Cocyclic Subshifts

(Revised and Expanded)

Jaroslaw Kwapisz[1]
Department of Mathematical Sciences
Montana State University
Bozeman MT 59717-2400
tel: (406) 994 5343
fax: (406) 994 1789
http://www.math.msu.edu/~jarek/

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Abstract: Motivated by the computations in the theory of cohomological Conley index, *cocyclic subshifts* are the supports of locally constant matrix cocycles on the full shift over a finite alphabet. They properly generalize sofic systems and topological Markov chains; and, via the Wedderburn-Artin theory of finite-dimensional algebras, admit a similar structure theory with a spectral decomposition into mixing components. These components have specification, which implies intrinsic ergodicity and entropy generation by sequences of horseshoes. Also, a zeta-like generating function for cocyclic subshifts leads to simple criteria for existence of a factor map onto the full two-shift — which gives practical tools for detecting chaos in general discrete dynamical systems.

1 Introduction.

An elementary question encapsulates the topic of this article: Given two square matrices Φ_0, Φ_1, what can one say about binary sequences $\sigma =$

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$(\sigma_1, ..., \sigma_n)$ for which the product $\Phi_\sigma = \Phi_{\sigma_1}...\Phi_{\sigma_n}$ is not zero? Concretely, when does the number a_n of such sequences of length n increase exponentially in n, i.e. $h := \lim_{n\to\infty} \ln a_n/n > 0$? We give a sharp answer in terms of certain algebras associated with the products $\Phi_{\sigma_1}...\Phi_{\sigma_n}$. Our approach leads through topological dynamics and yields results going far beyond answering the question. Indeed, the number h can be interpreted as the topological entropy of the shift map acting on the space of infinite binary sequences for which any finite segment is as above. This space is an example of what we call a *cocyclic subshift* — a new kind of subshift that generalizes topological Markov chains and sofic systems.

If only to justify the name (cocyclic subshifts), let us assume a broader perspective for a moment. Given a map $f : X \to X$, one may consider cocycles Φ with values in a semigroup $\mathcal{G}$ with zero 0. This is to say that $\Phi : \mathbf{N} \times X \to \mathcal{G}$ satisfies $\Phi(n+m, x) = \Phi(n, x) \cdot \Phi(m, f^n x)$, $n, m \in \mathbf{N}$, and $0 \in \mathcal{G}$ is such that $0 \cdot g = g \cdot 0 = 0$ for all $g \in \mathcal{G}$. *The support of the cocycle* Φ, $X_\Phi := \{x \in X : \Phi(n, x) \neq 0 \text{ for all } n \in \mathbf{N}\}$, is forward invariant under f, $f X_\Phi \subset X_\Phi$. Our problem is an instance of a general question about the relation between the properties of X_Φ and those of $\mathcal{G}$ and Φ.

The *cocyclic subshifts* are, by definition (Section 2), the spaces X_Φ obtained from the shift map f on $X := \{1, ..., m\}^{\mathbf{N}}$, $(fx)_i = x_{i+1}$, and from a locally constant[2] cocycle Φ into the semigroup $\mathcal{G} = \mathrm{End}(V)$ of all linear transformations of a finite dimensional vector space V. (Our initial question corresponds to $m = 2$ and Φ depending only on x_0.) This should be viewed as a generalization of [21], where B. Weiss introduced sofic systems by taking for $\mathcal{G}$ any finite semigroup (c.f. Section 10).

Besides the broader class of subshifts considered, what sets our work apart from the existing literature on sofic systems is the focus on the algebra generated by the cocycle: the algebra is less structured and more regular than the semigroup, thus allowing for more complete and constructive theory. Most importantly, by exploiting the classical Wedderburn-Artin theory of finitely-dimensional algebras, we are able to implement for cocycles the ideas of reducible, irreducible, and aperiodic such that the corresponding cocyclic subshifts have a structure very similar to that of topological Markov chains defined by reducible, irreducible, and aperiodic matrices.

In particular, the mixing cocyclic subshifts are those definable by aperi-

[2] i.e. $\Phi(1, x)$ depends on a finite initial block of x of fixed length.

odic irreducible (primitive) cocycles; and they satisfy the specification property. This is the key result of the paper with a corollary (via Bowen's theory) that the topologically transitive cocyclic subshifts are intrinsically ergodic (i.e. have a unique invariant probability measure of maximal entropy).

To reveal our motivation, we mention that our results are relevant to the symbolic dynamics built around the Conley index for maps ([15, 19, 18, 20, 13]). While the reader may consult [18] for a formal exposition, let us give a glimpse of how cocyclic subshifts enter the scene.

