placed on sufficiently many passes of $D^{(k)}$ near p_0 to affect the average return times, gentleness of slowing dictates suitably slow decay of the gap lengths in the Denjoy example; specifically, we take $|I_n^{(k)}| \sim \mathrm{e}^{-k}(|n| + k)^{-D}$ where $D > 1$ satisfies $D < b'' < b$ (see (4.1)).

2. The flow as a special representation

Given an orientation preserving circle diffeomorphism $f : \mathbb{T} \to \mathbb{T}$, we form a quotient $\mathbb{T}_f^2 := (\mathbb{T} \times \mathbb{R})/\langle D\rangle$ where $\langle D\rangle$ is the group generated by the (deck) transformation $D(x, y) := (f(x), y - 1)$. We shall often think of $\mathbb{T}_f^2$ as the square $[0, 1]^2$ with the appropriate boundary identifications and describe its points by the (standard) coordinates (x, y). $\mathbb{T}_f^2$ is to serve as a convenient model of the standard torus $\mathbb{T}^2 = \mathbb{T} \times \mathbb{R}/\mathbb{Z}$ in terms of which the rotation set was defined. Therefore, upon fixing a suitable isotopy $\{f^y\}_{y\in[0,1]}$ joining $f^0 = \mathrm{Id}$ and $f^1 = f$, we perform a C^1-diffeomorphic identification $\mathbb{T}^2 \to \mathbb{T}_f^2$ via the quotient of $H : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$, $H(x, y) := (f^y(x), y)$ where $f^y := f^{\lfloor y\rfloor} \circ f^{y-\lfloor y\rfloor}$. (Here $\lfloor y\rfloor$ is the largest integer not exceeding y.) Also, lifting the isotopy $\{f^y\}_{y\in[0,1]}$ to an isotopy $\{\tilde{f}^y : \mathbb{R} \to \mathbb{R}\}_{y\in[0,1]}$ on $\mathbb{R}$ so that $\tilde{f}^0 = \mathrm{Id}$ yields a lift $\tilde{f} := \tilde{f}^1$ of f; let $\alpha \in \mathbb{R}$ denote the rotation number of $\tilde{f}$.

Consider on $\mathbb{T}^2_f$ a (vertical) vector field of the form $V(x, y)\frac{\partial}{\partial y}$ where V is a C^1-smooth function on $\mathbb{T}^2_f$ that vanishes at a single point $p_0 = (s_0, y_0)$ and is positive otherwise. For now, we only ask that p_0 is located off the cross-section $\mathbb{T} \times 0$ but, ultimately, p_0 will be carefully placed with an eye on the dynamics of f (see the last paragraph of section 4). Denote the flow associated with $V\frac{\partial}{\partial y}$ by $(F^t)_{t\in\mathbb{R}}$. The return time $\phi(x)$ of $(x, 0)$ to the circle $\mathbb{T} \times 0$ is given by

$$\phi(x) := \int_0^1 \frac{1}{V(x, y)}\,\mathrm{d}y, \qquad x \in \mathbb{T}. \tag{2.1}$$

Clearly, ϕ is finite, positive and C^1 at all $x \neq s_0$ (where $\phi(s_0) = +\infty$); and for any ϕ with these three attributes and a small $\epsilon > 0$, one can produce V for which the return time function agrees with the given ϕ at $x \notin (s_0 - \epsilon, s_0 + \epsilon)$. The behaviour of ϕ near s_0 cannot be as freely prescribed but all we need is the following.

Fact 2.1. *Suppose $b, c > 1$. If $V(x, y)$ is of the form*

$$V(x, y) = |x - s_0|^b + |y - y_0|^c \tag{2.2}$$

on a neighbourhood of its zero (s_0, y_0), then $\phi(x)$ behaves like $|x - s_0|^{-b+b/c}$ near s_0; specifically, the one sided limits with $x \to s_0$ of $|x - s_0|^{b-b/c}\phi(x)$ are finite and non-zero.

Proof. It suffices to compute the divergent part of the integral (2.1); namely, $\phi_1(x) := \int_0^\delta (x^b + y^c)^{-1}\,\mathrm{d}y$ where we assumed for a moment (to avoid clutter) that $s_0 = y_0 = 0$ and $\delta > 0$ was taken so that $V(x, y) = |x|^b + |y|^c$ for $|x| + |y| < 2\delta$. The substitution $u := yx^{-b/c}$ yields

$$x^{b-b/c}\phi_1(x) = x^{b-b/c}\int_0^\delta x^{-b}(1 + (yx^{-b/c})^c)^{-1}\,\mathrm{d}y = \int_0^{\delta x^{-b/c}} (1 + u^c)^{-1}\,\mathrm{d}u,$$

which converges to $\int_0^\infty (1 + u^c)^{-1}\,\mathrm{d}u$ as $x \to 0^+$. □

Given $b, c > 1$, there are plenty of C^1-functions V generating ϕ that satisfies the assertions of fact 2.1. We shall employ one such V in section 6.

Let us now compute the rotation set of the time-one-map F^1 in terms of ϕ. By associating with every (x, y) of the fundamental domain $\mathbb{T} \times [0, 1]$ with $x \neq s_0$ the point $h(x, y) := (x, t)$, where $t = t(x, y)$ is the time it takes to flow $(x, 0)$ to (x, y), we can conjugate the dynamics of F^t that are not asymptotic to p_0 to the dynamics of the *special flow* $(T^t_\phi)_{t\in\mathbb{R}}$ constructed over f and under the *roof function* ϕ. This is to say that $T^t_\phi \circ h(x, y) = h \circ F^t(x, y)$ for all t, x, y with $x \notin \{f^n(s_0) : n \in \mathbb{Z}\}$, where the map T^t_ϕ is the time-t-map of the vector field $\frac{\partial}{\partial t}$ on the space $X_\phi := \{(x, t) : x \in \mathbb{T},\ 0 \leqslant t \leqslant \phi(x) < +\infty\}$ with the identifications $(x, \phi(x)) \sim (f(x), 0)$, $x \neq s_0$.

Consider the set of Birkhoff averages of ϕ,

$$\mathcal{A}_f(\phi) := \left\{ \lim_{n\to\infty} \frac{1}{n} S^n_f(\phi)(x) : x \in \mathbb{T},\ \text{the limit exists} \right\}, \tag{2.3}$$

where $S^n_f(\phi)(x) := \phi(x) + \cdots + \phi(f^{n-1}(x))$. (Taking $x = s_0$ places $+\infty$ in $\mathcal{A}_f(\phi)$.)

