Elements of the theory of unimodular Pisot substitutions with an application to β-shifts

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Abstract. We apply the geometric theory of unimodular Pisot substitutions to show that the natural extension of the Renyi β-transformation is canonically isomorphic to a toral automorphism for a certain new class of Pisot units β.

1. Introduction

We have two goals. First, we want to outline some key aspects of the geometric theory of the tiling spaces associated to substitutions that are unimodular Pisot. The full development with complete proofs can be found in [**4**]. Second, we want illustrate how [**4**] can be applied in the much studied context of Renyi's β-expansions by showing that the natural extension of the Renyi β-transformation is canonically isomorphic to a toral automorphism for a certain new class of Pisot units (Theorem 3.3).

2. Tiling Space of Unimodular Pisot Substitution

2.1. Hypotheses. We make the following standing hypotheses in this section. Let $\phi : \mathcal{A} \to \mathcal{A}^*$ be a substitution over a finite alphabet $\mathcal{A} = \{1, \ldots, d\}$ where $d \geq 2$. Denote by $A = (a_{ij})_{i,j=1}^{d}$ the matrix of ϕ; that is, a_{ij} is the number of occurrences of i in $\phi(j)$. We assume that A is unimodular Pisot: $\det(A) = 1$ and A has a simple Perron eigenvalue $\lambda > 1$ with all other eigenvalues of modulus strictly less than one. Substitutions ϕ with the above properties are called *unimodular Pisot.* (They are primitive and aperiodic, and A is irreducible over $\mathbb{Q}$.) See [**11, 26, 4**] for background.

We also fix Perron eigenvectors ω and ω^* of A and its transpose: $A\omega = \lambda\omega$ and $A^T\omega^* = \lambda\omega^*$, normalized so that $|\omega| = 1$ and $\langle\omega|\omega^*\rangle = 1$ (where $\langle\cdot|\cdot\rangle$ is the Euclidean scalar product and $|\cdot|$ is its norm).

Let $T^t : \mathcal{T}_\phi \to \mathcal{T}_\phi$, $t \in \mathbb{R}$, be the translation flow (with unit speed) on the tiling space of ϕ. We postpone the precise definition of T but mention that it can

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be thought of as the *special flow* of the left shift $\sigma : X_\phi \to X_\phi$ on the *substitutive system* $X_\phi \subset \mathcal{A}^{\mathbb{Z}}$ associated[1] to ϕ. The return time over the ith cylinder in X_ϕ is the length of the tile corresponding to the letter i and is taken equal to ω_i^*. T is uniquely ergodic [**26**].

2.2. Motivation. While we have some basic understanding of the ergodic properties of self-affine tiling spaces — like unique ergodicity, mixing and weak mixing [**26**] — their measure theoretical classification has hardly begun. In contrast to the case of ϕ of constant length — where generalized Riesz products have been successfully deployed in [**19**] — for substitutions of non-constant length, there are no sweeping results and the best hope for such is in the following:

Pure Discrete Spectrum Conjecture (PDSC): For ϕ that is unimodular Pisot, the flow T has pure discrete spectrum.

We note that, via Theorems 2.1 and 2.7 below, for the PDSC to be correct, T would have to be isomorphic to a translation flow on the d-dimensional torus $\mathbb{T}^d$. Also, via [**8**], T has pure discrete spectrum iff σ does. According to [**14**], where the case of $d = 2$ has been tackled (via [**3**]), the conjecture goes back at least to the 1980's, although its exact origin is hard to pin down. Beyond $d = 2$, the conjecture is known to hold for some concrete families of substitutions ϕ [**12, 1, 13**]. (Theorem 3.2, ahead, adds to the collection of such ϕ.) We mention that the pure discrete spectrum for T is equivalent to the pure discrete diffraction spectrum for the related quasi-crystal [**10**] (see also [**16**]).

2.3. Some Results. The extensive work [**4**] may serve as a launching pad for attacking the PDSC. It simplifies and unifies what is already known in a self-contained geometrical setup and develops a number of characterizations of the conjecture, including concrete algorithms (cf. [**23, 24**]). Let us advertise two results of the theory that shed some light on what the PDSC really entails. Recall that $\alpha \in \mathbb{R}$ is an eigenvalue of T iff there is an L^2-function g such that $g \circ T^t = e^{i2\pi\alpha t} g$ for all $t \in \mathbb{R}$.