Roughly, the phase space (of a discrete dynamical system) is divided into finitely many compact pieces labeled 1 through m. Each piece has associated an index which is a pointed topological space, and the dynamics induces on the cohomologies of the indices an action that generates the cocycle on $\{1, ..., m\}^{\mathbf{N}}$. An infinite sequence of pieces *codes* an orbit of the map (i.e. the orbit is selected from the sequence) provided the cocycle does not vanish: the sequence is in X_Φ. One may think of this as a common generalization of the Lefschetz fixed point theorem, where there is only one piece (the whole space), and the usual concept of a Markov partition, where there are many pieces but the way they map is very restricted. The role of the cocyclic subshifts is then analogous to that of subshifts of finite type in the standard symbolic dynamics.

The primary application of the technique is for confirming chaos in concrete dynamical systems, a problem that reduces to the question whether X_Φ factors onto the full two-shift ([14, 15, 20, 18, 3]). Our structure theory for cocyclic subshifts resolves the issue completely: the factor map exists iff in the spectral decomposition given by the Wedderburn-Artin theory of the appropriate algebras, there is an aperiodic component which is not a single point (Corollary 9.3); and this criterion admits efficient numerical implementation — see the appendix. In fact, we prove that all of topological entropy on a cocyclic subshift is realized by embedded horseshoes (Theorem 7.2). Moreover, the cocyclic subshifts with zero entropy stand out as those with particularly simple non-wandering dynamics concentrated on few periodic orbits captured by a certain zeta-like generating function (Section 9).

As this paper is aimed at both a solution of the chaos detection problem and an introduction to a new type of symbolic dynamics, we confined its scope in many respects. Restriction to the algebraically closed base field or the one-sided shifts is easy to overcome and helped to simplify presentation of the main ideas. More notable omission is that of ergodic theory of the intrinsic measure (including the computation of the entropy h), which is dealt with in the forthcoming [11]. Unresolved is also left the problem of factors of cocyclic subshifts, an uncharted class that brings out more exotic semigroups of subspaces of matrices, yet possibly coincides with cocyclic subshifts (see Section 11). Here, [16] instills some hope by picking up our new class of *subspace semigroups* for systematic study.

To end the introduction, we put together a quick guide to what follows. Sections 2 and 3 contain definitions and some basic properties of cocyclic subshifts as dynamical systems. The progression of Sections 4, 5, and 6 develops a decomposition of a cocyclic subshift into irreducible and primitive (irreducible and aperiodic) pieces, and shows that these are topologically transitive and mixing, correspondingly. Thus the stage is set for the proof that primitivity implies specification in Section 7, with intrinsic ergodicity of a topologically transitive cocyclic subshift and entropy generation by horseshoes obtained as easy corollaries. Section 8 digresses to show that, under a suitable non-degeneracy assumption on a cocycle, its irreducibility and aperiodicity follows from transitivity and mixing (correspondingly) of the underlying cocyclic subshift. Section 9 (together with the appendix) characterizes the cocyclic subshifts with zero entropy and then derives criteria for chaos; a certain zeta-like generating function is one notable tool here. Section 10 discusses the inclusion of sofic systems into cocyclic subshifts; in particular, it contains a concrete example of a non-sofic cocyclic subshift — perhaps worth inspecting just after reading Section 2. Section 11, in turn, contains an example (*the context free subshift*) of a subshift with specification that is not cocyclic nor is a factor of a cocyclic subshift. Finally, Section 12 introduces a useful way of presenting cocyclic subshifts by *graphs with propagation*, i.e. labeled (colored) graphs with matrix weights over the edges.

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2 The Definition.

Let $\mathcal{A}$ be a finite alphabet of m symbols, say $\mathcal{A} = \{1, ..., m\}$. Recall that *the (full) one-sided shift (over $\mathcal{A}$)* is the product space $\mathcal{A}^{\mathbf{N}}$ with *the shift map* $f : \mathcal{A}^{\mathbf{N}} \to \mathcal{A}^{\mathbf{N}}$ given by $f : (x_i)_{i \in \mathbf{N}} \mapsto (x_{i+1})_{i \in \mathbf{N}}$. Taken as a product of discrete spaces, $\mathcal{A}^{\mathbf{N}}$ is compact, and f is a continuous map. The standard metric d on X is $d((x_i), (y_i)) = 2^{-j}$ where $j \in \mathbf{N}$ is minimal such that $x_j \neq y_j$. By *a subshift of $\mathcal{A}^{\mathbf{N}}$* we understand any closed $X \subset \mathcal{A}^{\mathbf{N}}$ that is invariant under f (i.e. $f(X) \subset X$).

Denote by $\mathrm{End}(V)$ all the linear endomorphisms of a linear space V. The space V is always assumed to be finite-dimensional, non-zero, and over an algebraically closed field $\mathbf{C}$. Moreover, we shall compose linear maps in $\mathrm{End}(V)$ on the right: $\Phi(\Psi(v)) = v\Psi\Phi$ for $\Phi, \Psi \in \mathrm{End}(V)$ and $v \in V$. (Thereby we treat V as a right $\mathrm{End}(V)$-module.) The following is the central definition of this paper.