Fact 2.2.

$$\rho_{\mathrm{pp}}(\tilde{F}^1) = (1/\mathcal{A}_f(\phi)) \cdot (\alpha, 1). \tag{2.4}$$

Proof. To compute the rotation set, we have to work in the universal cover $\mathbb{R}^2$ of $\mathbb{T}^2$. This is achieved by conjugating the lifted flow $\tilde{F}^t$ on the universal cover $\mathbb{R} \times \mathbb{R}$ of $\mathbb{T}^2_f$ via the identification $\tilde{H} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ that is the (unique) lift of H fixing the x-axis. Without risk of confusion we shall write $\tilde{F}^t$ for that conjugated flow in this proof. Fix $p \in \mathbb{R}^2$ for which $v := \lim_{t\to\infty}(\tilde{F}^t(p) - p)/t$ exists. If $v = 0$, then $v \in (1/\mathcal{A}_f(\phi)) \cdot (\alpha, 1)$ due to $+\infty \in \mathcal{A}_f(\phi)$. Suppose then that $v \neq 0$. As v is unaffected by moving p along its flow line or by a deck transformation, we may well assume that $p = (x, 0)$ for some $x \in \mathbb{R}$. From $v \neq 0$, for every $n \in \mathbb{N}$, there is $t_n \in \mathbb{R}$ such that the y-coordinate of $\tilde{F}^{t_n}(p)$ is n. In fact, $t_n = S_f^n(\phi)(x)$ by the very definition of ϕ; and we can write

$$\frac{\tilde{F}^{t_n}(p) - p}{t_n} = \left(\frac{n}{t_n}\right) \frac{(\tilde{f}^n(x) - x, n)}{n}. \tag{2.5}$$

Taking the limit with $n \to \infty$ secures $v \in (1/\mathcal{A}_f(\phi)) \cdot (\alpha, 1)$.

Finally, any non-zero v in $(1/\mathcal{A}_f(\phi)) \cdot (\alpha, 1)$ is of the form $v = (\lim_{n\to\infty} \frac{1}{n} S_f^n(\phi)(x))^{-1} \cdot (\alpha, 1) \neq 0$ and thus belongs to $\rho(\tilde{F}^1)$ by again considering $p = (x, 0)$ in (2.5). □

3. Self-inducing

Computation of $\mathcal{A}_f(\phi)$ requires some understanding of the combinatorics (ordering) of the orbits for the circle map f or, equivalently, for the rotation $R_\alpha : \mathbb{T} \to \mathbb{T}$, $R_\alpha(x) := x + \alpha$, to which f is monotonically semi-conjugated. Recall that, to simplify this task, we take α to be the *golden mean*,

$$\alpha = \frac{\sqrt{5} - 1}{2} = 0.618\,033\,98\ldots, \tag{3.1}$$

thus making R_α self-similar under the *inducing map* $g : [0, 1] \to [0, 1]$ given by

$$g(x) = \begin{cases} 1 - \alpha^{-1}x & \text{if } x \in [0, \alpha], \\ 2 - \alpha^{-1}x & \text{if } x \in [\alpha, 1], \end{cases} \tag{3.2}$$

the graph of which is depicted in the upper right corner of figure 1. Below, we review the connection between g and R_α in detail necessary for our subsequent arguments (cf [4]).

The continued fraction expansion of α is $[1, 1, 1, \ldots]$ with the associated continued fraction approximants $\mathsf{p}_n/\mathsf{q}_n \to \alpha$ formed from the Fibonacci sequences $(\mathsf{p}_n)_{n=0}^\infty = (0, 1, 1, 2, 3, 5, \ldots)$ and $(\mathsf{q}_n)_{n=0}^\infty = (1, 1, 2, 3, 5, \ldots)$; the latter also given by

$$\mathsf{q}_n = \frac{1}{1 + \alpha^2}(\alpha)^{-n} + \frac{\alpha^2}{1 + \alpha^2}(-\alpha)^n. \tag{3.3}$$

Dynamically, upon fixing any $x_0 \in \mathbb{T}$, if we let $x_k := R_\alpha^k(x_0)$, the points x_{q_n} are characterized as the closest returns to x_0; that is $|x_{\mathsf{q}_n} - x_0| = \min\{|x_i - x_0| : 0 < i \leqslant \mathsf{q}_n\}$.[1]

It is convenient to cut $\mathbb{T}$ at x_1 and identify the resulting segment $[x_1^-, x_1^+]$ with $U_0 := [0, 1]$ (see figure 1). Note that x_0 gets identified with $\alpha^2 = 1 - \alpha \in [0, 1]$, a fixed point of g. For the most part, we will be dealing with a fixed x_0 and we suppress the dependence of this construction on x_0.

The segments J_n, $n \geqslant 0$, joining x_0 to x_{q_n} (with $J_0 = [x_0, x_1^+]$ and $J_1 = [x_1^-, x_0]$) have the property that the family (called the nth *dynamical partition* for R_α)

$$\mathcal{P}_n := \{J_n, \ldots, R_\alpha^{\mathsf{q}_{n+1}-1}(J_n), J_{n+1}, \ldots, R_\alpha^{\mathsf{q}_n-1}(J_{n+1})\} \tag{3.4}$$

covers $\mathbb{T}$ and has pairwise disjoint interiors. We choose to record the combinatorics of $\mathcal{P}_n$ by a mapping $f_n : \mathbb{T} \to J_{n+1} \cup J_n$ that translates each *long* segment of the partition (i.e. $R_\alpha^i(J_n)$) onto J_n and each *short* segment (i.e. $R_\alpha^i(J_{n+1})$) onto J_{n+1}. (Apart from f_n being possibly two-valued

[1] For $a, b \in \mathbb{T}$, we write $|a - b|$ for their standard distance on the circle $\mathbb{T} := [0, 1]/\sim$.

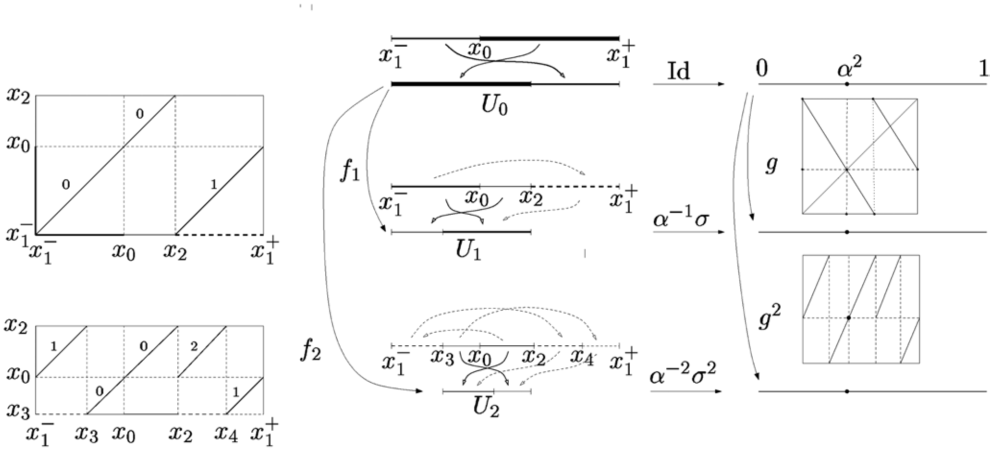

Figure 1. Centre: the dynamical partitions $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2$ and (in solid arrows) the first return interval exchanges R_0, R_1, R_2; R_i and R_{i+1} coincide up to scaling and orientation reversal by $\alpha^{-1}\sigma$. Left: f_1 and f_2 with laps labelled by the number of iterates of R_α^{-1} leading back to U_1 and U_2, respectively. Right: rescaled f_1 and f_2 coincide with g and g^2 where $g := \alpha^{-1}\sigma \circ f_1$.

