Poincaré rotation number for maps of the real line with almost periodic displacement

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October 4, 2000

Abstract

In generalizing the classical theory of circle maps, we study the rotation set for maps of the real line $x \mapsto f(x)$ with almost periodic displacement $f(x) - x$. Such maps are in one-to-one correspondence with maps of compact abelian topological groups with the displacement taking values in a dense 1-parameter subgroup. For homeomorphisms, we show existence of the analog of the Poincaré rotation number, which is the common rotation number of all orbits besides possibly those that have rotation zero. (The coexistence of zero and non-zero rotation numbers is the main new phenomenon as compared to the classical circle case.) For non-invertible maps, we prove results about realization of points of the rotation interval as the rotation numbers of orbits and ergodic measures. We also address the issue of practical computation of the rotation number.

AMS classification scheme numbers: 58F11, 58F13

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*This work has been supported by NSF EPSCoR grant MONTS-190729 and NSF grant DMS-9970725

1 Introduction

We are concerned with *rotation* — more accurately, average displacement — of points under an iterated map $f : \mathbf{R} \to \mathbf{R}$ of the form $f(x) = x + \phi(x)$ where ϕ is a bounded continuous function. Although we do not make this assumption just yet, for the most part, we will restrict ourselves to ϕ that is almost periodic in the sense of Bohner and Bohr, i.e. the family of translations $\{\phi(\cdot + \tau) : \tau \in \mathbf{R}\}$ is precompact in the topology of uniform convergence [4]. (The prototypical example is $f(x) = x + \gamma + a\sin(x) + b\sin(\sqrt{2}x)$.) *The rotation number of $x \in \mathbf{R}$ with respect to f* is defined as the limiting average displacement

$$\rho(f,x) := \lim_{n\to\infty} \frac{f^n(x) - x}{n}$$

provided the limit exists. By *the rotation set $\rho(f,K)$ of a subset $K \subset \mathbf{R}$* we understand the collection of all limiting average displacements

$$\lim_{i\to\infty} \frac{f^{n_i}(x_i) - x_i}{n_i}$$

where $n_i \in \mathbf{N}$ are such that $\lim_{i\to\infty} n_i = \infty$ and $x_i \in K$. Equivalently,

$$\rho(f,K) := \bigcap_{m\geq 1} \mathrm{cl}\left(\bigcup_{n>m} \left\{\frac{f^n(x) - x}{n} : x \in K\right\}\right). \tag{1.1}$$

(Here "cl" denotes for the topological closure.) This is an adaptation of the definition of Misiurewicz and Ziemian in [15]. It is easy to see (cf [15]) that $\rho(f,K)$ is always compact and that it is connected if K is connected. Thus *the rotation set of f, $\rho(f) := \rho(f,\mathbf{R})$*, is *a priori* a closed segment. We note that other *reasonable* formulations of rotation set of f are contained in $\rho(f)$ and include as a subset *the pointwise rotation set of f, $\rho_p(f) := \{\rho(f,x) : x \in \mathbf{R}$ for which $\rho(f,x)$ exists$\}$*. Which definition one chooses is to some degree a matter of taste and our choice of $\rho(f)$ should be understandable by the time we are done.

Much is known when ϕ is periodic, i.e. when f is a lift of a circle map (see e.g. [17, 1, 23]). We only mention that the title reaches back to the theory of orientation preserving circle homeomorphisms, for which $\rho(f)$ is a point: the celebrated Poincaré rotation number ([18]). However, periodicity is an idealization that may be disputed in some instances[1], and one is naturally lead to inquire: what if ϕ is not exactly periodic but merely almost periodic? Although apparently absent from the literature, this question poses a problem related to that about skew-products of circle maps, which appear in many works (e.g. [11, 13, 19]). The relation is established via flow equivalence and is perhaps most transparent from a classical standpoint: Circle homeomorphisms arise as Poincaré maps for doubly

[1]Heraclitus would say: "No-one can step into the same river twice".

periodic ODE's, e.g. $x' = F(x,t)$ where F is periodic in reals x and t. When the periodicity in t is relaxed to almost periodicity, the skew-products emerge. The homeomorphisms of $\mathbf{R}$ with almost periodic displacement correspond, in turn, to F periodic in t but almost periodic in x (cf [20]). Interchanging the roles of x and t roughly effects the said flow equivalence, and the standard tools of ergodic theory allow for rigorous implementation of this idea. In what follows, hardly straying from the traveled path and without any pretense of particular originality, we provide an introduction to the theory.

To do away with the obvious, we first consider the general situation of bounded ϕ and non-decreasing f. The preservation of order on $\mathbf{R}$ forces the orbits to march in "lock step"; for example, if $\rho(f,x) > 0$ and $f^n(y)$ is unbounded from above, then $\rho(f,y)$ exists and $\rho(f,y) = \rho(f,x)$. One easily discovers then that there are four possibilities for the pointwise rotation set $\rho_p(f)$:

Case 1) $\rho_p(f) = \{0\}$,

Case 2) $\rho_p(f) = \{\rho\}$ where $\rho \neq 0$,

Case 3) $\rho_p(f) = \{0, \rho\}$ where $\rho \neq 0$,

Case 4) $\rho_p(f) = \{\rho_-, 0, \rho_+\}$ where $\rho_- < 0 < \rho_+$.

Note that in case 2 actually $\rho(f,x) = \{\rho\}$ for all $x \in \mathbf{R}$, and in cases 3 and 4 f must have a fixed point; however, there may be no fixed points in Case 1. Also, easy examples show that $\rho_p(f)$ can be a proper subset of $\rho(f)$ in any of the cases.

From now on, we restrict ourselves to displacement functions ϕ that are almost periodic (in the sense of Bohner and Bohr). Any such ϕ is either of fixed sign or changes sign on an infinite set unbounded from above and below. The latter possibility leads to rather easy dynamics that fall under Case 1. When the sign is fixed, i.e. $\phi \geq 0$ or $\phi \leq 0$, Case 4 is clearly excluded but any of Cases 1, 2, 3 may occur — although Case 3 is a bit more subtle (see the example below). Less obvious are possibilities for $\rho(f)$.

Theorem 1 *Suppose that $f(x) = x + \phi(x)$, $x \in \mathbf{R}$, is non-decreasing and ϕ is almost periodic. Then there is a number $\rho \in \mathbf{R}$ such that either $\rho(f) = \{\rho\}$, or $\rho(f) = \mathrm{Conv}(\{0, \rho\})$. (Here* Conv *stands for the convex hull.)*
Moreover, in the former case, $\rho_p(f) = \{\rho\}$, and in the latter case, $\rho_p(f)$ is of the form $\{0\}$, $\{r\}$, or $\{0, r\}$ where r is some number in $\rho(f)$.

We shall refer to the number ρ as *the rotation number of* f. The astute reader will notice that the only non-trivial ingredient of the theorem is that $\rho(f) = [\rho_-, \rho_+]$ cannot happen for some $\rho_\pm$ that are both positive or both negative. In fact, Theorem 1 is merely a pale reflection of a less elementary and more revealing result — Theorem 3 below — asserting that ρ is the common rotation vector of all ergodic measures (on the appropriate compactification of $\mathbf{R}$).

The possibility that $\rho(f) = \text{Conv}(\{0, \rho\})$ with $\rho \neq 0$ allowed by the theorem is a manifestation of a phenomenon absent in the classical periodic context whereby $f(x)$ has an unbounded sequence of near tangencies with the diagonal and these tangencies *detach* from the diagonal fast enough so that they can be negotiated by some orbits with non-zero average speed. This behavior, reminiscent of *intermittency* in circle families, is best illustrated by a quintessential example of a toroidal flow due to A. Katok ([9, 7]).