Definition 2.1 *A cocyclic subshift of $\Phi = (\Phi_1, ..., \Phi_m) \in \mathrm{End}(V)^m$ is the subshift $X_\Phi \subset \mathcal{A}^{\mathbf{N}}$ given by*

$$X_\Phi := \{x \in \mathcal{A}^{\mathbf{N}} : \ \Phi_{x_1} \cdots \Phi_{x_n} \neq 0, \ \forall n \in \mathbf{N}\}.$$

A subshift $X \subset \mathcal{A}^{\mathbf{N}}$ is a cocyclic subshift iff $X = X_\Phi$ for some Φ.

Note that X_Φ can be empty.

Following [5], any finite sequence $\sigma \in \mathcal{A}^k$ will be referred to as *a block* (of *length* $|\sigma| := k$). In particular, given $x \in \mathcal{A}^{\mathbf{N}}$ and $k \in \mathbf{N}$, we have a block $[x]_k := (x_1, ..., x_k)$. (We will also use $[x]_{i,k} := (x_i, ..., x_{i+k-1})$.) Each block σ determines an open set $U_\sigma := \{x \in \mathcal{A}^{\mathbf{N}} : [x]_k = \sigma, \ k = |\sigma|\}$ and a product $\Phi_\sigma := \Phi_{\sigma_1} \cdots \Phi_{\sigma_k}$. We say that σ *occurs in* X_Φ iff $U_\sigma \cap X_\Phi \neq \emptyset$, and we say that σ is *allowed* (or $\Phi-$*allowed*) iff $\Phi_\sigma \neq 0$. All blocks occurring in X_Φ are allowed, but not vice versa: an allowed σ may not be a sub-block of any $x \in X_\Phi$. Nevertheless, the complement of X_Φ is the union of U_σ over all disallowed σ's; therefore, X_Φ is compact. Since $f(X_\Phi) \subset X_\Phi$, X_Φ indeed is a subshift.

As indicated in the introduction, Definition 2.1 can be recast in a more general context of cocycles. Consider $\Phi : \mathbf{N} \times \mathcal{A}^{\mathbf{N}} \to \mathrm{End}(V)$ that is *a locally constant cocycle* with values in the semigroup $\mathrm{End}(V)$. This is to say that there are $q \in \mathbf{N}$ and endomorphisms $\Phi_{i_1 \dots i_q} \in \mathrm{End}(V)$, $i_j \in \mathcal{A}$, $j = 1, ..., q$, such that

$$\Phi(n, x) = \Phi_{x_1 \dots x_q} \Phi_{x_2 \dots x_{q+1}} \cdots \Phi_{x_n \dots x_{n+q-1}}, \qquad x \in \mathcal{A}^{\mathbf{N}},\ n \in \mathbf{N}.$$

The minimal such q we call *the anticipation*[3] *of* Φ, and by *the support of* Φ we understand the set $\{x \in \mathcal{A}^{\mathbf{N}} :\ \Phi(n, x) \neq 0,\ \forall n \in \mathbf{N}\}$. In the case when $q = 1$, the support coincides with the cocyclic subshift X_Φ.

Proposition 2.1 (characterization via cocycles) *The class of cocyclic subshifts of $\mathcal{A}^{\mathbf{N}}$ coincides with that of the supports of locally constant cocycles on $\mathcal{A}^{\mathbf{N}}$ (with values in the endomorphism semigroup of a finite dimensional vector space).*

Proposition 2.1 is an immediate consequence of the following lemma.

Lemma 2.1 *If Φ is a locally constant cocycle in $\mathrm{End}(V)$, then there is a finite-dimensional linear space V' and a locally constant cocycle Φ' in $\mathrm{End}(V')$ with anticipation $q' \leq 1$ such that, for $x \in \mathcal{A}^{\mathbf{N}}$,*

$$\Phi(n, x) = 0,\ \forall n \in \mathbf{N} \iff \Phi'(n, x) = 0,\ \forall n \in \mathbf{N}. \tag{1}$$

Proof. It suffices to show that if the anticipation of Φ is $q > 1$, then Φ' satisfying (1) can be found with anticipation $q' < q$. Let $J_i :\ V \to V^m$ and $P_j :\ V^m \to V$ be the canonical injections and projections, so that $vJ_iP_j = \delta_{ij}v$ for $v \in V$ and $i, j \in \mathcal{A}$. Set, for any $i \in \mathcal{A}^{q-1}$ and $x \in \mathcal{A}^{\mathbf{N}}$,

$$\Phi'_{i_1 \dots i_{q-1}} := \sum_{k=1}^{m} P_{i_{q-1}} \Phi_{i_1 \dots i_{q-1} k} J_k \quad \text{and} \quad \Phi'(n, x) := \Phi'_{x_1 \dots x_{q-1}} \cdots \Phi'_{x_n \dots x_{n+q-2}}.$$