at the endpoints $\{x_1, \ldots, x_{\mathsf{q}_n+\mathsf{q}_{n+1}-1}\}$, $f_n(p)$ is uniquely defined as the first backward entry into $U_n := J_n \cup J_{n+1}$, i.e. $f_n(p) = R_\alpha^{-i}(p) \in U_n$, where $i \geqslant 0$ is minimal.) By construction, we have

$$f_n^{-1}(q) = \begin{cases} \{q, R_\alpha^1(q), \ldots, R_\alpha^{\mathsf{q}_n-1}(q)\} & \text{if } q \in J_{n+1}\backslash\{x_0\}, \\ \{q, R_\alpha^1(q), \ldots, R_\alpha^{\mathsf{q}_{n+1}-1}(q)\} & \text{if } q \in J_n\backslash\{x_0\}. \end{cases} \tag{3.5}$$

Now, R_α can be presented as the interval exchange $R_0 : U_0 \to U_0$ swapping $J_1 = [x_1^-, x_0]$ and $J_0 = [x_0, x_1^+]$ as depicted at the top centre of figure 1. The first return to $U_1 = J_1 \cup J_2 = [x_1^-, x_2]$ under R_0 is, in turn, the interval exchange $R_1 : U_1 \to U_1$ swapping J_1 and J_2. Using $\alpha^2 + \alpha = 1$, one checks that R_1 is conjugated to R_0 via rescaling by α^{-1} followed by an isometric orientation reversal σ:

$$R_0 \circ \sigma \circ \alpha^{-1} = \sigma \circ \alpha^{-1} \circ R_1. \tag{3.6}$$

An easy induction argument shows then that $\mathcal{P}_n$ results from repeating the subdivision process that refines $\mathcal{P}_0$ to $\mathcal{P}_1$, which can be expressed as

$$f_n = \alpha^n\sigma^n \circ g^n, \tag{3.7}$$

where $\alpha^n\sigma^n := \alpha^n \circ \sigma^n = \sigma^n \circ \alpha^n : [0, 1] \to J_n \cup J_{n+1}$ sends α^2 to x_0 and identifies $[0, \alpha^2]$ with J_{n+1} and $[\alpha^2, 1]$ with J_n. (Compare the left- and right-hand sides of figure 1.)

From (3.7), $g^{-n} = f_n^{-1} \circ \alpha^n\sigma^n$ and $g^{-n} \circ g^n = f_n^{-1} \circ f_n$. Therefore, from (3.5), we see that preimages of points under iterates of g constitute orbit segments of R_α; in particular, for $p \in [0, 1]\backslash\{x_0, \ldots, x_{\mathsf{q}_n+\mathsf{q}_{n+1}-1}\}$, (for which $g^n(p)$ is uniquely defined) we have

$$g^{-n} \circ g^n(p) = \begin{cases} \{q, R_\alpha^1(q), \ldots, R_\alpha^{\mathsf{q}_n-1}(q)\} & \text{if } g^n(p) \in [0, \alpha^2), \\ \{q, R_\alpha^1(q), \ldots, R_\alpha^{\mathsf{q}_{n+1}-1}(q)\} & \text{if } g^n(p) \in (\alpha^2, 1], \end{cases} \tag{3.8}$$

where $q := \alpha^n\sigma^n(g^n(p))$ is the first entry of $R_\alpha^{-i}(p)$, $i \geqslant 0$, into $U_n = J_n \cup J_{n+1}$.

4. Denjoy example and singularity placement

As explained before, the diffeomorphism $f : \mathbb{T} \to \mathbb{T}$ used in our ultimate construction (in section 6) is a *Denjoy example* with infinitely many distinct orbits of *gaps* (i.e. wandering intervals). Below, we recount the construction of such f (cf [3, 10]) with a goal of finding points s_0 (to later serve as the singularity locus of the return time function ϕ) that are not too closely approached by the wandering intervals.

Consider an infinite collection $\{x_m^{(k)} : m \in \mathbb{Z}\}$, $k \in \mathbb{N}$, of distinct orbits of R_α labelled so that $R_\alpha(x_m^{(k)}) = x_{m+1}^{(k)}$. We intend to *blow up* each point $x_m^{(k)}$ into a segment $I_m^{(k)}$ of length

$$l_m^{(k)} := (|m| + k)^{-D} \mathrm{e}^{-k}, \tag{4.1}$$

where $D > 1$ so that $L := \sum_{k,m} l_m^{(k)} < +\infty$. (Later, in (6.1), we shall further restrict D.) To this end, the formula

$$h_0(x) := \left(x + \sum_{m,k: 0 \leqslant x_m^{(k)} < x} l_m^{(k)} \right), \qquad x \in [0, 1], \tag{4.2}$$

defines a monotonic function $h_0 : [0, 1] \to [0, L + 1]$ which is continuous except for a jump of magnitude $l_m^{(k)}$ at each $x_m^{(k)}$. The segment $I_m^{(k)}$ is defined as the span of this jump:

$$I_m^{(k)} := [h(x_m^{(k)}), h(x_m^{(k)}) + l_m^{(k)}]. \tag{4.3}$$

To construct f, set

$$j_0(x) := 1 + \sum_{m,k} b_m^{(k)}(x), \ \ x \in [0, L + 1], \tag{4.4}$$

where $b_m^{(k)}$ is a continuous function supported on $I_m^{(k)}$ with $\int_{I_m^{(k)}} 1 + b_m^{(k)} = l_{m+1}^{(k)}$ and $\max |b_m^{(k)}| \leqslant 2(l_m^{(k)})^{-1} \int_{I_m^{(k)}} b_m^{(k)}$ (as is the case for a suitable *tent* function over $I_m^{(k)}$). The last inequality amounts to $\max |b_m^{(k)}| \leqslant 2(l_{m+1}^{(k)}/l_m^{(k)} - 1)$; so the maximum converges to 0 as $|m| + k \to \infty$ securing continuity of the function j_0. Assuming that α is not among the $x_m^{(k)}$ so that $h_0(\alpha)$ is a singleton, we set

$$f_0(x) := h_0(\alpha) + \int_0^x j_0(t)\,\mathrm{d}t \ \mathrm{mod}(L + 1), \qquad x \in [0, L + 1]. \tag{4.5}$$

After rescaling $[0, L + 1]$ to $[0, 1]$, h_0 and f_0 yield h and $f : \mathbb{T} \to \mathbb{T}$ such that f is a C^1-diffeomorphism, $h \circ R_\alpha = f \circ h$, and $f(I_m^{(k)}) = I_{m+1}^{(k)}$ for $k \in \mathbb{N}$, $m \in \mathbb{Z}$, as desired. The following bound on the jumps of h will be used in section 6.