THEOREM 2.1. *[Theorem 9.3 in* [**4**]*] The set of eigenvalues of T (discrete spectrum) coincides with the subgroup of $\mathbb{R}$ generated by the components of ω, that is*

$$\sigma_d(T) = \left\{ \sum_{i=1}^{d} k_i \omega_i : \ k_i \in \mathbb{Z} \right\}. \tag{2.1}$$

The inclusion "$\supset$" has been widely known before and traces back to [**7**]. In the parlance of [**15**], the inclusion "$\subset$" translates to "ϕ has no nontrivial coboundaries" (see also 7.3.2 in [**11**]).

THEOREM 2.2. *[Corollary 9.4 in* [**4**]*] T has pure discrete spectrum iff ϕ satisfies a certain combinatorial (and algorithmically decidable) condition called the Geometric Coincidence Condition (GCC).*

Sufficient conditions incorporating the idea of *coincidence* have been in use for quite some time, debuting for constant length substitutions in [**9**]. Our condition GCC is a strengthening of that in [**2**], and we shall explain it below in the context

[1] X_ϕ is the closure of the shift orbit of any bi-infinite word w fixed by ϕ, or ϕ^N for some $N > 0$.

of an unorthodox definition of $\mathcal{T}_\phi$ via a certain space of broken lines in $\mathbb{R}^d$ called *strands*. We mention that one of the consequences of the above theorems is that pure discrete spectrum of T is equivalent with model set representation of $\mathcal{T}_\phi$ (cf. [**17**]).

2.4. Strand Space. Denote by e_i the ith unit basis vector in $\mathbb{R}^d$ and let $I_i := \{te_i : \ t \in [0,1]\}$ be the unit segment on the corresponding axis. Any segment J congruent by translation to one of the basic segments I_i is called an *edge*. The ends of J are naturally ordered and denoted by $\min J$ and $\max J$. A *strand* is a subset γ in $\mathbb{R}^d$ obtained as a union of a sequence of edges J_n where $\max J_n = \min J_{n+1}$ for all n (see Figure 1). Let $\mathcal{F}$ be the space of all bi-infinite strands in $\mathbb{R}^d$ with the compact-open topology. There is an obvious map

$$h : \mathcal{F} \to \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d \tag{2.2}$$

given by $h(\gamma) = v \bmod \mathbb{Z}^d$ where v is a vertex of γ (i.e. an endpoint of some J_n). Any finite strand γ determines a word $[\gamma]$ over $\mathcal{A}$ in an obvious way, e.g. $[I_i] = i$. The substitution ϕ induces a map Φ on edges such that $\Phi(x + I_i)$ begins at Ax and determines the word $\phi(i)$. One readily extends Φ to all strands by applying it edge by edge, as illustrated by Figure 1. By construction, Φ factors via h to the automorphism f_A of $\mathbb{T}^d$ induced by $x \mapsto Ax$, $x \in \mathbb{R}^d$,

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\Phi} & \mathcal{F} \\ {\scriptstyle h}\downarrow & & {\scriptstyle h}\downarrow \\ \mathbb{T}^d & \xrightarrow{f_A} & \mathbb{T}^d. \end{array} \tag{2.3}$$

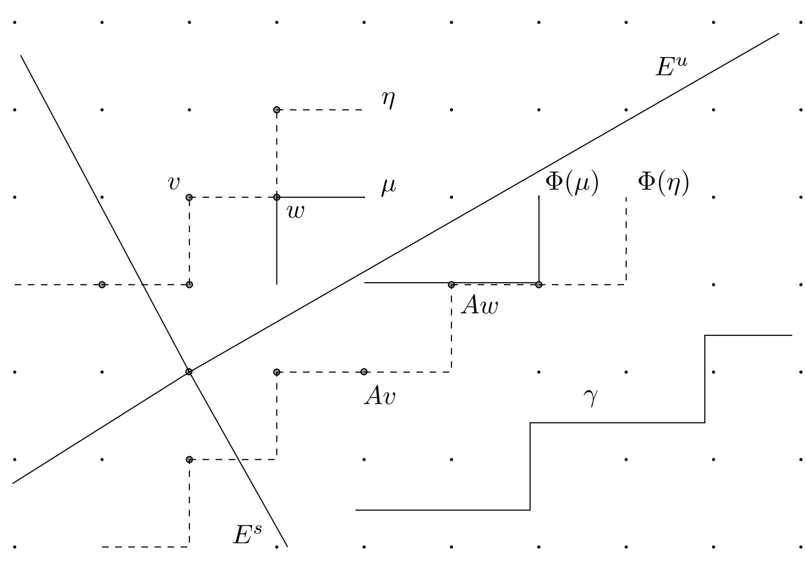

FIGURE 1. The map Φ for the Fibonacci substitution $\phi : 1 \mapsto 12,\ 2 \mapsto 1$. μ, η and γ are example strands; μ and η are coincident, $\mu \sim \eta$.