By using $J_i P_j = \delta_{ij}$, it is easy to see the corresponding cocycle to be

$$\begin{aligned} \Phi'(n, x) \quad = \quad & \sum_{k_1, \dots, k_n = 1}^{m} P_{x_{q-1}} \Phi_{x_1 \dots x_{q-1} k_1} J_{k_1} P_{x_q} \Phi_{x_2 \dots x_q k_2} J_{k_2} \cdots \Phi_{x_n \dots x_{n+q-2} k_n} J_{k_n} = \\ & \sum_{k_n = 1}^{m} P_{x_{q-1}} \Phi_{x_1 \dots x_q} \Phi_{x_2 \dots x_{q+1}} \cdots \Phi_{x_n \dots x_{n+q-2} k_n} J_{k_n} \end{aligned}$$

[3] c.f. [12]

The last sum contains the factor $\Phi(n-1,x)$ in each term, so $\Phi(n-1,x)=0$ for some $n>1$ implies $\Phi'(n,x)=0$. Also, $J_{x_{q-1}}\Phi'(n,x)P_{x_{n+q-1}}=\Phi(n,x)$ so that $\Phi'(n,x)=0$ implies $\Phi(n,x)=0$. The equivalence (1) follows. □

We finish this section with a couple of remarks. It may be convenient at times to talk about possibly infinite blocks $\sigma=(\sigma_i)_{i=a}^{i=b}$, where $a,b\in \mathbf{Z}\cup\{-\infty,\infty\}$, $a\le b$. Note that, even though Φ_σ may be undefined, one can unambiguously define the kernel $\ker(\Phi_\sigma)$ if a is finite, the image $\mathrm{im}(\Phi_\sigma)$ if b is finite, and non-vanishing of Φ_σ in any case.

Also, rather than X_Φ, it is often more suitable to consider *the two-sided cocyclic subshift* consisting of all bi-infinite ($-a=b=\infty$) allowed blocks,

$$\tilde{X}_\Phi := \{(x_i)_{i\in\mathbf{Z}}:\ \Phi_{x_n}\cdots\Phi_{x_m}\neq 0,\ n<m,\ n,m\in\mathbf{Z}\}.$$

Translation between $\tilde{X}_\Phi$ and X_Φ is standard: one views $\tilde{X}_\Phi$ as the natural extension of X_Φ by identifying each $(x_i)_{i\in\mathbf{Z}}\in\tilde{X}_\Phi$ with the corresponding *full orbit* $(a_n)_{n\in\mathbf{Z}}$ in X_Φ, $a_{n+1}=fa_n$; the two are related via $x_i:=[a_i]_1$, $i\in\mathbf{Z}$ ([5]).

3 Preliminary properties.

In order to establish cocyclic subshifts as a natural class of dynamical systems, we shall verify that cocyclicity of a subshift $X\subset\mathcal{A}^{\mathbf{N}}$ is an intrinsic property of its shift dynamics $f:X\to X$ and that it is preserved under the basic set theoretic operations.

Recall that a subshift X is *conjugate* to another subshift Y (possibly over a different finite alphabet $\tilde{\mathcal{A}}$) iff there is a homeomorphism $h:Y\to X$ such that $h\circ\tilde{f}=f\circ h$ where f and $\tilde{f}$ are the shift maps on X and Y, respectively.

Theorem 3.1 (conjugacy invariance) *A subshift conjugate to a cocyclic subshift is a cocyclic subshift.*

Before a proof, recall that any subshift $X\subset\mathcal{A}^{\mathbf{N}}$ determines for $r\in\mathbf{N}$ a subshift $X^{[r]}=\{([x]_{i,r})_{i\in\mathbf{N}}:\ x\in X\}$ over the refined alphabet $\mathcal{A}^r$, (recall $[x]_{i,r}=(x_i,...,x_{i+r-1})$). This $X^{[r]}$, so called *r-block presentation of X*, is conjugate to X via the map $\gamma_{\mathcal{A}}^{[r]}:\mathcal{A}^{\mathbf{N}}\to(\mathcal{A}^r)^{\mathbf{N}}$ given by $(x_i)_{i\in\mathbf{N}}\mapsto([x]_{i,r})_{i\in\mathbf{N}}$ (see [12]).