Fact 4.1. *For a.e. $s \in \mathbb{T}$, the derivative $h'(s)$ exists and equals 1 and we have*

$$\forall_{D' \in (0,D)}\ \exists_{C>0}\ \forall_{\tau \in \mathbb{T}}\ |h(\tau_+) - h(\tau_-)| \leqslant C|\tau - s|^{D'}. \tag{4.6}$$

(Here $h(\tau_\pm) := \lim_{t \to \tau^\pm} h(t)$.)

Proof. The Cantor set Ω obtained by removing the interiors of the segments $I_m^{(k)}$ is the unique minimal set for f. Since its measure $|\Omega|$ is positive and $f'(x) = 1$ for $x \in \Omega$ by (4.4), the unique invariant measure for f is just the Lebesgue measure restricted to Ω, $Leb|_\Omega$. Under a semiconjugacy of f to R_α, $Leb|_\Omega$ must map to the Lebesgue measure on $\mathbb{T}$, i.e. $h_*^{-1}(Leb|_\Omega) = Leb$ and so $(h^{-1})'(x)$ exists and equals 1 for a.e. $x \in \Omega$. As h is also a.e. continuous, $h'(s) = 1/(h^{-1})'(h(s)) = 1$ for a.e. $s \in \mathbb{T}$.

For the proof of (4.6), we fix $D' \in (0, D)$ and consider a jump locus $\tau = x_m^{(k)}$ since for other τ there is nothing to prove. Denote $s_n := R_\alpha^n(s)$, $n \in \mathbb{Z}$. We note that $|x_m^{(k)} - s_0| = |x_0^{(k)} - s_{-m}|$, rewrite (4.6) as

$$\left| x_0^{(k)} - s_{-m} \right| \geqslant (C^{-1} \mathrm{e}^{-k} (|m| + k)^{-D})^{\frac{1}{D'}}, \tag{4.7}$$

and see that it suffices to prove that a.e. $s_0 \in [0,1]$ belongs to all but finitely many sets

$$G_m^{(k)} := \left\{ s_0 : \ \mathrm{dist}\left(x_0^{(k)}, \{s_{-m}, \ldots, s_m\}\right) > \frac{\mathrm{e}^{\frac{-k}{D'}}}{|m|^{\frac{D}{D'}}} \right\}, \qquad (k,m) \in \mathbb{N} \times \mathbb{Z}. \tag{4.8}$$

Because $\mathsf{q}_n \sim \alpha^{-n}$ (i.e. $\{\mathsf{q}_n/\alpha^{-n}\}_{n\in\mathbb{N}} \subset [c^{-1}, c]$ for some $c > 0$, cf (3.3)), to each $m \in \mathbb{Z}$ we can associate q_n comparable to $|m|$ so that $\lfloor \mathsf{q}_n/2 \rfloor > |m|$ and the complement of $G_m^{(k)}$ satisfies

$$(G_m^{(k)})^c \subset \left\{ s_0 : \ \mathrm{dist}\left(x_0^{(k)}, \{s_{-\lfloor \mathsf{q}_n/2\rfloor+1}, \ldots, s_{\lceil \mathsf{q}_n/2\rceil -1}\}\right) \leqslant C\mathrm{e}^{\frac{-k}{D'}} \alpha^{n\frac{D}{D'}} \right\}, \tag{4.9}$$

where $C > 0$ is some constant independent of k, m, n. The right-hand side of (4.9) coincides in measure with that of its image under $R_\alpha^{\lceil \mathsf{q}_n/2\rceil -1}$,

$$D_n^{(k)} := \left\{ s_0 : \ \mathrm{dist}\left(x_0^{(k)}, \{s_{-\mathsf{q}_n+2}, \ldots, s_0\}\right) \leqslant C\mathrm{e}^{\frac{-k}{D'}} \alpha^{n\frac{D}{D'}} \right\}, \tag{4.10}$$

and to be done (via Borell–Cantelli lemma) all we need is the summability

$$\sum_{k,n\in\mathbb{N}} |D_n^{(k)}| < +\infty. \tag{4.11}$$

To prove (4.11), after fixing $k \in \mathbb{N}$, we shall use the *inducing scheme* introduced in the previous section with $x_0 := x_0^{(k)}$ to estimate $|D_n^{(k)}|$. Note that, since the return time to $U_n = J_n \cup J_{n+1}$ (under R_α) is at least q_n, among the points $\{s_{-\mathsf{q}_n+2}, \ldots, s_0\}$ at most one can belong to U_n; and if such a point exists it equals $f_n(s_0)$, the first backward entry into U_n. It follows that $|x_0^{(k)} - f_n(s_0)| \leqslant C_1\, \mathrm{dist}(x_0^{(k)}, \{s_{-\mathsf{q}_n+2}, \ldots, s_0\})$ for a suitable C_1 that depends on α only (concretely, $C_1 = \alpha^{-1}$). Thus, factoring in that $|x_0^{(k)} - g^n(s_0)| = \alpha^{-n}|x_0^{(k)} - f_n(s_0)|$ (from (3.7)), we see that

$$D_n^{(k)} \subset \left\{ s_0 : \ \left| x_0^{(k)} - g^n(s_0)\right| \leqslant C_2 \mathrm{e}^{-\frac{k}{D'}} \alpha^{(\frac{D}{D'}-1)n} \right\} = g^{-n}(B_n^{(k)}), \tag{4.12}$$

where $C_2 := C_1 C$ and

$$B_n^{(k)} := \left\{ y : \ \left| x_0^{(k)} - y \right| < C_2 \mathrm{e}^{-\frac{k}{D'}} \alpha^{(\frac{D}{D'}-1)n} \right\}. \tag{4.13}$$

Now, the map g has an invariant measure μ_* that is absolutely continuous with respect to the Lebesgue measure with the density equal to

$$\psi_*(x) := \begin{cases} \dfrac{1}{1+\alpha^2} & \text{for } x \in [0, \alpha^2], \\[2ex] \dfrac{\alpha^{-1}}{1+\alpha^2} & \text{for } x \in [\alpha^2, 1]. \end{cases} \tag{4.14}$$