Katok's Example. On the two-dimensional torus $\mathbf{T^2}$, realized by the usual identification of the sides of the unit square $[0,1]^2$, consider a function $\Phi : \mathbf{T^2} \to \mathbf{R}$ such that $\Phi \geq 0$ and $\Phi^{-1}(0) = \{p_0\}$ for a single point $p_0 \in \mathbf{T^2}$. Let X be a vectorfield on $\mathbf{T^2}$ that arises from a constant vectorfield on $[0,1]^2$ with irrational slope, and let F be the time-one-map of $\Phi \cdot X$. A whole family of different maps $f : \mathbf{R} \to \mathbf{R}$ with quasi-periodic displacement is obtained by restricting F to particular flowlines of X. These f's share the same rotation set $\rho(f)$ that necessarily contains 0. At the same time, the mean return time to a circular cross-section of the torus is proportional to $\int_{\mathbf{T^2}} 1/\Phi(p)\nu(dp)$ (where the integral is taken with respect to the surface area). Thus if $\int_{\mathbf{T^2}} 1/\Phi(p)\nu(dp) < +\infty$, we actually have $\rho(f) = [0, \rho]$ where $\rho = \mathrm{C}/\int_{\mathbf{T^2}} 1/\Phi(p)\nu(dp)$ with C depending on X only — otherwise, $\rho(f) = \{0\}$. By the Birkhoff ergodic theorem, $\rho_p(f) = \rho(f)$ for ν-almost all flowlines. At the same time, Φ can be adjusted so that on some flowlines $\rho_p(f)$ is $\{0\}$, $\{r\}$, or $\{0, r\}$ for any value r in the range $[0, \rho]$ — (which reveals a disturbing nonrobustness of $\rho_p(f)$). Also note that, even if $\rho(f, x) = \rho$, typically $\sup_{n\in\mathbf{N}} |f^n(x) - x - n\rho| = \infty$ (cf Furstenberg's example invoked in Section 4). □

We mention that whether intermittent behavior with non-zero rotation (à la the above example) is typical or not depends on the frequency module of ϕ. For instance, a typical quasi-periodic ϕ with r-frequencies and $\inf \phi = 0$ — by which we mean a ϕ arising as above from a smooth $\Phi : \mathbf{T}^r \to \mathbf{R}$ with a generic zero — has rotational intermittency exactly when $r > 2$. This can be shown via the usual approximation of the map by a flow near the tangency.

We shift our attention now to the situation that generalizes the degree-one circle maps when f is no longer increasing and $\rho(f)$ is typically a non-trivial interval. We prove the following result by adopting the "cutting technique" described in [1].

Theorem 2 *Suppose that $f(x) = x + \phi(x)$ where $\phi : \mathbf{R} \to \mathbf{R}$ is an almost periodic function and $\inf_{x\in\mathbf{R}} \phi(x) > 0$, then $\rho_p(f) = \rho(f)$, i.e. for any $\rho \in \rho(f)$ there is $x \in \mathbf{R}$ such that $\rho(f, x) = \rho$.*

Again, Katok's example shows that the hypothesis $\inf \phi > 0$ is necessary. After we prove Theorem 2 in Section 3, we shall see what the possibilities are without this assumption. Also, for any $\rho \in \rho(f)$ in Theorem 2, we actually find a whole locally compact *almost periodic* invariant set K of x with $\rho(f, x) = \rho$ and an *almost periodic* invariant probability measure ν with $\int \phi(x)\, \nu(dx) = \rho$. At the same time, we should emphasize that, even if $\rho(f)$ is just a point $\{\rho\}$, there may as well be that $\sup_{n\in\mathbf{N}} |f^n(x) - x - n\rho| = \infty$ for almost all $x \in \mathbf{R}$, unlike in the periodic case (see Section 4).

Before leaving the introduction we recast the problem in a way that is both conceptually and technically more appealing. By standard results in topological dynamics (see e.g. [6, 2]), given an almost periodic $\phi : \mathbf{R} \to \mathbf{R}$, there is a compact abelian topological group $\mathcal{G}$, a dense homomorphic immersion $v : \mathbf{R} \to \mathcal{G}$, and a continuous function $\Phi : \mathcal{G} \to \mathbf{R}$ such that $\phi = \Phi \circ v$. We shall adopt additive notation for the group operation in $\mathcal{G}$ and write suggestively $x \cdot v := v(x)$, $x \in \mathbf{R}$. (To give a simple example, when $\phi(x) = \sin x + \sin(\sqrt{2}x)$, $\mathcal{G}$ is the two dimensional torus $\mathbf{T^2} := \{(x \pmod{2\pi}, y \pmod{2\pi})\}$, $x \cdot v = (x, \sqrt{2}x)$, and $\Phi(x, y) = \sin x + \sin y$.) The map f uniquely *extends* to $F : \mathcal{G} \to \mathcal{G}$ called *the hull of f* and given on $p \in \mathcal{G}$ by

$$F(p) = p + \Phi(p) \cdot v. \tag{1.2}$$

Writing $\mathbf{S}$ for the unit complex circle, we now fix a character $\chi : \mathcal{G} \to \mathbf{S}$ such that $\chi \circ v \neq 1$. Clearly, $\chi \circ v : \mathbf{R} \to \mathbf{S}$ is of the form $\chi \circ v(x) = e^{i\lambda x}$, $\lambda \neq 0$. We may as well require that $\lambda > 0$. (The existence of χ is assured by the Peter–Weyl–van Kampen theorem, see [16].) By lifting $\chi|_{\mathbf{R}\cdot v}$ through the exponential $\exp : \mathbf{R} \to \mathbf{S}$, $\exp(x) := e^{i2\pi x}$, we obtain a coordinate $\theta : \mathbf{R} \cdot v \to \mathbf{R}$ on the dense subgroup $\mathbf{R} \cdot v \subset \mathcal{G}$; let us take it so that $\theta(0) = 0$. Since $\theta \circ v(x) = \theta(x \cdot v) = \lambda x$, we are led to abuse the notation and write $\theta(v)$ for λ so that the two coordinates x on $\mathbf{R}$ and θ on $\mathbf{R} \cdot v$ are related[2] by

$$\theta = \theta(v)x. \tag{1.3}$$

The group $\mathcal{G}$ is foliated by F-invariant immersed lines $p + \mathbf{R} \cdot v$, and the θ-displacement of points under the application of F is measured by $\Phi^\theta : \mathcal{G} \to \mathbf{R}$,

$$\Phi^\theta(p) := \Phi(p)\theta(v) = \theta(F(p) - p).$$

The appropriate rotation set is

$$\rho^\theta(F) := \bigcap_{m\geq 1} \mathrm{cl}\left(\bigcup_{n>m} \left\{\frac{\theta(F^n(p) - p)}{n} \,:\, p \in \mathcal{G}\right\}\right). \tag{1.4}$$

The definitions of the rotation number $\rho^\theta(F, p)$ and the pointwise rotation set $\rho^\theta_p(F)$ should be easy to guess. The important fact is that the rotation sets $\rho^\theta(F)$ and $\rho(f)$ coincide up to the obvious scaling:

$$\rho^\theta(F) = \theta(v)\rho(f).$$

This follows from the density of $\mathbf{R} \cdot v$ in $\mathcal{G}$ and the continuity of Φ. At the same time, the obvious inclusion $\theta(v)\rho_p(f) \subset \rho^\theta_p(F)$ is often proper; although, clearly

[2] By rescalling x, we could achieve $\theta(v) = 1$; however, we prefer to keep distinction between x and θ.