Lemma 3.1 *If $X \subset \mathcal{A}^{\mathbf{N}}$ is a cocyclic subshift, then so is its r-block presentation $X^{[r]} \subset (\mathcal{A}^r)^{\mathbf{N}}$ for $r \in \mathbf{N}$.*

Proof. Suppose that $X = X_\Phi$ for some $\Phi \in \mathrm{End}(V)^m$. Consider the cocycle $\Psi : \mathbf{N} \times (\mathcal{A}^r)^{\mathbf{N}} \to \mathrm{End}(V)$ given by

$$\Psi(1, ((x_{1,1}, ..., x_{1,r}), (x_{2,1}, ..., x_{2,r}), ...)) := \begin{cases} \Phi_{x_{1,1}} & \text{if } x_{1,2} = x_{2,1}, ..., x_{1,r} = x_{2,r-1}, \\ 0 \text{ otherwise.} \end{cases}$$

Roughly, Ψ is Φ on the image of $\gamma_{\mathcal{A}}^{[r]}$ and zero on the complement, (where *the progressive overlap condition*, see [12], is violated). The anticipation of Ψ does not exceed two by definition. It is also easy to verify that $X^{[r]}$ is the support of Ψ, which makes $X^{[r]}$ a cocyclic subshift. □

Proof of Theorem 3.1. Suppose a subshift $Y \subset \tilde{\mathcal{A}}^{\mathbf{N}}$ is conjugate to a cocyclic subshift $X \subset \mathcal{A}$ via $h : Y \to X$. Denote by $B_X^{[r]}$ the set $\{[x]_r : x \in X\}$ of all r-blocks occurring in X, with the analogous definition for Y. It is well known that the conjugacy h and its inverse h^{-1} are *sliding block codes* ([12]), meaning that there are $r, s \in \mathbf{N}$ and maps $\lambda : B_Y^{[s]} \to B_X^{[r]}$ and $\mu : B_X^{[r]} \to B_Y^{[1]}$ such that, for $y \in Y$ and $x \in X$,

$$x = h(y) \iff [x]_{i,r} = \lambda([y]_{i,s}), \forall i \in \mathbf{N} \iff y_i = \mu([x]_{i,r}), \forall i \in \mathbf{N}.$$

The maps λ and μ on the symbols induce $\lambda^\infty : (B_Y^{[s]})^{\mathbf{N}} \to (B_X^{[r]})^{\mathbf{N}}$ and $\mu^\infty : (B_X^{[r]})^{\mathbf{N}} \to (B_Y^{[1]})^{\mathbf{N}}$. From $h^{-1} \circ h = \mathrm{Id}$, $\mu \circ \lambda(y_1, ..., y_s) = y_1$ for any $(y_1, ..., y_s) \in B_Y^{[s]}$. It follows that, $\mu^\infty \circ \lambda^\infty \circ \gamma_{\tilde{\mathcal{A}}}^{[s]}(y) = y$ for any $y \in \tilde{\mathcal{A}}^{\mathbf{N}}$ such that $[y]_{i,s} \in B_Y^{[s]}$ for all $i \in \mathbf{N}$.

By Lemma 3.1, there are V and $\Phi \in \mathrm{End}(V)^{m^r}$ such that $X^{[r]} = X_\Phi \subset (\mathcal{A}^r)^{\mathbf{N}}$. We shall prove that Y is the support of the cocycle $\Psi : \mathbf{N} \times \tilde{\mathcal{A}}^{\mathbf{N}} \to \mathrm{End}(V)$ given by

$$\Psi(1, y) := \begin{cases} \Phi(1, \lambda([y]_s)) \text{ if } [y]_s \in B_Y^{[s]}, \\ 0 \text{ otherwise.} \end{cases}$$

Assume that $y \in \tilde{\mathcal{A}}^{\mathbf{N}}$ is such that $\Psi(n, y) \neq 0$ for all $n \in \mathbf{N}$. Then $[y]_{i,s} \in B_Y^{[s]}$ for all $i \in \mathbf{N}$, i.e. $\gamma_{\tilde{\mathcal{A}}}^{[s]}(y) \in (B_Y^{[s]})^{\mathbf{N}}$. Moreover, $\Psi(n, y) = \Phi(n, x)$ for $x := \lambda^\infty \circ \gamma_{\tilde{\mathcal{A}}}^{[s]}(y) \in (B_X^{[r]})^{\mathbf{N}}$ and all $n \in \mathbf{N}$. From the assumption $x \in X^{[r]}$, and so $\mu^\infty(x) \in Y$. Since $\mu^\infty(x) = \mu^\infty \circ \lambda^\infty \circ \gamma_{\tilde{\mathcal{A}}}^{[s]}(y) = y$, we have proven that $y \in Y$.

On the other hand, given $y \in Y$, we have $[y]_{i,s} \in B_Y^{[s]}$ for all $i \in \mathbf{N}$, so that $\Psi(n,y) = \Phi(n,x) \neq 0$ for $x = \lambda^\infty \circ \gamma_{\tilde{\mathcal{A}}}^{[s]}(y)$ and all $n \in \mathbf{N}$. □

Out of the multitude of possible algebraic operations on cocycles, we summon the direct sum and the tensor product to observe the following:

Fact 3.1 *The sum, intersection, and Cartesian product of two cocyclic subshifts are cocyclic subshifts.*

Proof. Let $X_\Phi \subset \mathcal{A}^{\mathbf{N}}$ and $X_{\tilde{\Phi}} \subset \tilde{\mathcal{A}}^{\mathbf{N}}$ be cocyclic subshifts, $\Phi \in \mathrm{End}(V)^m$ and $\tilde{\Phi} \in \mathrm{End}(\tilde{V})^{\tilde{m}}$.