We can therefore write

$$\begin{aligned} \min \psi_* \, |D_n^{(k)}| &\leqslant \mu_*(D_n^{(k)}) \leqslant \mu_*(g^{-n}(B_n^{(k)})) = \mu_*(B_n^{(k)}) \leqslant \max \psi_* |B_n^{(k)}| \\ &= \max \psi_* C_2 \mathrm{e}^{-\frac{k}{D'}} \alpha^{(\frac{D}{D'}-1)n}. \end{aligned} \tag{4.15}$$

Since $\sum_{n,k} \mathrm{e}^{-\frac{k}{D'}} \alpha^{(\frac{D}{D'}-1)n} < \infty$, the desired summability condition (4.11) follows. $\square$

The above argument shows that only a null measure set of s_0 can belong to infinitely many sets in the middle of (4.12). As a consequence, there is a full measure set of $s_0 \in \mathbb{T}$ such that

$$\forall_{\zeta>0,\ k\in\mathbb{N}}\ \exists_{\delta>0}\ \forall_{n\in\mathbb{N}} |x_0^{(k)} - g^n(s_0)| > \delta \alpha^{\zeta n}. \tag{4.16}$$

(Note that the action of g on $\mathbb{T}$ above depends on k because so does the identification of $[0,1]$ with $\mathbb{T}$ effected by cutting $\mathbb{T}$ at the point $x_1^{(k)}$ and placing $x_k^{(0)}$ at a fixed point of g, see section 3.)

Therefore, it is possible to select a point s_0 satisfying the assertions of fact 4.1 (including $h'(s_0) = 1$ and (4.6) with $s = s_0$) as well as the inequality (4.16). Because h can be precomposed with a rotation, we may additionally arrange that $h(s_0) = s_0$ (just for notational expediency). In what follows, we shall assume that the point $p_0 = (s_0, y_0)$ at which the speed function V vanishes is placed so that the sole singularity of the return time function ϕ occurs at such s_0.

5. Pre-perturbation rotation set

This section is devoted to the proof of the following proposition implying (via fact 2.2) that the pointwise rotation set consists of two points only provided the return time function ϕ is majorized near the singularity locus s_0 by $|x - s_0|^{-\gamma}$ with $\gamma \in (0, 1)$; as it is the case (with $\gamma := b - b/c$) for the ϕ in fact 2.1 as soon as ϕ is integrable (i.e. when $1/b + 1/c > 1$).

Proposition 5.1. *Suppose that ϕ is positive continuous on $\mathbb{T}\backslash\{s_0\}$ and that there exists $C_1 > 0$ and $\gamma \in (0, 1)$ so that*

$$\phi(x) \leqslant C_1|x - s_0|^{-\gamma}, \qquad x \in \mathbb{T}\backslash\{s_0\}. \tag{5.1}$$

Let f be either the golden rotation R_α or the Denjoy example constructed in section 4. Then the set of Birkhoff averages of ϕ satisfies

$$\mathcal{A}_f(\phi) = \{\bar{\psi}, \infty\}, \tag{5.2}$$

where $\bar{\psi}$ is the average of ϕ with respect to the unique ergodic measure of f. Moreover, in the Denjoy case, if z belongs to one of the wandering segments $I_0^{(k)}$, $k \in \mathbb{N}$, then

$$\lim_{m\to\infty} \frac{1}{m} S_f^m(\phi)(z) = \bar{\psi}. \tag{5.3}$$

Proof. In the course of this proof, C_2, C_3, C_4, etc denote some suitable constants that depend on f and ϕ only. We focus on the Denjoy case which requires a superset of the arguments needed for $f = R_\alpha$ (in which case we set $h := \mathrm{Id}$).

Recall that, at each $x \in \mathbb{T}$, h is continuous or has a jump with well defined one sided limits $h(x_\pm) := \lim_{y\to x^\pm} h(y)$. Set, for $x \in \mathbb{T} \setminus \{s_0\}$,

$$\psi(x) := \max_{z\in[h(x_-),h(x_+)]} \phi(z) \qquad \text{and} \qquad \psi_-(x) := \min_{z\in[h(x_-),h(x_+)]} \phi(z). \tag{5.4}$$

For $x \notin \{x_m^{(k)}\}_{k\in\mathbb{N},m\in\mathbb{Z}}$ (the locus of jumps of h), ψ_- and ψ are continuous at x and coincide: $\psi(x) = \psi_-(x) = \phi\circ h(x)$. As a consequence of the monotonicity of h together with $s_0 = h(s_0)$ and $h'(s_0) = 1$ (per the last paragraph of section 4), we have

$$\lim_{x\to s_0} \max_{z\in[h(x_-),h(x_+)]} \frac{z - s_0}{x - s_0} = \lim_{x\to s_0} \frac{h(x) - s_0}{x - s_0} = h'(s_0) = 1,$$

which guarantees that (5.1) has an analogue for ψ:

$$\psi(x) \leqslant \left\{ \max_{z\in[h(x_-),h(x_+)]} C_1 \left|\frac{z - s_0}{x - s_0}\right|^{-\gamma} \right\} |x - s_0|^{-\gamma} \leqslant C_2|x - s_0|^{-\gamma}, \qquad x \in \mathbb{T}\backslash\{s_0\}. \tag{5.5}$$

In particular, ψ is integrable:

$$\bar{\psi} := \int_{\mathbb{T}} \psi_- = \int_{\mathbb{T}} \psi < +\infty.$$

(Actually, from the proof of fact 4.1, $\int_{\mathbb{T}} \psi = \int_\Omega \phi$ where Ω is the minimal set of f.)

By the definition of ψ and ψ_-, if $x \in \mathbb{T}$ and $z \in [h(x_-), h(x_+)]$ (which includes $z = h(x)$ for $x \notin \{x_m^{(k)}\}_{k\in\mathbb{N},m\in\mathbb{Z}}$), then the Birkhoff sums over f and over R_α are related by

$$S_{R_\alpha}^m(\psi_-)(x) \leqslant S_f^m(\phi)(z) \leqslant S_{R_\alpha}^m(\psi)(x), \qquad m \in \mathbb{N}. \tag{5.6}$$

Another preliminary observation is that

$$\liminf_{m\to\infty} \frac{1}{m} S^m_{R_\alpha}(\psi_-)(x) \geqslant \bar{\psi}, \qquad x \in \mathbb{T}, \tag{5.7}$$

as can be seen by approximating[2] ψ_- by continuous $\psi_N \leqslant \psi_-$ so that $\int \psi_N \to \int \psi_- = \bar{\psi}$ as $N \to \infty$ and then writing $\liminf_{m\to\infty} \frac{1}{m} S^m_{R_\alpha}(\psi_-)(x) \geqslant \lim_{m\to\infty} \frac{1}{m} S^m_{R_\alpha}(\psi_N)(x) = \int \psi_N$, where the second pointwise limit exists by the unique ergodicity of R_α.