$\theta(v)\rho(f,x) = \rho^\theta(F, x\cdot v)$ for any $x \in \mathbf{R}$. We should also mention that each point ρ of $\rho^\theta(F)$ is realized by some invariant probabilistic measure μ on $\mathcal{G}$, i.e. $\rho = \int_\mathcal{G} \Phi^\theta(p)\,\mu(dp)$. (If $\rho = \lim_{i\to\infty} \frac{\theta(F^{n_i}(p_i)-p_i)}{n_i}$, then μ can be found as the weak*-limit of the atomic measures equidistributed on orbit segments $p_i, \dots, F^{n_i-1}(p_i)$.) Moreover, if ρ is the extreme point of $\rho^\theta(F)$, μ can be selected to be ergodic. At the same time, given any ergodic measure μ, $\rho^\theta(F,p) = \int \Phi^\theta(p)\,\mu(dp)$ for all μ-generic points $p \in \mathcal{G}$ (— these points may all be off the line $\mathbf{R}\cdot v$.) In particular, we have

$$\rho^\theta(F) = \left\{ \int \Phi^\theta(p)\,\mu(dp) :\ \mu \in \mathcal{M}(F) \right\} = \mathrm{Conv}(\rho^\theta_p(F)) \tag{1.5}$$

where $\mathcal{M}(F)$ is the set of all probabilistic invariant measures on $\mathcal{G}$. The proofs of the last three facts are easily obtained by using the arguments similar to those in [15, 21].

To summarize — conforming with the *de rigueur* approach of the topological theory of almost periodic functions, see e.g. [19] — we made a transition from the the elementary context of a map f of the real line with an almost periodic displacement to the abstract context of a self-map F of a compact abelian group $\mathcal{G}$ with the displacement $F - \mathbf{Id}$ taking values in a dense 1-parameter subgroup $\mathbf{R}\cdot v$. (Such F are always in the form (1.2).) From this perspective, Theorem 1 is a manifestation of the following more revealing result.

Theorem 3 *Suppose that $F : \mathcal{G} \to \mathcal{G}$ is the hull of a non-decreasing continuous map $f : \mathbf{R} \to \mathbf{R}$ with almost periodic displacement (as defined above). There exists a unique $\rho \in \mathbf{R}$ (depending on f only) such that*

$$\int_\mathcal{G} \Phi^\theta(p)\,\mu(dp) = \theta(v)\rho$$

for any ergodic invariant probability measure μ on $\mathcal{G}$. Moreover, any such measure with $\int_\mathcal{G} \Phi^\theta(p)\,\mu(dp) = 0$ is supported on the set of fixed points of F.

The key to the proof of Theorems 1 and 3 is in observing that the part of the dynamics of F that contributes non-zero rotation is flow equivalent to a skew-product over a minimal translation (map) on a subgroup of $\mathcal{G}$. This will facilitate application of the subadditive ergodic theorem to compute the average rotation — details follow in Section 2. Section 3 is devoted to the proof of Theorem 2. Section 4, based on [22], contains a formula for the rotation number that is more suitable for numerical approximation than the definition. (We add that the constructions of Section 2 are referred to in Sections 3 and 4.)

To close, we stress again the introductory character of this note. We are silent, for instance, on existence of conjugacy with the rigid translation, which is far more delicate than for circle diffeomorphisms (e.g. a semi-conjugacy does not exist in general as exemplified by Furstenberg's example in Section 4). Ultimately, one should explore the interplay between smoothness, the rotation number, and the frequency module of the displacement function. This however leads (via flow equivalence) to a well known open question about existence of global analogues of the local KAM-type results in dimensions higher than one (the circle case). Appreciation of this deep problem can be developed by reading Herman's [12] and following the references therein.

2 Non-decreasing Maps

In this section, we prove Theorems 1 and 3, and we establish continuity of the rotation number as a function of the map. We assume thruought this section that f is non-decreasing.

Proof of Theorems 1 and 3. If ϕ changes sign then we are in the trivial situation with $\rho(f) = \rho_p(f) = \{0\}$ whereby all orbits tend to fixed points and thus all invariant measures are carried on the fixed point set. Therefore, we assume from now on that that ϕ does not change sign, say $\phi \geq 0$, and that $\rho(f) \neq \{0\}$. In view of equality (1.5), the first assertion of Theorem 1 will be established if we show that $\rho(f)$ has only one positive extremal point, and this follows from the Claim 2.0 below. The assertion about $\rho_p(f)$ in Theorem 1 is then a consequence of the elementary discussion in the introduction.

Claim 2.0 *Consider as above $\phi \geq 0$ and all F-invariant ergodic probability measures μ on $\mathcal{G}$ with $\lim_{n\to\infty} \theta(F^n(p) - p) = +\infty$ for μ-almost all points. (These include μ's with $\int_{\mathcal{G}} \Phi(p)\mu(dp) > 0$ by the Birkhoff ergodic theorem.) The rotation vector of μ, $\int_{\mathcal{G}} \Phi(p)\mu(dp)$, is positive and independent on which μ is taken.*

This claim also implies Theorem 3. Indeed, the main assertion is clear; and to see "the moreover part" suppose that $\int_{\mathcal{G}} \Phi(p)\,\mu(dp) = 0$ for an ergodic measure μ. From the claim, for a positive μ-measure set of p, the non-decreasing sequence $\theta(F^n(p) - p)$ is bounded and thus convergent, which implies that $F^n(p)$ converges to a fixed point of F. By ergodicity, this actually happens for μ-almost all p.

To complete the proof, we focus now on showing Claim 2.0. Recall that *the suspension semi-flow of F*, $\mathcal{S} : \mathcal{G}_F \times \mathbf{R}^+ \to \mathcal{G}_F$, $\mathbf{R}^+ := [0, \infty)$, is abstractly defined as the constant unit speed flow along the segments $p \times [0,1]$ on $\mathcal{G}_F := \mathcal{G} \times [0,1]/\sim$ with the identification $(p,1) \sim (F(p), 0)$, $p \in \mathcal{G}$. We shall prefer however a more concrete realization of $\mathcal{S}$. To *interpolate* between the identity and F, we set $F^\tau(p) := p + \tau\Phi(p)\cdot v$, $\tau \in [0,1]$, $p \in \mathcal{G}$; and we consider a semi-flow $\tilde{\mathcal{F}}$ on $\mathcal{G} \times \mathbf{R}$ obtained by patching together infinitely many copies of this homotopy: $\tilde{\mathcal{F}} : (\mathcal{G} \times \mathbf{R}) \times \mathbf{R}^+ \to (\mathcal{G} \times \mathbf{R})$ is given on $(p,t) \in \mathcal{G} \times \mathbf{R}$ by

$$\tilde{\mathcal{F}}^\tau(p,t) := \left(F^{s-\lfloor s\rfloor} \circ F^{\lfloor s\rfloor} \circ F^{-t+\lfloor t\rfloor}(p), t+\tau\right) \tag{2.1}$$

where $s := \tau + t - \lfloor t\rfloor$ (and $\lfloor a\rfloor := \max\{k \in \mathbf{Z} : \ k < a\}$). Note that, given $p \in \mathcal{G}$, the points of $p \times \mathbf{R}$, move under the semi-flow $\tilde{\mathcal{F}}$ confined to the invariant plane $(p+\mathbf{R}\cdot v)\times\mathbf{R}$ along broken lines of positive (possibly infinite) slope. These flow lines are ordered (i.e. cannot cross — although they may merge at points of $(p+\mathbf{R}\cdot v)\times\mathbf{Z}$). We record the following easy facts.