Let $\mathbb{R}^d = E^s \oplus E^u$ be the stable/unstable decomposition for A and denote by $\mathcal{F}^R$ the subset of the strands in $\mathcal{F}$ that are contained in the cylinder of radius

R about the one-dimensional space E^u (with respect to the A-*adapted norm*). It is easy to see that, for sufficiently large $R > 0$, $\Phi(\mathcal{F}^R) \subset \mathcal{F}^R$ and the following definition does not depend on the choice of such R.

DEFINITION 2.3. *The strand space of ϕ is*

$$\mathcal{F}_\phi := \bigcap_{n \geq 0} \Phi^n(\mathcal{F}^R). \tag{2.4}$$

Thus $\mathcal{F}_\phi$ is the global attractor of Φ. We have two dynamical systems on $\mathcal{F}_\phi$, a map induced by Φ and a flow along E^u given by $T^t(\gamma) := \gamma + t\omega$.

If we identify E^u with $\mathbb{R}$ via $t \mapsto t\omega$, then any $\gamma \in \mathcal{F}_\phi$ determines a tiling of $\mathbb{R}$ obtained by projecting the edges of γ onto E^u along E^s. The tiling space $\mathcal{T}_\phi$, by definition, consists of all thus obtained tilings of E^u, i.e., $\mathcal{T}_\phi = \mathcal{F}_\phi / E^s$.[2] Of course, Φ and T induce a map and a flow on $\mathcal{T}_\phi$; little harm is done by denoting these with the same letters. Mossé's recognizability result [**18**] (see also [**27, 4**]) guarantees that $\Phi : \mathcal{T}_\phi \to \mathcal{T}_\phi$ is a homeomorphism.

PROPOSITION 2.4 (Theorem 6.1 in [**4**]). *The natural map $\gamma \mapsto \gamma$ (mod E^s) is a homeomorphism $\mathcal{F}_\phi \to \mathcal{T}_\phi$.*

This is so fundamental and simple we repeat the proof here.

Proof: The surjectivity is a tautology, so we concentrate on the injectivity. Suppose that $\gamma, \gamma' \in \mathcal{F}_\phi$ are such that $\gamma \equiv \gamma' \mod E^s$. Fix $n \in \mathbb{N}$. There are $\gamma_n, \gamma'_n \in \mathcal{F}_\phi$ such that $\gamma = \Phi^n(\gamma_n)$ and $\gamma' = \Phi^n(\gamma'_n)$. Φ^{-1} being well defined on $\mathcal{T}_\phi$ (not $\mathcal{F}_\phi$), $\gamma_n \equiv \gamma'_n \mod E^s$ so that $\gamma'_n = \gamma_n + x$ where $x \in E^s$ and $|x| \leq R$. Hence, $\gamma' = \gamma + A^n x$. As $n \to \infty$, $|A^n x| \to 0$ showing $\gamma' = \gamma$. □

The advantage of $\mathcal{F}_\phi$ over $\mathcal{T}_\phi$ is that it allows the following effortless definition.

DEFINITION 2.5. *The canonical torus of $\mathcal{F}_\phi$ is the restriction of h to $\mathcal{F}_\phi$,*

$$h_\phi := h|_{\mathcal{F}_\phi} : \mathcal{F}_\phi \to \mathbb{T}^d.$$

We also refer to h_ϕ as the *geometric realization* of $\mathcal{F}_\phi$. The latter is justified inasmuch as, for all known examples, h_ϕ constitutes a measure theoretical isomorphism between Φ and f_A, and T and the Kronecker flow T_ω on $\mathbb{T}^d$ given by $T^t_\omega : p \mapsto p + t\omega$.