In view of (5.6) and (5.7), to establish (5.2), it suffices to show that, for any $x_0 \in \mathbb{T}$ whose forward R_α-orbit misses s_0, there is a sequence $(n_j) \subset \mathbb{N}$, $n_j \to \infty$, such that

$$\limsup_{j\to\infty} \frac{1}{\mathsf{q}_{n_j}} S^{\mathsf{q}_{n_j}}_{R_\alpha}(\psi)(x_0) \leqslant \bar{\psi}. \tag{5.8}$$

In order to prove (5.8), fix an arbitrary $x_0 \in \mathbb{T} \setminus \{R_\alpha^{-n}(s_0) : n \geqslant 0\}$. If $x_0 = x_0^{(k)}$ for some $k \in \mathbb{N}$, take $\delta, \zeta \in (0, 1)$ as in (4.16) with ζ small enough so that

$$1 - (1+\zeta)\gamma > 0. \tag{5.9}$$

We use the *inducing map* g defined in section 3 to determine (n_j) as follows. (Recall that x_0 is identified with $\alpha^2 \in [0, 1]$ and is fixed by g.) We have $g^n(s_0) \neq x_0$ for all $n \geqslant 0$ as otherwise $s_0 \in g^{-n}(x_0)$ putting (via (3.8)) s_0 on the forward R_α-orbit of x_0, contrary to our assumption on x_0. The expansiveness of g supplies $n_j \to \infty$ and $\delta > 0$ (which we may take the same as above) such that

$$|x_0 - g^{n_j}(s_0)| > \delta \qquad \text{for all } j \in \mathbb{N}. \tag{5.10}$$

Now, fix an arbitrary $n \in \mathbb{N}$ of which we require that $n = n_j$ for some j unless $x_0 = x_0^{(k)}$ for some k. This is so that the conjunction of (4.16) and (5.10) secures the following key ingredient of our subsequent estimates

$$|x_0 - g^n(s_0)| > \delta\alpha^{\zeta n}. \tag{5.11}$$

We are ready to majorize the Birkhoff average $\frac{1}{m} S^m_{R_\alpha}(\psi)(x_0)$ for any $m \in \mathbb{N}$ such that $\mathsf{q}_{n-1} \leqslant m \leqslant \mathsf{q}_n$. For an arbitrary (small) $\Delta > 0$, we shall estimate separately the two terms of the sum

$$\frac{1}{m} S^m_{R_\alpha}(\psi)(x_0) = \frac{1}{m} \sum_{\tau\in\{x_0,\ldots,x_{m-1}\},\ |\tau-s_0|\geqslant\Delta} \psi(\tau) + \frac{1}{m} \sum_{\tau\in\{x_0,\ldots,x_{m-1}\},\ |\tau-s_0|<\Delta} \psi(\tau). \tag{5.12}$$

The first term equals $\frac{1}{m} S^m_{R_\alpha}(\psi^{(\Delta)})(x_0)$ where $\psi^{(\Delta)}$ is a *truncation* of ψ:

$$\psi^{(\Delta)}(t) := \begin{cases} \psi(t) & \text{if } |t - s_0| \geqslant \Delta, \\ 0 & \text{if } |t - s_0| < \Delta, \end{cases} \qquad (t \in [0, 1]). \tag{5.13}$$

The point of this observation is that

$$\limsup_{m\to\infty} \frac{1}{m} S^m_{R_\alpha}(\psi^{(\Delta)})(x) \leqslant \int \psi^{(\Delta)} \leqslant \bar{\psi}, \qquad x \in \mathbb{T}, \tag{5.14}$$

as can be seen by appoximating with continuous $\psi_N \geqslant \psi^{(\Delta)}$ so that $\int \psi_N \to \int \psi^{(\Delta)}$ as $N \to \infty$ and arguing as for (5.7).

To estimate the second term in (5.12), we use the properties of g. From (3.8), the preimage $g^{-n}(\alpha^2) = \{x_0, \ldots, x_{\mathsf{q}_{n+1}-1}\}$ is the union of $g^{-n}(\alpha^2_-) := \lim_{t\to\alpha^2_-} g^{-n}(t) = \{x_0, \ldots, x_{\mathsf{q}_n-1}\}$ and $g^{-n}(\alpha^2_+) := \lim_{t\to\alpha^2_+} g^{-n}(t) = \{x_0, \ldots, x_{\mathsf{q}_{n+1}-1}\}$. Since g is a *Markov map* with a constant slope α^{-1} (cf figure 1), there is $C_3 > 1$ such that $g^{-n}(\alpha^2)$ consists of at most $C_3\alpha^{-n}$ points

[2] This hinges on the measure of the set of discontinuity points of ψ_- being zero.

no two closer to each other than $C_3^{-1}\alpha^n$. Also, (5.11) amounts to $|\alpha^2 - g^n(s_0)| > \delta\alpha^{\zeta n}$ and implies that $\mathrm{dist}(s_0, g^{-n}(\alpha^2)) > C_3^{-1}\delta\alpha^{(1+\zeta)n}$. Thus, using (5.5), we can estimate the second term in (5.12) by

$$\frac{1}{m}\sum_{\tau\in g^{-n}(\alpha_-^2),\ |\tau-s_0|<\Delta}\psi(\tau) \leqslant \frac{1}{m}2C_2(C_3^{-1}\delta\alpha^{(1+\zeta)n})^{-\gamma} + \frac{1}{m}2\sum_{k=1}^{\lceil C_3\Delta\alpha^{-n}\rceil} C_2(kC_3^{-1}\alpha^n)^{-\gamma}, \tag{5.15}$$

where the first term accounts for the two points $\tau \in g^{-n}(\alpha_-^2)$ closest to s_0 and the second term accounts for all other $\tau \in g^{-n}(\alpha_-^2)$ with $|\tau - s_0| < \Delta$ by exploiting their linear ordering on both sides of s_0 with the spacing bounded below by $C_3^{-1}\alpha^n$. Therefore, since also $\sum_{k=1}^{\lceil C_3\Delta\alpha^{-n}\rceil} k^{-\gamma} \sim \int_{k=1}^{C_3\Delta\alpha^{-n}} k^{-\gamma}\,dk \sim (C_3\Delta\alpha^{-n})^{1-\gamma}$ and $m^{-1} \sim \mathsf{q}_n^{-1} \sim \alpha^n$, we have

$$\begin{aligned}\frac{1}{m}\sum_{\tau\in g^{-n}(\alpha_-^2),\ |\tau-s_0|<\Delta}\psi(\tau) &\leqslant C_4\alpha^{(1-(1+\zeta)\gamma)n} + C_5\alpha^n\alpha^{-\gamma n}(\alpha^{-n})^{1-\gamma}\Delta^{1-\gamma} \\ &\leqslant C_4\alpha^{(1-(1+\zeta)\gamma)n} + C_5\Delta^{1-\gamma}\end{aligned}. \tag{5.16}$$

In view of (5.9), the inequalities (5.16) and (5.14) guarantee that given $\epsilon > 0$ one can take $\Delta > 0$ small enough so that, once $\mathsf{q}_{n-1} \leqslant m \leqslant \mathsf{q}_n$ are sufficiently large, we have

$$\frac{1}{m}S_{R_\alpha}^m(\psi)(x_0) \leqslant \bar{\psi} + \epsilon. \tag{5.17}$$

As this applies, in particular, to $m = \mathsf{q}_n$ for $n = n_j$ and all large $j \in \mathbb{N}$, we obtain our goal inequality (5.8) (and thus secure (5.2)).