Fact 2.1 *(i) F^τ is a homeomorphisms for $\tau\in[0,1)$;*
(ii) $\tilde{\mathcal{F}}$ is equivariant under the $\mathbf{Z}$-action generated by $(p,t)\mapsto(p,t+1)$;
(iii) The quotient $\mathcal{F}:(\mathcal{G}\times\mathbf{S})\times\mathbf{R}^+\to\mathcal{G}\times\mathbf{S}$ of $\tilde{\mathcal{F}}$ by the $\mathbf{Z}$-action is conjugate to $\mathcal{S}$ via $h:\mathcal{G}_F\to\mathcal{G}\times\mathbf{S}$ induced by $(p,t)\mapsto(F^t(p),t)$, $p\in\mathcal{G}$, $t\in[0,1]$.

The F-invariant measure μ yields a semi-flow invariant measure $\kappa:=h_*\circ\pi_*(\mu\otimes dt)$ on $\mathcal{G}\times\mathbf{S}$, where $\pi:\mathcal{G}\times[0,1]\to\mathcal{G}_F$ is the natural projection (and h is provided by (iii) of Fact 2.1.) The rotation set of F may be recovered from $\tilde{\mathcal{F}}$ via

$$\lim_{\tau\to\infty}\theta\left(\mathrm{pr}_{\mathcal{G}}(\tilde{\mathcal{F}}^\tau(p,t))-p\right)\Big/\tau=\lim_{n\to\infty}\theta(F^n(\overline{p})-\overline{p})/n$$

where $\overline{p}:=\tilde{\mathcal{F}}^{-t+\lfloor t\rfloor}(p)$ so that $(\overline{p},\lfloor t\rfloor)$ and (p,t) are on the same $\tilde{\mathcal{F}}$-orbit. (Indeed, for $\tau\in[n,n+1]$, the difference between the expressions under the limits on both sides is of order $O(1/n)$.) Recall that the measure μ is carried on the set

$$\mathcal{G}^+:=\{p\in\mathcal{G}:\ \lim_{n\to\infty}\theta(F^n(p)-p)=+\infty\},$$

that is $\mu(\mathcal{G}^+)=1$. Likewise, $\kappa(\mathcal{G}^+\times\mathbf{S})=1$, where we used the fact that if $\lim_{\tau\to\infty}\theta(\mathrm{pr}_{\mathcal{G}}(\tilde{\mathcal{F}}^\tau(p,0))-p)=\infty$, then $\lim_{\tau\to\infty}\theta(\mathrm{pr}_{\mathcal{G}}(\tilde{\mathcal{F}}^\tau(p,t))-p)=\infty$ for any $t\in\mathbf{R}$ (due to the order preservation and the $\mathbf{Z}$-equivariance of $\tilde{\mathcal{F}}$).

Denote by Λ the kernel of the character χ (defined in the introduction),

$$\Lambda:=\chi^{-1}(1)\lhd\mathcal{G},$$

and set

$$\Lambda^+:=\mathcal{G}^+\cap\Lambda.$$

Observe that there is a well defined first return map $\tilde{R}:\Lambda^+\times\mathbf{R}\to\Lambda^+\times\mathbf{R}$ for $\tilde{\mathcal{F}}$. Since all the flow-lines of points in $\Lambda^+\times\mathbf{R}$ have finite positive slope — as vieved in the plane $(p+\mathbf{R}\cdot v)\times\mathbf{R})$ — it follows that $\tilde{R}$ is continuous and that $\Lambda^+\times\mathbf{R}$ is a G_δ set.

Another thing to notice is that $\tilde{R}$ is a skew-product over the translation $T:\Lambda\to\Lambda$, $T(p):=p+\omega$ where $\omega:=\theta^{-1}(1)$, i.e. the diagram below commutes,

$$\begin{array}{ccc}\Lambda^+\times\mathbf{R} & \xrightarrow{\tilde{R}} & \Lambda^+\times\mathbf{R}\\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow\\ \Lambda^+ & \xrightarrow{T} & \Lambda^+\end{array}.$$

Here, T is minimal because $\{T^n(0):\ n\in\mathbf{N}\}$ is dense in Λ. Also, $\tilde{R}$ maps the fibers $p\times\mathbf{R}$ one to another via $\mathbf{Z}$-equivariant maps and thus descends to a skew product $R : \Lambda^+ \times \mathbf{S} \to \Lambda^+ \times \mathbf{S}$ of non-decreasing circle maps:

$$\begin{array}{ccc} \Lambda^+ \times \mathbf{S} & \xrightarrow{R} & \Lambda^+ \times \mathbf{S} \\ \pi \downarrow & & \pi \downarrow \\ \Lambda^+ & \xrightarrow{T} & \Lambda^+ \end{array} .$$

Much like in the standard theory of special representation flows ([5]), the semi-flow invariant measure κ on $\mathcal{G}^+ \times \mathbf{S}$ induces an R-invariant probabilistic measure η on $\Lambda^+ \times \mathbf{S}$. Here, for lack of a good reference, we indicate the main steps. First, a time-change of $\mathcal{F}$ is made so that the return time to $\Lambda^+ \times \mathbf{S}$ is constant and one; here, the Jacobian of the time change is piecewise constant along the orbits of $\tilde{\mathcal{F}}$ and is given by

$$w(p,t) = \frac{d\theta}{dt} := \frac{d}{d\tau}\bigg|_{\tau=0} \theta\left(\mathrm{pr}_{\mathcal{G}}(\tilde{\mathcal{F}}^\tau(p,t)) - p\right),$$

which can be seen to be integrable with respect to κ. The measure κ_1 of density with respect to κ proportional to w is invariant for the time-changed semi-flow. In the second step, the natural projection $r : \mathcal{G}^+ \times \mathbf{S} \to \Lambda^+ \times \mathbf{S}$ that sends a point (p,t) to its first return to $\Lambda^+ \times \mathbf{S}$ is used to push κ forward to the desired measure $\eta := r_*(\kappa)$ invariant under R.

We note that T (as a minimal group translation) is uniquely ergodic so that the push forward $\nu := \pi_*(\eta)$ must be the unique probabilistic Haar measure on Λ.

The rotation vectors can be expressed in terms of R by averaging the return time, denoted $\Delta t : \Lambda^+ \times \mathbf{S} \to \mathbf{R}^+$. Namely, for η-almost all $(p,t) \in \Lambda^+ \times \mathbf{S}$,

$$\rho^\theta(F, \overline{p})^{-1} = \lim_{n\to\infty} \Delta t(n,p,t)/n = \int_{\Lambda^+ \times \mathbf{R}} \Delta t(p,t)\, \eta(dp,dt), \tag{2.2}$$

where we abused (*overloaded*) the notation to write $\Delta t : \mathbf{N} \times \Lambda^+ \times \mathbf{S} \to \mathbf{R}$ for the additive cocycle of time displacements:

$$\Delta t(n,p,t) := \mathrm{pr}_{\mathbf{R}}(R^n(p,t)) - t = \Delta t(p,t) + \ldots + \Delta t(R^{n-1}(p,t)).$$

Indeed, we assumed that $\lim_{k\to\infty} \theta(F^k(\overline{p}) - \overline{p}) = \lim_{n\to\infty} \Delta t(n,p,t) = \infty$ for almost all p; and simple geometry in the $(p+\mathbf{R}\cdot v)\times\mathbf{R}$ plane of the flow $\tilde{\mathcal{F}}$ shows that for any n there is k (and vice versa) such that $|\Delta t(n,p,t) - k| < 2$ and $|\theta(F^k(\overline{p}) - \overline{p}) - n| < 2$, where $\overline{p} := \tilde{\mathcal{F}}^{-t+\lfloor t \rfloor}$. This yields the first equality in (2.2); and the second follows from the Birkhoff ergodic theorem.