2.5. Coincidence. We say that two edges J and K are *coincident*, $J \sim K$, iff there is $n \in \mathbb{N}$ such that $\Phi^n(J)$ and $\Phi^n(K)$ have an edge in common (Figure 1). For $t \in \mathbb{R}$, two edges J, K are *coincident along* $t\omega + E^s$ iff there is $n \in \mathbb{N}$ such that $\Phi^n(J)$ and $\Phi^n(K)$ have a common edge L that intersects $E^s + \lambda^n t\omega$ at a point other than $\max L$. Unlike $\sim$, $\sim_{t\omega}$ is an equivalence relation; and $J \sim K$ iff there is t such that $J \sim_{t\omega} K$. An edge J with a vertex in $\mathbb{Z}^d$ intersecting E^s at a point other than $\max J$ is called an *integer state*.

Geometric Coincidence Condition (GCC_ϕ)**:** Any two integer states are coincident.

Geometric Coincidence Conjecture (GCC): The GCC_ϕ holds for any unimodular Pisot ϕ.

As long as this is open, we have to contend with the following definition.

[2] Those already familiar with the tiling space of ϕ will rightfully disagree: our $\mathcal{T}_\phi$ may have a finite set of extra T-orbits that would have to be removed to get the orthodox definition.

DEFINITION 2.6. *The coincidence rank of ϕ is*

$$cr_\phi := \max\{\#S : \ S \text{ is a family of mutually non-coincident integer sates}\}. \tag{2.5}$$

Note that cr_ϕ is finite because it suffices to consider only the states within radius R from E^u.

THEOREM 2.7 (Coincidence Theorem, Theorem 7.3 in [**4**]). *The cardinality of the fiber $h_\phi^{-1}(p)$ is bounded uniformly for all $p \in \mathbb{T}^d$, and there is a dense full measure G_δ-set $G_\phi^u \subset \mathbb{T}^d$ such that $\#h_\phi^{-1}(p) = cr_\phi$ for $p \in G_\phi^u$. Moreover, if $p \in G_\phi^u$ and $\gamma, \eta \in h_\phi^{-1}(p)$ are distinct, then no edge of γ is coincident with an edge of η.*

The moreover part implies that, for $p \in G_\phi^u$, no two different strands in $h_\phi^{-1}(p)$ share an edge. Thus, upon discarding a negligible set, $\mathcal{F}_\phi$ is a cr_ϕ-to-1 covering of $\mathbb{T}^d$.

2.6. Elements of the Proof of Theorem 2.1. For computation of the discrete spectrum of T we employ *homoclinic return times*, that is $t \in \mathbb{R}$ for which there is a tiling $\gamma \in \mathcal{T}_\phi$ such that $\lim_{n\to\infty} \mathrm{dist}(\Phi^n \circ T^t(\gamma), \Phi^n(\gamma)) = 0$ (i.e. $T^t(\gamma)$ is in the stable set of γ). This concept is closely related to the return times as used in [**15, 26**] but is of utility in the general context of self-similar flows, i.e., T and Φ satisfying

$$\Phi \circ T^t = T^{\lambda t} \circ \Phi, \quad t \in \mathbb{R}. \tag{2.6}$$

As a model example, in the place of Φ and T, consider f_A and the Kronecker flow T_ω. The homoclinic return times (of any $p \in \mathbb{T}^d$) are easily seen to be exactly the numbers of the form $t = \langle \omega^* | v \rangle$ where $v \in \mathbb{Z}^d$ — as dictated by $E^s + t\omega = E^s + v$. Going back to $\mathcal{F}_\phi$, since Φ and T^t factor via h_ϕ to f_A and T_ω^t, respectively, a return on $\mathcal{T}_\phi$ implies one on $\mathbb{T}^d$; and we conclude that the homoclinic return times of $\gamma \in \mathcal{F}_\phi$ are of the form

$$t = \langle \omega^* | v \rangle, \quad v \in Z_\gamma,$$

where Z_γ is a certain subset of $\mathbb{Z}^d$.

PROPOSITION 2.8 (Lemma 13.2 and Corollary 13.2 in [**4**]). *A number α is an eigenvalue of T iff it can be written in the form*

$$\alpha = \langle k | \omega \rangle$$

where

$$k \in \left(\bigcup_{\gamma \in \mathcal{F}_\phi} Z_\gamma \right)^* .$$

(Here, for $D \subset \mathbb{R}^d$, $D^* := \{k \in \mathbb{R}^d : \ \langle k | x \rangle \in \mathbb{Z} \ \forall x \in D\}$.)