Finally, in the case when $x_0 = x_0^{(k)}$ for some $k \in \mathbb{N}$, we placed no restrictions on $n \in \mathbb{N}$ so (5.17) holds for all sufficiently large $m \in \mathbb{N}$. This yields the 'moreover part' (5.3) of the proposition if one also brings to bear (5.7) and (5.6). □

6. Perturbation and conclusion

We finalize the construction of the example and the proof of theorem 1.1 by describing the perturbation that is the object of the following proposition.

Proposition 6.1. *Suppose that a C^1 function V satisfies the hypotheses of fact 2.1, its associated return time function ϕ and the circle map f satisfy the hypotheses of proposition 5.1 (so that $\mathcal{A}_f(\phi) = \{\bar{\psi}, \infty\}$), and that the exponents $b, c, D > 1$ are such that*

$$1/b + 1/c > 1 \quad \text{and} \qquad b > D \quad \text{and} \quad cD > b. \tag{6.1}$$

Given a bounded increasing sequence $(c_k)_{k\in\mathbb{N}} \subset (\bar{\psi}, \infty)$ such that $c_\infty := \lim_{k\to\infty} c_k$ does not belong to the sequence, it is possible to perturb V to a C^1 function V_{new} for which the return time ϕ_{new} satisfies

$$\{\bar{\psi}, c_1, c_2, \ldots, \infty\} \subset \mathcal{A}_f(\phi_{\mathrm{new}}) \subset [\bar{\psi}, c_\infty) \cup \{\infty\}. \tag{6.2}$$

Proof of proposition 6.1. Fix a sequence $(c_k)_{k\in\mathbb{N}} \subset (\bar{\psi}, \infty)$ as in the hypothesis. We shall modify ϕ on some of the gaps $I_m^{(k)}$ to secure (6.2) and only then describe how to construct a C^1 function V_{new} that realizes this modified ϕ_{new}.

We seek ϕ_{new} in the form

$$\phi_{\mathrm{new}}(x) := \phi(x) + \beta(x), \qquad \beta(x) := \sum_{k\in\mathbb{N}}\sum_{j=j_0^{(k)}}^{\infty} a_j^{(k)}\beta_{I_{n_j}^{(k)}}(x), \tag{6.3}$$

where $j_0^{(k)}, n_j^{(k)}$ and $a_j^{(k)}$ are yet to be chosen and β_J is a non-negative *bump* function supported on the length $|J|/2$ central subsegment of J, symmetric about the centre of J, with maximum 1, and with the C^1-norm proportional to $|J|^{-1}$.

Fix $\sigma > 0$ small and $\eta > 1$ large so that

$$b'' := (D+\sigma)(1/\eta + 1) < b \tag{6.4}$$

as made possible by $D < b$. In the course of the proof, we shall use C_6, C_7, etc to denote suitable constants that depend only on f and ϕ. Let

$$N_j := \lfloor j^{\eta+1} \rfloor.$$

Because $N_{j+1} - N_j \geqslant C_6 j^\eta$, the R_α-orbit segment $x_{N_j}^{(k)}, \ldots, x_{N_{j+1}}^{(k)}$ contains a point; denote it by $x_{n_j^{(k)}}^{(k)}$, $N_j \leqslant n_j^{(k)} < N_{j+1}$, such that

$$\left| x_{n_j^{(k)}}^{(k)} - s_0 \right| < C_7 j^{-\eta} \quad \text{for } j \in \mathbb{N}. \tag{6.5}$$

(That an R_α-orbit segment of length N is C_8/N dense is apparent for $N = \mathsf{q}_n$ from the dynamical partition $\mathcal{P}_N$ (see (3.4)), and it follows for any N by considering $\mathsf{q}_n \leqslant N < \mathsf{q}_{n+1}$.)

Having selected $n_j^{(k)}$, we take $a_j^{(k)} > 0$ so that $\frac{1}{n_j^{(k)}+1}(a_1^{(k)} + \cdots + a_j^{(k)}) = c_k - \bar{\psi}$ ($j \in \mathbb{N}$). In view of (5.3), at the centre point $z_0^{(k)}$ of $I_0^{(k)}$, we then have

$$\lim_{j\to\infty} \frac{1}{n_j^{(k)}+1} S_f^{n_j^{(k)}+1}(\phi_{\text{new}})(z_0^{(k)}) = \bar{\psi} + (c_k - \bar{\psi}) = c_k \tag{6.6}$$

because $S_f^{n_j^{(k)}+1}(\phi_{\text{new}})(z_0^{(k)}) = S_f^{n_j^{(k)}+1}(\phi)(z_0^{(k)}) + a_{j_0^{(k)}}^{(k)} + \cdots + a_j^{(k)}$.

The subsequential limit (6.6) already guarantees existence of the limit

$$\lim_{n\to\infty} \frac{1}{n} S_f^n(\phi_{\text{new}})(z_0^{(k)}) = c_k \tag{6.7}$$

since, for any sufficiently large $n \in \mathbb{N}$, we can find j with $n_j^{(k)} < n \leqslant n_{j+1}^{(k)}$, and we have

$$S_f^{n_j^{(k)}+1}(\phi_{\text{new}})(z_0^{(k)}) \leqslant S_f^n(\phi_{\text{new}})(z_0^{(k)}) \leqslant S_f^{n_{j+1}^{(k)}+1}(\phi_{\text{new}})(z_0^{(k)}), \tag{6.8}$$

where $\lim_{j\to\infty} \frac{n_{j+1}^{(k)}+1}{n_j^{(k)}+1} \to 1$ due to $\frac{N_{j+1}}{N_j} \sim (1+1/j)^{\eta+1} \to 1$. (Note that the choice of $j_0^{(k)}$ is immaterial for (6.7); we shall commit to it only later to assure C^1-smoothness of V_{new}.)