To compute the averages, we introduce a subadditive cocycle $\overline{\Delta t} : \mathbf{N} \times \Lambda^+ \to \mathbf{R}$ over T that majorizes Δt:

$$\overline{\Delta t}(n,p) := \sup_{t\in\mathbf{R}} \mathrm{pr}_{\mathbf{R}}\left(\tilde{R}^n(p,t)\right) - t.$$

Because $\tilde{R}$ maps the fibers $p \times \mathbf{R}$ in a non-decreasing and $\mathbf{Z}$-equivariant fashion, the following inequalities hold for any $(p,t) \in \Lambda^+ \times \mathbf{S}$ and $n \in \mathbf{N}$

$$\overline{\Delta t}(n,p) - 1 \leq \Delta t(n,p,t) \leq \overline{\Delta t}(n,p). \tag{2.3}$$

It follows that $\overline{\Delta t}$ is ν-integrable:

$$\int_\Lambda \overline{\Delta t}(n,p)\, \nu(dp) \leq \int_{\Lambda\times\mathbf{R}} (\Delta t(n,p,t)+1)\, \eta(dp,dt) = n\rho(F,\mu)^{-1} + 1 < \infty.$$

Thus the subadditive ergodic theorem can be applied to get, for ν-almost all $p \in \Lambda$ and all $t \in \mathbf{R}$,

$$\lim_{n\to\infty} \Delta t(n,p,t)/n = \lim_{n\to\infty} \overline{\Delta t}(n,p)/n = \lambda := \inf_n \int_\Lambda \overline{\Delta t}(n,p)/n\, \nu(dp) \tag{2.4}$$

where we used (2.3) and λ depends only on f — not on μ. One concludes, via (2.2), that $\int \Phi^\theta(p)\, \mu(dp) = 1/\lambda$ for all ergodic measures μ with non-zero average displacement. This finishes the proof of Claim 2.0. □

The above proof shows, in particular, existence of the rotation number for skew-products of circle homeomorphisms, which has been already established in [11, 13] (see also [22]). Our argument is more in the spirit of Herman; although, he does not invoke the subadditive ergodic theorem preferring to work exclusively with invariant measures and their average displacements.

We shall show now that, where $\rho(f) = \{\rho\}$, the rotation number ρ behaves continuously with respect to almost periodic perturbations of f. The proof hinges on the continuity of the rotation number for skew-products of circle maps, which again goes back to [11, 13].

Proposition 2.1 *Suppose that $F = \mathbf{Id} + \Phi \cdot v$ with $\rho^\theta(F) = \{\rho\}$ for some $\rho \neq 0$ and that $\Phi_k : \mathcal{G} \to \mathbf{R}$, $k \in \mathbf{N}$, are continuous and converge to Φ uniformly. For $F_k := \mathbf{Id} + \Phi_k$, there is k_0 and numbers ρ_k such that $\rho(F_k) = \{\rho_k\}$ for $k \geq k_0$ and $\lim_{k\to\infty} \rho_k = \rho$.*

Proof of Proposition 2.1. Since $\rho^\theta(F) = \{\rho\}$, $\rho \neq 0$, at the cost of reversing the orientation of $\mathbf{R}$, we can assume that there is $\kappa > 0$ and k_0 such that $\Phi > \kappa$ and $\Phi_k > \kappa$ for $k \geq k_0$. As in the proof of Theorem 1, we get the return maps $\tilde{R}, \tilde{R}_k : \Lambda \times \mathbf{R} \to \Lambda \times \mathbf{R}$ and subadditive cocycles $\overline{\Delta t}, \overline{\Delta t}_k : \Lambda \times \mathbf{N} \to \mathbf{R}$ so that $\rho^{-1} = \inf_n \int \overline{\Delta t}(n,p)/n\, \nu(dp)$ and $\rho_k^{-1} = \inf_n \int \overline{\Delta t}_k(n,p)/n\, \nu(dp)$ just like (2.4). Similarly, we have superadditive cocycles

$$\underline{\Delta t}(n,p) := \inf_{t\in\mathbf{R}} \mathrm{pr}_{\mathbf{R}}(R^n(p,t)) - t$$

for which the analog of (2.3) holds:

$$\underline{\Delta t}(n,p) \leq \Delta t(n,p,t) \leq \underline{\Delta t}(n,p) + 1.$$

The subadditive ergodic theorem applied to $-\underline{\Delta t}$ yields that $\rho^{-1} = \sup_n \int \underline{\Delta t}(n,p)/n\, \nu(dp)$ — with similar formulas for ρ_k's. Here we silently used that Φ_k's are bounded away from zero by κ so that $\Lambda^+ = \Lambda$ and the functions $\overline{\Delta t}(n,\cdot)$, $\overline{\Delta t}_k(n,\cdot)$, $\underline{\Delta t}(n,\cdot)$, and $\underline{\Delta t}_k(n,\cdot)$ are uniformly bounded (hence integrable) for each $n \in \mathbf{N}$. It follows also that R_k's converge uniformly to R and $\underline{\Delta t}_k(n,\cdot)$ and $\overline{\Delta t}_k(n,\cdot)$ converge uniformly to $\underline{\Delta t}(n,\cdot)$ and $\overline{\Delta t}(n,\cdot)$ for any fixed n. Now, given $\epsilon > 0$, we can fix n large enough so that the difference of the extreme sides in the following inequalities is less than ϵ,

$$\int (\underline{\Delta t}(n,p)/n - 1/n)\, \nu(dp) \le \rho \le \int (\overline{\Delta t}(n,p)/n + 1/n)\, \nu(dp).$$

At the same time we can pass to the limit with $k \to \infty$ in

$$\int (\underline{\Delta t}_k(n,p)/n - 1/n)\, \nu(dp) \le \rho_k \le \int (\overline{\Delta t}_k(n,p)/n + 1/n)\, \nu(dp)$$

and see that ρ_k's are eventually contained in the ϵ neighborhood of ρ. This finishes the proof by the arbitrariness of ϵ. □

The hypothesis $\Phi_0 > 0$ in Proposition 2.1 is essential as can be easily seen by tinkering with the way Φ tends to zero near p_0 in Katok's example. (Integrability of $1/\Phi$ is clearly sensitive to arbitrarily small perturbations, say of class C^∞.)

Remark 2.1 *The majorization with $\underline{\Delta t}$ and $\overline{\Delta t}$ guarantees uniform convergence of $(f^n(x) - x)/n$ to the rotation number in the case when $\inf \phi > 0$ — c.f. Section 4.*

3 Non-monotonic Maps

We turn our attention to non-monotonic maps in order to demonstrate Theorem 2 as stated in the introduction. The idea is to adapt the "cutting technique" described in [1] to our almost periodic context. As before, $\mathcal{G}$ is a compact topological group with a dense homomorphisms $\mathbf{R} \ni x \mapsto x \cdot v \in \mathcal{G}$.