The basic premise is that of *Fourier duality* between periods (return times) and frequencies (eigenvalues). In the elementary context, for a periodic motion with frequency α, t is a period iff $\alpha \cdot t \in \mathbb{Z}$. In our case — where the motions are at best just quasi-periodic — t ranges over all homoclinic return times, α ranges over all the eigenvalues of T, and $\alpha \cdot t \in \mathbb{Z}$ is taken with respect to a suitable scalar product: $\alpha \cdot t := \langle k | v \rangle$. [3]

In view of $(\mathbb{Z}^d)^* = \mathbb{Z}^d$ and the last proposition, Theorem 2.1 is a consequence of the following.

[3] This is the *trace* scalar product on $\mathbb{Q}(\lambda) = \langle \mathbb{Z}^d | \omega^* \rangle$.

Proposition 2.9 (Theorem 12.1 in [**4**]). *The submodule of* $\mathbb{R}$ *generated by all homoclinic return times coincides with* $\langle \mathbb{Z}^d | \omega^* \rangle$*, i.e.,*

$$\left\langle \bigcup_{\gamma \in \mathcal{F}_\phi} Z_\gamma \right\rangle = \mathbb{Z}^d.$$

2.7. Proof of Theorem 2.2. If $cr_\phi = 1$ then $h_\phi : \mathcal{F}_\phi \to \mathbb{T}^d$ is a measure theoretical isomorphism by Theorem 2.7 and thus T has pure discrete spectrum.

For the other implication, it suffices to show that if $cr_\phi > 1$ then there is an L^2-function on $\mathcal{F}_\phi$ that is not in the closure of the linear span of the eigenfunctions. From Theorem 2.7, if we take a sufficiently small ball $B \subset \mathbb{T}^d$, then $B \cap G^u_\phi$ is well covered by h_ϕ so that $h_\phi^{-1}(B \cap G^u_\phi) = B_1 \cup \dots B_c$ where $c = cr_\phi$ and B_i's have pairwise disjoint closures. From Theorem 2.1, all the eigenfunctions factor through h_ϕ. No linear combination of such functions can approximate the characteristic function χ_{B_1} of B_1 (since $c > 1$).

2.8. Checking GCC$_\phi$**.** Given $p \in \mathbb{T}^d$, let $\mathbb{S}_p$ stand for all *states over* p, i.e., the edges with a vertex in the lattice $p + \mathbb{Z}^d$ that intersect E^s at a point other than their max vertex.

Lemma 2.10 (Lemma 10.1 in [**4**]). *There is a full measure dense* G_δ *set* $G^s_\phi \subset \mathbb{T}^d$ *such that, for* $p \in G^s_\phi$, $\mathbb{S}_p$ *consist of exactly* cr_ϕ *equivalence classes of* $\sim_0$*.*

We mention that the appropriate completion of the space of all equivalence classes of $\sim_0$ in $\bigcup_{p \in G^s_\phi} \mathbb{S}_p$ constitutes the dual tiling space $\mathcal{F}^*_\phi$ of ϕ.

Since $\sim_0$ is invariant under translations along E^s, so is G^s_ϕ. (Thus E^s naturally acts on $\mathcal{F}^*_\phi$.) In particular, the lemma assures that $\{t \in \mathbb{R} : -t\omega (\text{mod } \mathbb{Z}^d) \in G^s_\phi\}$ is a full measure dense G_δ subset of $\mathbb{R}$. In what follows, we shall call such t *generic*. To explain the minus sign, note that $cr_\phi = 1$ implies that, for a generic t and any two edges K, L with vertices in $\mathbb{Z}^d$ that intersect $E^s + t\omega$, we have $K - t\omega \sim_0 L - t\omega$, i.e., $K \sim_{t\omega} L$. The opposite implication — which is also immediate — can be sharpened to the following useful criterion for $cr_\phi = 1$. Fix any $i, j \in \mathcal{A}$ and let $\Gamma_{i,j}$ be the boundary of the unit square with vertices 0, e_i, e_j, $e_i + e_j$.

Proposition 2.11 (Proposition 17.3 in [**4**]). $cr_\phi = 1$ *provided, for generic* $t \in \mathbb{R}$*, any two edges* $I, J \subset \Gamma_{ij}$ *intersecting* $E^s + t\omega$ *satisfy* $I \sim_{t\omega} J$*.*

The proof relies on the characterization of $cr_\phi = 1$ in terms of *partial commutation* of the translation actions on $\mathcal{F}_\phi$ and $\mathcal{F}^*_\phi$ (see Theorem 16.3 in [**4**]).[4] Also, Proposition 2.11 can be reformulated as a balanced pair algorithm (see Proposition 17.4 in [**4**]).