We claim that $X_\Phi \cup X_{\tilde{\Phi}} = X_{\Phi \oplus \tilde{\Phi}}$ where we assume that $\mathcal{A} = \tilde{\mathcal{A}}$ and the cocycle $\Phi \oplus \tilde{\Phi} \in \mathrm{End}(V \oplus \tilde{V})^m$ is given by $(v \oplus \tilde{v})(\Phi \oplus \tilde{\Phi})(n,x) = v\Phi(n,x) \oplus \tilde{v}\tilde{\Phi}(n,x)$ for $x \in \mathcal{A}^{\mathbf{N}}$, $n \in \mathbf{N}$. The simple reason is that $a \oplus b = 0$ iff $a = 0$ and $b = 0$.

On the other hand, $X_\Phi \times X_{\tilde{\Phi}} \subset (\mathcal{A} \times \tilde{\mathcal{A}})^{\mathbf{N}}$ coincides with $X_{\Phi \otimes \tilde{\Phi}}$ where the cocycle $\Phi \otimes \tilde{\Phi} \in \mathrm{End}(V \otimes \tilde{V})^{m\tilde{m}}$ is given on simple tensors by $(v \otimes \tilde{v})(\Phi \otimes \tilde{\Phi})(n,(x,y)) = v\Phi(n,x) \otimes \tilde{v}\tilde{\Phi}(n,y)$ for $(x,y) \in \mathcal{A}^{\mathbf{N}} \times \tilde{\mathcal{A}}^{\mathbf{N}}$, $n \in \mathbf{N}$. This hinges on the fact that $a \otimes b = 0$ iff $a = 0$ or $b = 0$.

Finally if $\mathcal{A} = \tilde{\mathcal{A}}$, to get $X_\Phi \cap X_{\tilde{\Phi}}$ as a cocyclic subshift one can use $\Phi \otimes \tilde{\Phi}$ restricted to the diagonal in $\mathcal{A}^N \times \mathcal{A}^N$. By abusing notation we still write for it $\Phi \otimes \tilde{\Phi} \in \mathrm{End}(V \otimes \tilde{V})^m$ but now $(v \otimes \tilde{v})(\Phi \otimes \tilde{\Phi})(n,x) = v\Phi(n,x) \otimes \tilde{v}\tilde{\Phi}(n,x)$ for $x \in \mathcal{A}^N$, $n \in \mathbf{N}$. □

Another useful property is that cocyclic subshifts are closed under taking powers and roots (of the shift map f). Recall, for a subshift $X \subset \mathcal{A}^{\mathbf{N}}$ and $l \in \mathbf{N}$, the map $\pi_{\mathcal{A}}^{(l)} : (x)_{i\in\mathbf{N}} \mapsto ([x]_{(k-1)l+1,l})_{k\in\mathbf{N}}$ conjugates $f^l : X \to X$ to what is called a *power subshift* $X^{(l)} \subset (\mathcal{A}^l)^{\mathbf{N}}$.

Proposition 3.1 (powers) *Suppose that $l \in \mathbf{N}$ and $X \subset \mathcal{A}^{\mathbf{N}}$ is a subshift. Then X is cocyclic iff $X^{(l)} \subset (\mathcal{A}^l)^{\mathbf{N}}$ is cocyclic.*

Proof. One implication is simple. If $X = X_\Phi$, then tautologically $X^{(l)} = X_{\Phi^{(l)}}$ where $\Phi^{(l)} \in \mathrm{End}(V)^{m^l}$ is *the power cocycle*, $\Phi_\sigma^{(l)} := \Phi_\sigma$ for $\sigma \in \mathcal{A}^l$.

For the opposite implication, let $\Phi : \mathbf{N} \times (\mathcal{A}^l)^{\mathbf{N}} \to \mathrm{End}(V)$ be a cocycle realizing $X^{(l)}$ as its support. Set $\tilde{V} := \bigoplus_{i=0}^{l-1} \bigotimes_{j=0}^{l-1} V_{i,j}$ where $V_{i,j}$'s are disjoint copies of V. The indexing is considered cyclic modulo l. For $i = 0, ..., l-1$, $k \in \mathbf{N}$, and $y \in \mathcal{A}^{\mathbf{N}}$, let $\phi_i(k,y) \in \mathrm{End}(V)$ be given by

$$\phi_i(k,y) := \begin{cases} \Phi(k, \pi^{(l)}_{\mathcal{A}}(y)) & \text{if } i = 0, \\ \mathrm{Id} & \text{otherwise.} \end{cases}$$