Definition 3.1 *Given a continuous $\Phi : \mathcal{G} \to \mathbf{R}$, the plateau set of Φ is*

$$\mathrm{Const}(\Phi) := \{p \in \mathcal{G} : \; x \mapsto x + \Phi(p + x\cdot v) \text{ is constant on a neighborhood of } x = 0\}.$$

We define the upper Φ as $\Phi_+ : \mathcal{G} \to \mathbf{R}$ given on $p \in \mathcal{G}$ by

$$\Phi_+(p) := \sup_{s \le 0} \{s + \Phi(p + s \cdot v)\}$$

and the lower Φ as $\Phi_- : \mathcal{G} \to \mathbf{R}$ given on $p \in \mathcal{G}$ by

$$\Phi_-(p) := \inf_{s \ge 0} \{s + \Phi(p + s \cdot v)\}.$$

For $\lambda \geq 0$ we also define $\Phi_\lambda : \mathcal{G} \to \mathbf{R}$ by

$$\Phi_\lambda(p) := (\min\{\Phi, \Phi_- + \lambda\})_+ ,$$

and we denote by Const_λ *the plateau set of* Φ_λ.

Note that the plateau set may not be open or closed in $\mathcal{G}$. The definitions of $\Phi_\pm$ are best understood by observing that, in the periodic case, the functions $x + \Phi_+(x \cdot v)$ and $x + \Phi_-(x \cdot v)$ coincide with $f_u(x)$ and $f_l(x)$ in [1]. We omit a number of facts that can be easily generalized from [1] via this correspondence; we shall need the following:

(i) The mappings $\Phi_\pm$ and Φ_λ are continuous and depend continuously on Φ and λ. (In fact, they are Lipschitz with constant 1 with respect to the sup metric);

(ii) $\Phi_- \leq \Phi_\lambda \leq \Phi_+$, $\inf \Phi_- = \inf \Phi$, and $\sup \Phi_+ = \sup \Phi$;

(iii) $\Phi_0 = \Phi_-$ and there is $\lambda_+ > 0$ such that $\Phi_{\lambda_+} = \Phi_+$;

(iv) $\Phi_\lambda(p)$ is non-decreasing in $\lambda \geq 0$ for any fixed $p \in \mathcal{G}$;

(v) $\Phi_\lambda(p) = \Phi(p)$ for p in the complement of the interior $\mathrm{int}(\mathrm{Const}_\lambda)$.

The following is an analog of Lemma 3.7.15 in [1] with a very similar proof.

Lemma 3.1 *Suppose that $\Phi : \mathcal{G} \to \mathbf{R}$ is continuous and such that the function $x \mapsto F(x \cdot v)$ is non-decreasing taking $F = \mathbf{Id} + \Phi \cdot v$. There exists a non-empty compact invariant set $\Omega \subset \mathcal{G}$ such that $F^n(p) \notin \mathrm{int}(\mathrm{Const}(\Phi))$ for all $p \in \Omega$ and $n \in \mathbf{N}$.*

Proof of Lemma 3.1. The set $\Omega := \mathcal{G} \setminus \bigcup_{n \in \mathbf{N}} F^{-n}(\mathrm{int}(\mathrm{Const}(\Phi)))$ has the required properties provided we can show that it is non-empty. Otherwise, if $\Omega = \emptyset$, compactness yields that $\mathcal{G} = \bigcup_{n \in \mathcal{N}} F^{-n}(\mathrm{int}(\mathrm{Const}(\Phi)))$ for some finite set $\mathcal{N} \subset \mathbf{N}$, and it follows that $\mathcal{G}$ is the plateau set for the iterate F^{n_*} where $n_* := \max \mathcal{N}$. However, $\mathrm{int}(F^{n_*}(\mathcal{G})) = \mathcal{G} \neq \emptyset$ by surjectivity of F and observe that if $q \in \mathrm{int}(F^{n_*}(\mathcal{G}))$, then there must be some non-plateau point $p \in \mathcal{G}$ such that $q = F^{n_*}(p)$. This is a contradiction. □

Note that Lemma 3.1 does not assure that $\Omega \cap p + \mathbf{R} \cdot v \neq \emptyset$ for all $p \in \mathcal{G}$ (c.f. Remark 3.2). This prompted us to give a yet another generalization of Lemma 3.7.15 in [1], which shows that the compactness hypothesis is superfluous.

Lemma 3.2 *Suppose that $f : \mathbf{R} \to \mathbf{R}$ is non-decreasing and that $f(x) - x$ does not change sign over all $x \in \mathbf{R}$ (i.e. is either non-negative or non-positive). Then there exists an orbit $(x_k)_{k \in \mathbf{Z}}$, $x_k = f(x_{k-1})$, such that $x_k \notin \mathrm{int}(\mathrm{Const}(f))$ for all $k \in \mathbf{Z}$. (Here $\mathrm{Const}(f) := \{x \in \mathbf{R} : f$ is locally constant at $x\}$.)*

An easy example with a single attracting plateau shows that the hypothesis on the sign of $f(x) - x$ is essential.

Corollary 3.1 *Suppose that $f : \mathbf{R} \to \mathbf{R}$ is non-decreasing with almost periodic displacement, then there exists an orbit $(x_k)_{k\in\mathbf{Z}}$, $x_k = f(x_{k-1})$, such that $x_k \notin \mathrm{int}(\mathrm{Const}(f))$ for all $k \in \mathbf{Z}$.*

Proof of Corollary 3.1. If $f(x) - x$ does not change sign, we are done by the lemma. Otherwise $f(x)$ changes sign and does it so infinitely many times by almost periodicity. There will be in particular an x_* where $f(x) - x$ changes sign from negative to positive. Note that x_* is a fixed point of f that is not in the interior of $\mathrm{Const}(f)$. $\square$

Proof of Lemma 3.2. To fix attention let us assume that $f(x) \geq x$ for all $x \in \mathbf{R}$. If there is a point at which $f(x) = x$ then $x_k := x$ is the sought after orbit as x cannot be in the interior of $\mathrm{Const}(f)$. Suppose then that $f(x) > x$ for all $x \in \mathbf{R}$. Since orbits are monotone sequences we have $\lim_{k\to+\infty} x_k = +\infty$ and $\lim_{k\to-\infty} x_k = -\infty$; otherwise, the finite limit would constitute a fixed point of f contradicting $f(x) > x$.