Finally, the main result in [**3**] asserts that there are always $i, j \in \mathcal{A}$ such that $i \neq j$ and $I_i \sim I_j$. In fact, from Fact 19.1 and Corollary 19.2 in [**4**], we extract the following more detailed statement.

Lemma 2.12. *Suppose that for* $p \in G^u_\phi$ *the* cr_ϕ *strands in* ${h_\phi}^{-1}(p)$ *do not intersect. Then there exist* $\epsilon > 0$ *and* $i, j \in \mathcal{A}$ *such that* $i \neq j$ *and* $I_i \sim_{t\omega} I_j$ *for all generic* $t \in [0, \epsilon)$*.*

[4]The two actions live on the same space since $\mathcal{F}_\phi$ and $\mathcal{F}^*_\phi$ are a.e. identified via duality.

3. β-shifts

Fix $\beta > 1$. The β-transformation is $f_\beta : [0,1] \to [0,1]$, given by $x \mapsto \beta x - \lfloor \beta x \rfloor$. It is widely conjectured (see e.g. [**22, 21**]) that

CONJECTURE 3.1. *If β is a Pisot number, then the natural extension of f_β is canonically almost homeomorphically conjugated to an automorphism of a compact Abelian group.*

Here, an *almost homeomorphism* refers to a measure preserving isomorphism that maps homeomorphically a full measure set to a full measure set. The measures considered are the measure of maximal entropy for f_β on $[0,1]$ and the Haar measure on the group. The canonical map can be arrived at in a number of different ways. In particular, we shall exploit the fact that it can be viewed as a special instance of the geometric realization h_ϕ. Pure discrete spectrum of the *adic transformation* associated to β [**25**] would be one of the consequences of the conjecture. Without going into details, we mention that the adic transformation can be viewed as a global cross-section of the tiling flow for a natural substitution associated to β. The conjecture is also equivalent to several purely number theoretical properties of the β-expansions.

For β that is Pisot the forward orbit of 1 is pre-periodic [**6, 20**]:

$$1 \mapsto \cdots \mapsto {f_\beta}^p(1) \mapsto \cdots \mapsto {f_\beta}^{n+p}(1) = {f_\beta}^p(1).$$

Thus f_β has a Markov partition into finitely many intervals and can be understood by studying appropriate substitutions.

Concretely, in connection with β-expansions (see e.g. [**25, 5**] and the references therein), one considers the substitution ϕ_β over the alphabet $\mathcal{A} = \{1, \ldots, d\}$, $d := n + p$, given by

$$\phi_\beta(i) = \underbrace{1 \ldots 1}_{a_i} k, \quad {f_\beta}^i(1) = {f_\beta}^{k-1}(1), \quad i, k \in \mathcal{A},$$

where a_i is the number of times f_β *wraps* $R_i := [0, {f_\beta}^{i-1}(1)]$ around the whole circle. Thus any letter j of $\phi_\beta(i)$ naturally corresponds to a subsegment of R_i that maps homeomorphically onto R_j. Given $\gamma \in \mathcal{T}_{\phi_\beta}$, we have the bi-infinite orbit $(\Phi^n(\gamma))_{n\in\mathbb{Z}}$ and a bi-infinite word $(i_n)_{n\in\mathbb{Z}}$ where i_n is the letter of the edge of γ_n intersecting E^s at a point other than its max vertex. From the definition of Φ, i_{n+1} corrsponds to a unique subletter of $\phi_\beta(i_n)$. Thus we can associate to γ a bi-infinite sequence of segments $(V_n)_{n\in\mathbb{Z}}$ such that $V_n \subset R_{i_n}$ and $f_\beta : V_n \to R_{i_{n+1}}$ is an expanding homeomorphism. Hence, there is a unique orbit $(x_n)_{n\in\mathbb{Z}}$ of f_β such that $x_n \in V_n$. This defines a map from $\mathcal{T}_{\phi_\beta}$ to the space of bi-infinite orbits of f_β, the natural extension of f_β. It is easy to see that this mapping is a.e. 1-1.

In this way, Conjecture 3.1 holds for β if the GCC holds for ϕ_β. Of course, at this point, we have to restrict attention to β that are Pisot units with d being their algebraic degree over $\mathbb{Q}$. The general case requires a suitable extension of the GCC condition and will be dealt with elsewhere.