Consider the cocycle $\Phi' : \mathbf{N} \times \mathcal{A}^{\mathbf{N}} \to \mathrm{End}(\tilde{V})$ that is given on simple tensors by[4]

$$\left(\bigoplus_{i=0}^{l-1} \bigotimes_{j=0}^{l-1} v_{i,j} \right) \Phi'(1,y) := \bigoplus_{i=0}^{l-1} \bigotimes_{j=0}^{l-1} v_{i+1,j} \phi_{i+j \pmod l}(1,y).$$

It is a routine calculation to verify that

$$\left(\bigoplus_{i=0}^{l-1} \bigotimes_{j=0}^{l-1} v_{i,j} \right) \Phi'(n,y) =$$
$$\bigoplus_{i=0}^{l-1} \bigotimes_{j=0}^{l-1} v_{i+n,j} \phi_{i+j+n-1}(1,y) ... \phi_{i+j+n-q}(1, f^{q-1}y) ... \phi_{i+j}(1, f^{n-1}y).$$

Thus, for $n = kl$, $k \in \mathbf{N}$, we have exactly k non-trivial ϕ's in the product above (when $i + j \equiv q \pmod l$) so that

$$\left(\bigoplus_{i=0}^{l-1} \bigotimes_{j=0}^{l-1} v_{i,j} \right) \Phi'(kl,y) = \bigoplus_{i=0}^{l-1} \bigotimes_{j=0}^{l-1} v_{i,j} \phi_0(k, f^{i+j-1 \pmod l} y). \tag{2}$$

Now, if $y \in X$ then also $f^{i+j-1 \pmod l} y \in X$, so that $\phi_0(k, f^{i+j-1 \pmod l} y) \neq 0$ for all i,j; and consequently, $\Phi'(kl,y) \neq 0$. Hence, X is contained in the support of Φ'.

On the other hand, if $y \notin X$ then $\phi_0(k,y) = \Phi(k, \pi^l_{\mathcal{A}}(y)) = 0$ for some $k \in \mathbf{N}$. It follows that $\Phi'(kl,y) = 0$, because, for each i, we have a tensor factor $\phi_0(k, f^{i+j-1 \pmod l} y) = \phi_0(k,y) = 0$ for $j = 1 - i \bmod l$. Hence, the support of Φ' is contained in X. □

[4]The idea is to *suspend* Φ by twisting the cyclic permutation of $\bigoplus_{i=0}^{l-1} V_{i,j}$ with Φ acting on the $-j^{\text{th}}$ place. Also, the tensor product would be superfluous if not for the possibility that $x \notin X$ but $f^k x \in X$ for some $x \in \mathcal{A}^{\mathbf{N}}$ and $k \in \mathbf{N}$.

4 Irreducibility and Topological Transitivity.

We start in this section our main theme of correlating the structure of X_Φ with the algebraic properties of the cocycle Φ.

The set of all blocks can be thought of as a free semigroup with the concatenation as multiplication. (For $\sigma = (\sigma_1, ..., \sigma_n)$ and $\eta = (\eta_1, ..., \eta_m)$, their concatenation is $\sigma\eta = (\sigma_1, ..., \sigma_n, \eta_1, ..., \eta_m)$.) The *semigroup of* $\Phi \subset \mathrm{End}(V)^m$ is, by definition,

$$\mathcal{G}_\Phi := \{\Phi_\sigma : \ \sigma \text{ is a block}\}$$

treated as a sub-semigroup of $\mathrm{End}(V)$ generated by the components of Φ. The map $\sigma \mapsto \Phi_\sigma$ is a homomorphism between the two semigroups.

Less structured (and more penetrable) is *the algebra of* Φ, by definition equal to the linear span of $\mathcal{G}_\Phi$ in $\mathrm{End}(V)$,

$$\mathcal{E}_\Phi := \left\{ \sum_\sigma a_\sigma \Phi_\sigma : \ a_\sigma \in \mathbf{C} \text{ almost all zero} \right\}.$$

The algebra $\mathcal{E}_\Phi$ acts on V on the right, which is a finite dimensional faithful representation. A particularly nice situation arises if this representation is irreducible, that is $v\mathcal{E}_\Phi = V$ for any non-zero $v \in V$. Existence of such faithful representation (primitivity) is equivalent to $\mathcal{E}_\Phi$ being simple (no proper bi-ideals exist and $\mathcal{E}_\Phi^2 \neq 0$). The Wedderburn-Artin theory (p 421 in [8] or [6]) asserts that a simple algebra is the full endomorphism algebra over a division ring, which means that $\mathcal{E}_\Phi = \mathrm{End}(V)$ because the field is algebraically closed.