Fix one orbit $(a_k)_{k\in\mathbf{Z}}$. Consider the product space $X := \prod_{k\in\mathbf{Z}}[a_k, a_{k+1}]$ with the product topology and its closed subspace consisting of orbits, $Y := \{(x_k) \in X : \; x_k = f(x_{k-1}), \; k \in \mathbf{Z}\}$. We have the manifestly open sets $U_m := \{(x_k) \in X : \; x_m \in \mathrm{int}(\mathrm{Const}(f))\}$, $m \in \mathbf{N}$. It suffices to demonstrate that $Y \setminus \bigcup_{m\in\mathbf{Z}} U_m \neq \emptyset$. For an indirect proof let us suppose it is otherwise. Then, by compactness, $Y \subset U_{m_1} \cup \ldots \cup U_{m_q}$ for some $m_1 < \ldots < m_q \in \mathbf{Z}$. We may as well assume that $m_1 = 0$ at the cost of renumbering a_k's. Set $n = m_q + 1$ and consider $g := f^n|_{[a_0,a_1]} : [a_0, a_1] \to [a_n, a_{n+1}]$. For any $x_0 \in [a_0, a_1]$, f is locally constant at one of the points $x_k = f^k(x_0)$, $k = 0, 1, \ldots, m_q = n - 1$, which implies that g is locally constant at x_0. Being locally constant at all points of $[a_0, a_1]$, g is constant on $[a_0, a_1]$. Hence, $a_n = g(a_0) = g(a_1) = a_{n+1}$, which contradicts strict monotonicity of a_k's as guaranteed by $f(x) > x$. $\square$

Proof of Theorem 2. Let $\rho(f) = [\rho_-, \rho_+]$. We have that $\Phi_\lambda > \inf \Phi \geq \kappa > 0$, so that Proposition 2.1 and (i) yield continuous dependence on $\lambda \geq 0$ of the rotation number ρ_λ of $F_\lambda = \mathbf{Id} + \Phi_\lambda$. Invoking (iii) yields $\rho_0 = \rho_-$ and $\rho_{\lambda_+} = \rho_+$ so that there must exist λ, $0 \leq \lambda \leq \lambda_+$, with $\rho_\lambda = \rho$. Take Ω provided by Lemma 3.1 applied to F_λ. For $p \in \Omega$, $F_\lambda^n(p) \notin \mathrm{int}(\mathrm{Const}_\lambda)$ for all $n \in \mathbf{N}$, and (iv) implies $F_\lambda^n(p) = F^n(p)$ so that $\rho(F, p) = \rho$. Define $K := \{x \in \mathbf{R} : \; x \cdot v \in \Omega\}$. Because clearly $\mathrm{int}(\mathrm{Const}(F)) \cap \mathbf{R} \cdot v \subset \mathrm{int}(\mathrm{Const}(f))$, Corollary 3.1 assures that K is nonempty. For $x \in K$, we have $\rho(f, x) = \rho(F, x \cdot v) = \rho$. $\square$

Remark 3.2 *When $\rho(F) = \{\rho\}$ with $\rho \neq 0$, the fact that $\Omega \cap p + \mathbf{R} \cdot v \neq \emptyset$ for all $p \in \mathcal{G}$ can be seen from the minimality of $T : \Lambda \to \Lambda$. Indeed, consider $F|_\Omega : \Omega \to \Omega$ and the associated skew product $R : \Lambda_0 \times \mathbf{R} \to \Lambda_0 \times \mathbf{R}$ constructed as in the proof of*

Theorem 1 with $\Lambda_0 := \Lambda \cap \Omega$. Now, Λ_0 is closed and invariant under the minimal translation $T : \Lambda \to \Lambda$ so $\Lambda_0 = \Lambda$. For $p \in \mathcal{G}$, the line $p + \mathbf{R} \cdot v$ clearly intersects Λ, and thus it intersects Ω.

As mentioned in the introduction, Katok's example shows that the theorem fails without the assumption that $\inf \phi > 0$ or $\sup \phi < 0$. When Φ has zeros, the function $\lambda \mapsto \rho_\lambda$ is only continuous over the set of λ's for which Φ_λ has no tangencies with zero. The possibility of jumps of ρ_λ at the first and last such tangency as λ traverses $[0, \lambda_+]$ allows one only to deduce that there are $\rho_- \leq b_- \leq c_- \leq 0 \leq c_+ \leq b_+ \leq \rho_+$ so that $\rho(f) = [\rho_-, \rho_+]$ and the set of rotation vectors realized by orbits or by ergodic measures on $\mathcal{G}$ is of the form $[\rho_-, b_-) \cup \{c_-\} \cup \{0\} \cup \{c_+\} \cup (b_+, \rho_+]$. It should be possible to construct examples showing that there are no extra restrictions on $\rho_\pm, b_\pm, c_\pm$ although we did not attempt to do so.

4 Approximation of the Rotation Number

We address briefly the issue of practical approximation of the rotation number for a non-decreasing map $f : \mathbf{R} \to \mathbf{R}$ with almost periodic displacement $\phi(x) = f(x) - x$. In the periodic case, one can determine the rotation number to within $O(1/n)$ by taking n iterates of any single point. (Faster algorithms exist if one is willing to a priori preclude some *bad* irrational rotation numbers, see [3].) As we shall see, this is generally no longer true in the almost periodic situation; however, we have the following result closely related to the main theorem of [22].

Theorem 4 *Suppose that $f(x) = x + \phi(x)$, $x \in \mathbf{R}$, is non-decreasing and ϕ is almost periodic with $\inf |\phi| > 0$. If $\rho \neq 0$ is the rotation number of f (i.e. $\rho(f) = \{\rho\}$), then we have for $l \in \mathbf{R}^+$ that*

$$\left| \frac{1}{2T} \lim_{T \to \infty} \int_{-T}^{T} \min\{n \in \mathbf{N} : \ |f^n(x) - x| \geq l\} \, dx - l\rho^{-1} \right| \leq 1 + 1. \tag{4.1}$$

(Here the first "1" is the length of $\mathbf{S}$ and the second "1" is the distance between two consecutive natural numbers.)

The functions $x \mapsto \min\{n \in \mathbf{N} : \ |f^n(x) - x| \geq l\}$ can be easily computed numerically. Also, the theorem does not cover the intermittent case when $\inf |\phi| = 0$ as then singularities develop under the integral in (4.1) yielding numerical integration tricky — c.f. Remark 4.3 below.

Before we prove the theorem, let us elaborate on the difficulty it is devised to bypass. When ϕ is periodic, we have $|f^n(x) - x - \rho(f)n| \leq 1$ for any point $x \in \mathbf{R}$ so that $(f^n(x) - x)/n$ approximates $\rho(f)$ with error bounded by $1/n$. At the same time, in the almost periodic case, it is possible that $\sup_{x \in \mathbf{R}} |f^n(x) - x - n\rho| = \infty$ for almost all $x \in \mathbf{R}$ and all $\rho \in \mathbf{R}$, and the obvious method of computing ρ fails. To illustrate this point, one can draw upon the classical Furstenberg example ([14, 8, 10]), which is a mapping of the $\mathbf{T^2}$ with a lift $\tilde{R} : \mathbf{R}^2 \to \mathbf{R}^2$ given by $\tilde{R}(x, y) = (x + \alpha, y + h(x) + 1)$. Here α is a lacunary irrational number and $h(x)$ is a continuous periodic function with average zero that fails to be cohomologous to zero in the continuous category but $h(x) = k(x + \alpha) - k(x)$ for some *essentially unbounded* periodic and locally integrable function k. By inverting the flow equivalence construction in the proof of Theorem 1, one produces from $\tilde{R}$ a mapping $\tilde{F} : \mathbf{R}^2 \to \mathbf{R}^2$ that preserves the foliation into *irrational* lines with slope α. The map $\tilde{F}$ covers a homeomorphisms $F : \mathbf{T^2} \to \mathbf{T^2}$, which is the hull of certain $f(x) = x + \phi(x)$ with a quasi-periodic ϕ obtained as the restriction of $\Phi(x, y) = \tilde{F}(x, y) - (x, y)$ to the irrational line. As in the proof of Theorem 1, c.f. (2.2), the rotation number of f is $\rho(f) = (\int_0^1 h(x) + 1\,dx)^{-1} = 1$. However, for almost all x and all y,

$$\tilde{R}^n(x, y) - (x, y) - n(\alpha, 1) = (0, k(x + (n - 1)\alpha) - k(x)) \tag{4.2}$$

gets arbitrarily big as n varies over $\mathbf{N}$ because of the essential unboundedness of k. This corresponds to unboundedness of $f^n(x) - x - n$.