We are ready to prove (6.2). The first inclusion is clear by design (combine (5.2) and (6.7)). To see the other inclusion, consider $x \in \mathbb{T}$ with a finite limit $c := \lim_{n\to\infty} \frac{1}{n} S_f^n(\phi_{\text{new}})(x)$ that is not $\bar{\psi}$. By proposition 5.1, such x has to be in one of the gaps $I_m^{(k)}$, and we may well assume that $x \in I_0^{(k)}$ for some $k \in \mathbb{N}$ as the limit is insensitive to replacing x by $R_\alpha^{-m}(x)$. By utilizing (5.3) of proposition 5.1 and the fact that the bump functions $\beta_{I_m^{(k)}}$ have the maxima at the centres $z_m^{(k)}$ of the gaps, we can write

$$\begin{aligned} c - \bar{\psi} &= \lim_{n\to\infty} \frac{1}{n} S_f^n(\phi_{\text{new}})(x) - \lim_{n\to\infty} \frac{1}{n} S_f^n(\phi)(x) = \lim_{n\to\infty} \frac{1}{n} S_f^n(\beta)(x) \\ &\leqslant \lim_{n\to\infty} \frac{1}{n} S_f^n(\beta)(z_0^{(k)}) = \lim_{n\to\infty} \frac{1}{n} S_f^n(\phi_{\text{new}})(z_0^{(k)}) - \lim_{n\to\infty} \frac{1}{n} S_f^n(\phi)(z_0^{(k)}) = c_k - \bar{\psi}. \end{aligned}$$

Hence, $c \leqslant c_k < c_\infty$, and the second inclusion in (6.2) follows.

We are left with showing that the ϕ_{new} constructed above can arise from a C^1-flow $(F^t)_{t\in\mathbb{R}}$ on $\mathbb{T}_f^2$ of the type described in section 2. We have to modify the speed function V, originally generating ϕ as its return time, to account for the series β of bump functions added to form

$\phi_{\rm new}$ in (6.3). Recall that V has its sole zero at $p_0 = (s_0, y_0)$. Consider the family of squares $R_j^{(k)} = I_{n_j^{(k)}}^{(k)} \times J_j^{(k)}$, $k, j \in \mathbb{N}$, with centres on the line $y = y_0$. By using (4.1) for the lengths $|I_m^{(k)}|$, we see that the side length of the square compares to the distance

$$d_j^{(k)} := \operatorname{dist}((s_0, y_0), R_j^{(k)})$$

as follows:

$$l_{n_j^{(k)}}^{(k)} = \mathrm{e}^{-k}(n_j^{(k)} + k)^{\sigma}(n_j^{(k)} + k)^{-D-\sigma} \geqslant \tfrac{1}{2}(j^{(1+\eta)})^{-D-\sigma} = \tfrac{1}{2}(j^{-\eta})^{(D+\sigma)(1/\eta+1)}$$
$$\geqslant C_8(d_j^{(k)})^{(D+\sigma)(1/\eta+1)} = C_8(d_j^{(k)})^{b''}, \tag{6.9}$$

where the first inequality holds for $j \geqslant j_0^{(k)}$ with $j_0^{(k)}$ chosen at this point suitably large for each $k \in \mathbb{N}$ and the second inequality follows from (6.5) coupled with $h'(s_0) = 1$ (as stipulated by the choice of s_0 in the last paragraph of section 4).

On the other hand, upon fixing $D' \in (1, D)$ with $D - D'$ small enough so that $b < D'c$ (as secured by the hypothesis (6.1)), the inequality (4.6) and $h'(s_0) = 1$ imply that

$$l_{n_j^{(k)}}^{(k)} < C_9(d_j^{(k)})^{D'}. \tag{6.10}$$

Therefore, when $d_j^{(k)}$ is sufficiently small and the points (x, y) of $R_j^{(k)}$ are close enough to p_0 so that $V(x, y)$ is given by (2.2), we see that the term $|x - s_0|^b$ is of order $(d_j^{(k)})^b$ and thus dominates the term $|y - y_0|^c \leqslant C_9'(l_{n_j^{(k)}}^{(k)})^c < C_9'(C_9(d_j^{(k)})^{D'})^c \sim (d_j^{(k)})^{D'c}$. This yields a majorization of the maximum value of V on $R_j^{(k)}$:

$$\max_{R_j^{(k)}} V \leqslant C_{10}(d_j^{(k)})^b. \tag{6.11}$$

The inequalities (6.9), (6.10) and (6.11) give us all necessary control of the geometry involved in the perturbation.

Looking at (6.3), our task is to slow the flow down inside $R_j^{(k)}$ in such a way that the time of flight through $R_j^{(k)}$ along the vertical segment with ordinate x is increased by $a_j^{(k)}\beta_{I_{n_j^{(k)}}^{(k)}}(x)$. We shall not burden the reader with explicit formulae for how to achieve that but rather explain why this can be done so that the resulting function $V_{\rm new}$ is still C^1. It is a fortunate fact that the size of the C^1 perturbation can be solely controlled by the distance $d_j^{(k)}$ of $R_j^{(k)}$ to (s_0, y_0) and the C^0-norm of $V|_{R_j^{(k)}}$; it is insensitive to the magnitude of the $a_j^{(k)}$ (i.e. the severity of the slowing). Indeed, imagine that we want to stop the flow completely on the central subsquare $Q_j^{(k)}$ of $R_j^{(k)}$ with the side length $\frac{1}{2}l_{n_j^{(k)}}^{(k)}$. That would entail lowering the speeds to zero from values not exceeding the order of $(d_j^{(k)})^b$ (see (6.11)) in a distance at least of order $(d_j^{(k)})^{b''}$ (see (6.9)). This can be done by adding to V a C^1 function supported on $R_j^{(k)}$ and of the C^1-norm majorized by $C_{11}(d_j^{(k)})^{b-b''}$. Since $b - b'' > 0$ (cf (6.4)), as $d_j^{(k)} \to 0$, the C^1-norms of the perturbations on the $R_j^{(k)}$ tend to 0. Because the squares $\{R_j^{(k)}\}_{k\in\mathbb{N}, j\geqslant j_0^{(k)}}$ have p_0 as its only accumulation point (as it follows from (6.9)), this guarantees that the perturbations they support constitute a series of functions converging in the C^1-norm. The limit function $V_{\rm new}$ is guaranteed to be C^1. □

Proof of theorem 1.1. Fix $b, c, D > 1$ so that (6.1) holds. (For instance, taking any $b = c \in (1, 2)$ and then picking $D \in (1, b)$ will do.) It is easy to produce a function V for which the hypotheses of proposition 6.1 are satisfied, and then the flow associated with the C^1 function $V_{\rm new}$ supplied by proposition 6.1 has the desired rotation set. □

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