Since our arguments are more palatable for the reverse ϕ of the substitution ϕ_β, we state our main result in terms of ϕ. Of course, $cr_\phi = cr_{\phi_\beta}$.

THEOREM 3.2. *Suppose that ϕ is a substitution over an alphabet with d letters that is unimodular Pisot and of the form*

$$\begin{cases} 1 & \mapsto 21^{a_1} \\ 2 & \mapsto 31^{a_2} \\ \vdots & \\ d-1 & \mapsto d1^{a_{d-1}} \\ d & \mapsto 1 \end{cases} \tag{3.1}$$

where $a_k \geq 0$, $1 \leq k < d$. If d is prime and the following hypotheses are satisfied

(H) *$a_1 > 0$ and $a_r > 0$ where $r := \max\{k : \ a_{d-k+1} = \cdots = a_{d-1} = 0\}$,*

then GCC_ϕ holds.

THEOREM 3.3. *Suppose that β is a Pisot unit of a prime degree d and that ${f_\beta}^d(1) = 0$. Then the minimal polynomial of β has the form*

$$p(x) = x^d - a_1 x^{d-1} - \cdots - a_{d-1}x - 1 \tag{3.2}$$

where $a_k \geq 0$, $1 \leq k < d$. If the hypotheses (H) hold then the natural extension of f_β is canonically[5] almost homeomorphically isomorphic to an automorphism of the torus.

Reduction of Theorem 3.3 to Theorem 3.2: Because, under f_β, $1 \mapsto f_\beta(1) \mapsto \cdots \mapsto {f_\beta}^d(1) = 0 \mapsto 0$, the reverse of the substitution ϕ_β is of the form (3.1). The polynomial (3.2) is the characteristic polynomial the matrix A of ϕ (or ϕ_β). By the assumption on the degree of β, p is the minimal polynomial of β. Thus the hypotheses of Theorem 3.2 are satisfied and we have $cr_\phi = cr_{\phi_\beta} = 1$. The map $h_{\phi_\beta} : \mathcal{T}_{\phi_\beta} \cong \mathcal{F}_{\phi_\beta} \to \mathbb{T}^d$ constitutes the sought after isomorphism. $\square$

The hypothesis that d is prime can be removed but at the expense of complicating the proof, so we shall not do that here. In [**12**], the analogous result is shown for β that is a leading root of a polynomial of the form (3.2) with $a_1 \geq a_2 \geq \cdots \geq a_{d-1} \geq 1$. Such β constitute an infinite family satisfying our hypotheses. Another family, with $a_1 > \sum_{i=2}^d a_i$ is dealt with in [**13, 1**]. There are examples that are of neither of these forms, for instance, the Pisot unit $\beta = 2.67344\ldots$ with the minimal polynomial

$$x^5 - 2x^4 - x^3 - 2x^2 - 1$$

and the substitution ϕ given by

$$\begin{cases} 1 & \mapsto 211 \\ 2 & \mapsto 31 \\ 3 & \mapsto 411 \\ 4 & \mapsto 5 \\ 5 & \mapsto 1. \end{cases} \tag{3.3}$$

(The digits of the Renyi expansion of 1 are $2, 1, 2, 1, 0, 0, \ldots$.)

LEMMA 3.4. *Suppose that ϕ is of the form (3.1) with $a_1 > 0$. There is $N \in \mathbb{N}$ such that the last letter of $\phi^N(i)$ is 1 for all $i \in \mathcal{A}$.*

[5] A measure theoretical isomorphism at large exists because this is a Bernoulli shift.

Proof: Let $A := \{j : a_j > 0\}$. Denoting by $\phi_-(i)$ the last letter of $\phi(i)$, we have $\phi_-(i) = 1$ for every $i \in A$. For $i \notin A$, $\phi_-(i) = i + 1$. Thus, for $i \in \mathcal{A}$, $\phi_-^{k_i}(i) = 1$ where $k_i := 1 + \min\{k : i + k \in A\}$. Since $\phi_-(1) = 1$ due to $a_1 > 0$, setting $N := \max\{k_i : i \in \mathcal{A}\}$ yields $\phi_-^N(i) = \phi_-^{N-k_i} \circ \phi_-^{k_i}(i) = \phi_-^{N-k_i}(1) = 1$. □

COROLLARY 3.5. *There exist $\epsilon > 0$ and $k, l \in \mathcal{A}$ such that $k \neq l$ and $I_k \sim_{t\omega} I_l$ for all generic $t \in [0, \epsilon)$ (i.e. when $-t\omega \in G_\phi^s$).*