The route to remedy the situation is suggested by observing that, even though the right hand side of (4.2) is rather ill behaved, its average is zero. This sheds some light on the role of the integral in Theorem 4. The proof below is an adaptation of the argument in [22].

Proof of Theorem 4. We shall use the suspension semi-flow $\mathcal{F}$ on $\mathcal{G} \times \mathbf{S}$ and its lift $\tilde{\mathcal{F}}$ to $\mathcal{G} \times \mathbf{R}$ as constructed in the proof of Theorem 1. We shall also need a continuous version of the cocycle $\Delta t(n, p, t)$, namely: $\mathcal{T} : \mathbf{R}^+ \times \mathcal{G} \times \mathbf{R} \to \mathbf{R}^+$,

$$\mathcal{T}(\Delta, p, t) := \min\{s > 0 :\ \theta(\mathrm{pr}_{\mathcal{G}} \circ \tilde{\mathcal{F}}^s(p, t) - p) \geq \Delta\}.$$

Thus $\mathcal{T}(\Delta, p, t)$ is simply the time required for the point (p, t) to cover θ-distance equal to Δ. From (1.3), the integrand in (4.1) is given by

$$\min\{n \in \mathbf{N} :\ |f^n(x) - x| \geq l\} = \lceil \mathcal{T}(l/\theta(v), x \cdot v, 0) \rceil \tag{4.3}$$

(where $\lceil a \rceil := \min\{k \in \mathbf{Z} :\ a \leq k\}$). Also,

$$\lim_{\Delta \to \infty} \mathcal{T}(\Delta, p, t)/\Delta = \rho^{-1}$$

for all $p \in \mathcal{G}$ — c.f. (2.4).

Fix $\Delta \in \mathbf{R}^+$ and set for convenience $\tau = \Delta/\theta(v)$ just so that $\theta(T^\tau(p) - p) = \Delta$ for all $p \in \mathcal{G}$, where T is the translation flow on $\mathcal{G}$, $T^s(p) = p + s \cdot v$. Consider the sum of the form

$$\Sigma^k(t_0, \ldots, t_{k-1}) := \sum_{j=0}^{k-1} \mathcal{T}(\Delta, T^{j\tau}(p), t_j) \tag{4.4}$$

where $k \in \mathbf{N}$ and $t \in \mathbf{R}^k$. By analogy to (2.3), the $\mathbf{Z}$-equivariance of $\tilde{\mathcal{F}}$ yields, for any $t, \tilde{t} \in \mathbf{R}$,

$$|\mathcal{T}(\Delta, T^{j\tau}(p), t) - \mathcal{T}(\Delta, T^{j\tau}(p), \tilde{t})| \leq 1 \tag{4.5}$$

so that

$$\left|\Sigma^k(t_0, \ldots, t_{k-1}) - \Sigma^k(\tilde{t}_0, \ldots, \tilde{t}_{k-1})\right| \leq k \tag{4.6}$$

for any $t, \tilde{t} \in \mathbf{R}^k$. Fix $k \in \mathbf{N}$ and consider a piece of orbit of $\tilde{\mathcal{F}}$, $((p(t), t))_{0 \leq t \leq t_*}$ where t_* is such that $\theta(p(t_*) - p(0)) = k\Delta$, i.e. $\mathcal{T}(k\Delta, p, 0) = t_*$ or $T^{k\tau}(p(0)) = p(t_*)$. Take $t_0 = 0$ and t_j's for $j = 1, \ldots, k-1$ so that $p(t_j) = T^{j\tau}(p(0))$. Also, set $\tilde{t}_0 = \ldots = \tilde{t}_{k-1} = 0$. The inequality (4.6) becomes

$$\left|\mathcal{T}(k\Delta, p, 0) - \sum_{j=0}^{k-1} \mathcal{T}(\Delta, T^{j\tau}p, 0)\right| \leq k. \tag{4.7}$$

Now, there is a dense set of Δ's in $\mathbf{R}^+$ for which $T^\tau : \mathcal{G} \to \mathcal{G}$ is ergodic. For any such Δ, after dividing by k and passing to the limit with $k \to \infty$, we get

$$\left|\Delta\rho^{-1} - \int_{\mathcal{G}} \mathcal{T}(\Delta, p, 0)\, \nu(dp)\right| \leq 2\pi$$

where ν is the Haar measure on $\mathcal{G}$. Taking into account (4.3) and the fact that

$$\int_{\mathcal{G}} \mathcal{T}(\Delta, p, 0)\, \nu(dp) = \lim_{T\to\infty} \frac{1}{2T} \lim_{T\to\infty} \int_{-T}^{T} \mathcal{T}(\Delta, x \cdot v, 0)\, dx \tag{4.8}$$

by strict ergodicity of the translation flow on $\mathcal{G}$, we arrive at (4.1) for a dense set of l. The extension to all l is easily obtained by continuity. $\square$

Remark 4.3 *In the intermittent case, (4.8) may fail since the integrand is no longer continuous; however, Theorem 4 remains valid if one replaces the integral over* $\mathbf{R}$ *by the corresponding integral over* $\mathcal{G}$ *with respect to the Haar measure. Without this modification one can only assert that (4.1) holds for almost all* $p \in \mathcal{G}$ *once* $x \mapsto f(x) = x + \Phi(x \cdot v)$ *is replaced by* $x \mapsto x + \Phi(p + x \cdot v)$.

We comment that the argument for Theorem 4 leads to an alternative proof of the existence of the rotation number for f. Let us only argue that $\lim_{\Delta\to\infty} \frac{\mathcal{T}(\Delta,p,0)}{\Delta}$ exists and is constant at Haar almost all p. Then one can conclude Theorem 1 by considering the invariant measures for the flow $\tilde{\mathcal{F}}$ with the time changed to θ. All those measures project to the Haar measure on $\mathcal{G}$. To show our claim, note that the limits

$$\lambda^+(p) := \limsup_{k\to\infty} \frac{\mathcal{T}(k\Delta, p, 0)}{k\Delta}$$

and

$$\lambda^-(p) := \liminf_{k\to\infty} \frac{\mathcal{T}(k\Delta, p, 0)}{k\Delta}$$

are independent of $\Delta > 0$ for any $p \in \mathcal{G}$. Indeed, given $\Delta, \tilde{\Delta} > 0$, and large $k \in \mathbf{N}$, there are $k_1, k_2 \in \mathbf{N}$ such that $k - \max\{\Delta, \tilde{\Delta}\} \le k_1 \le k \le k_2 \le k + \max\{\Delta, \tilde{\Delta}\}$ and $\mathcal{T}(k_1\tilde{\Delta}, p, 0) \le \mathcal{T}(k\Delta, p, 0) \le \mathcal{T}(k_2\tilde{\Delta}, p, 0)$. At the same time, (4.7) yields, for almost all p,

$$\left|\lambda^{\pm}(p) - \frac{1}{\Delta}\int_{\mathcal{G}} \mathcal{T}(\Delta, p, 0)\,\nu(dp)\right| \le 1/\Delta.$$

By taking Δ arbitrarily large, we conclude that $\lambda_-(p) = \lambda_+(p)$, that is $\lim_{\Delta\to\infty} \frac{\mathcal{T}(\Delta,p,0)}{\Delta} = \lambda$ exists and is constant at almost all p — as claimed.

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