Proof: Via Lemma 2.12, it suffices to see that, for $p \in G_\phi^u$, the strands making up ${h_\phi}^{-1}(p)$ do not intersect. Suppose that two such different strands γ_1 and γ_2 intersect at a vertex v. Let K_i be the edge of γ_i with $\max K_i = v$. Lemma 3.4 implies that $K_1 \sim K_2$. This contradicts the last assertion in Theorem 2.7 to the effect that $\gamma_1 \not\sim_{t\omega} \gamma_2$ for all $t \in \mathbb{R}$. □

COROLLARY 3.6. *If ϕ is of the form (3.1) with $a_1 > 0$ and d is prime, then there exists $\epsilon > 0$ such that, for any $k, l \in \mathcal{A}$, $I_k \sim_{t\omega} I_l$ for all generic $t \in [0, \epsilon)$.*

Proof: Let k, l, ϵ be as in the previous corollary. With the addition of indices in $\mathcal{A}$ taken modulo d, I_k and I_l map over I_{k+1} and I_{l+1} under Φ, respectively, i.e., $I_{k+1} \subset \Phi(I_k)$ and $I_{l+1} \subset \Phi(I_l)$. It follows that $I_{k+1} \sim_{t\omega} I_{l+1}$ for all generic $t \in [0, \lambda\epsilon)$ such that $E^s + t\omega$ intersects both of I_{k+1} and I_{l+1} — which is easily assured by taking $\epsilon > 0$ sufficiently small. Since $\lambda\epsilon > \epsilon$, we can repeat this argument to get $I_{k+j} \sim_{t\omega} I_{l+j}$ for all generic $t \in [0, \epsilon)$. In particular, for such t,

$$I_k \sim_{t\omega} I_l \sim_{t\omega} I_{l+l-k} \sim_{t\omega} I_{l+2(l-k)} \sim_{t\omega} \cdots \sim_{t\omega} I_{l+m(l-k)}$$

for any $m \geq 0$. Since d is prime, $\{l + m(l-k) : m \geq 0\} = \{1, \ldots, d\}$. The corollary follows by the transitivity of $\sim_{t\omega}$. □

LEMMA 3.7. *Suppose that ϕ is of the form (3.1) with $a_{d-r} > 0$, $a_{d-r+1} = a_{d-r+2} = \cdots = 0$, and $a_r > 0$. Then $\phi^d((r+1)1) = (r+1)1\ldots$ and $\phi^d(1(r+1)) = 1(r+1)\ldots$. (Here $\phi^d(a) = b\ldots$ means that b is a prefix of $\phi^d(a)$.)*

Proof: We apply ϕ d times to the word $(r+1)1$:

$$\underbrace{(r+1)1 \mapsto (r+2)\ldots \mapsto \cdots \mapsto r\ldots}_{\phi^{d-1}} \mapsto (r+1)1\ldots.$$

Likewise, we apply ϕ d times to the word $1(r+1)$:

$$\underbrace{1(r+1) \mapsto 2\ldots \mapsto \cdots \mapsto (d-r)\ldots}_{\phi^{d-r-1}} \mapsto$$

$$\underbrace{(d-r+1)1\ldots \mapsto (d-r+2)2\ldots \mapsto \cdots \mapsto dr\ldots \mapsto 1(r+1)\ldots}_{\phi^r}$$

□

COROLLARY 3.8. *For generic $t \in \mathbb{R}$, we have that*

(3.4) *if $I, J \subset \Gamma_{1,r+1}$ are two edges intersecting $E^s + t\omega$ then $I \sim_{t\omega} J$.*

Proof: By Corollary 3.6, $I_1 \sim_{t\omega} I_{r+1}$ for generic $t \in [0, \epsilon)$, that is the property (3.4) holds for such t. On the other hand, Lemma 3.7 asserts that $\Gamma_{1,r+1}$ maps over itself, i.e., $\Phi(\Gamma_{1,r+1}) \supset \Gamma_{1,r+1}$. It follows that the property (3.4) holds for all generic $t \in [0, \lambda\epsilon)$. It is left to repeat this argument n times where n is such that all t for which $E^s + t\omega$ intersects $\Gamma_{1,r+1}$ belong to $[0, \lambda^n\omega)$. □

Conclusion of Proof of Theorem 3.2: The last corollary verifies the hypothesis of Proposition 2.11. Hence $cr_\phi = 1$. □

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