Definition 4.1 *A cocycle $\Phi \in \mathrm{End}(V)^m$ is irreducible iff $V \neq 0$ and $\mathcal{E}_\Phi = \mathrm{End}(V)$. A cocyclic subshift is irreducible iff it can be represented as X_Φ for some irreducible Φ.*

The definition differs from the one in [21] where simplicity of the semigroup (not the algebra) is postulated[5]. This will ultimately allow for more

[5] The two are not equivalent: $\mathbf{Z}$ is not simple as a semigroup but has a primitive representation as multiplication in $\mathbf{C}$ (with $\mathcal{E} = \mathbf{C}$ simple), and the cyclic multiplicative semigroup $\mathbf{Z}_p$ is simple but its faithful complex representations are all diagonizable (p. 443, [1]).

complete description of the dynamics in terms of the underlying algebra. Recall that X_Φ is *topologically transitive* iff the orbit $\{f^n x\}_{n\in\mathbf{N}}$ is dense in X_Φ for some $x \in X_\Phi$. Also, if a block σ has non-nilpotent Φ_σ, then the infinite concatenation $x = \sigma^\infty$ (i.e. $x_i := \sigma_{i \bmod |\sigma|}$) is a periodic point in X_Φ; the period is equal to $|\sigma|$ iff σ is not a power (i.e. $\sigma = \eta^l$ implies $l = 1$). All *periodic points* in X_Φ arise in this way, and we will denote their union by $\mathrm{Per}(X_\Phi)$.

Theorem 4.1 (transitivity) *If Φ is irreducible, then X_Φ is non-empty, topologically transitive, and the set of periodic points $\mathrm{Per}(X_\Phi)$ is dense in X_Φ.*

The following frequently invoked lemma uncovers the mechanism behind the theorem.

Lemma 4.1 (connecting) *Suppose that Φ is irreducible. If σ and ω are two allowed blocks, that is $\Phi_\sigma, \Phi_\omega \neq 0$, then there is a block β for which $\Phi_{\sigma\beta\omega} \neq 0$. Moreover, such β exists with*

$$|\beta| \leq \max\{0, \dim(\ker(\Phi_\omega)) - \dim(\mathrm{im}(\Phi_\sigma)) + 1\} \leq d = \dim(V).$$

Note one easy corollary: all allowed blocks occur in X_Φ for irreducible Φ.

Proof of Lemma 4.1. Let $\mathbf{b}$ be a finite set of blocks such that $(\Phi_\beta)_{\beta\in\mathbf{b}}$ forms a basis of $\mathcal{E}_\Phi = \mathrm{End}(V)$ as a linear space over $\mathbf{C}$. There is $C \in \mathrm{End}(V)$ such that $\Phi_\sigma C \Phi_\omega \neq 0$, and so one must have $\Phi_\sigma\Phi_\beta\Phi_\omega \neq 0$ for some $\beta \in \mathbf{b}$. Note that $|\beta|$ has an upper bound that is uniform in σ and ω — a fact that suffices for much of what follows in this paper. The "moreover part" needs another argument though.

Suppose that $\Phi_{\sigma\omega} = 0$, as otherwise there is nothing to prove, and set $k := \min\{|\eta| : \Phi_\sigma\Phi_\eta\Phi_\omega \neq 0, \eta \text{ a block}\}$, $k \geq 1$. Consider $\eta = (i_k, ..., i_1)$ with $\Phi_\sigma\Phi_\eta\Phi_\omega \neq 0$. Set $V_1 := \ker(\Phi_\omega)$. Observe that

$$(i)_1 \quad \mathrm{im}(\Phi_\sigma\Phi_{i_k}...\Phi_{i_1}) \not\subset V_1,$$

and, by minimality of k, we have

$$(ii)_1 \quad \mathrm{im}(\Phi_\sigma\Phi_{i_k}...\Phi_{i_l}) \subset V_1, \; 1 < l \leq k,$$

since otherwise $\Phi_\sigma\Phi_{i_k}...\Phi_{i_l}\Phi_\omega \neq 0$. It follows that $V_1\Phi_{i_1} \not\subset V_1$, that is $V_2 := \{v \in V_1 : \ v\Phi_{i_1} \in V_1\}$ is properly contained in V_1. A similar argument as for V_1 yields

$$(i)_2 \ \ \mathrm{im}(\Phi_\sigma\Phi_{i_k}...\Phi_{i_2}) \not\subset V_2,$$

and, by minimality of k, we have

$$(ii)_2 \ \ \mathrm{im}(\Phi_\sigma\Phi_{i_k}...\Phi_{i_l}) \subset V_2, \ 2 < l \leq k.$$

Again, $V_3 := \{v \in V_2 : \ v\Phi_{i_2} \in V_2\}$ must be strictly contained in V_2. By iterating this process, we get a strictly descending sequence of linear spaces $V_1 \supset V_2 \supset ... \supset V_k$ all of which contain $\mathrm{im}(\Phi_\sigma)$. It follows that $k - 1 \leq \dim(V_1) - \dim(\mathrm{im}(\Phi_\sigma))$, which ends the proof. □

For $x \in X$, *the eventual rank of x (with respect to Φ)* is defined as

$$q(x) := \lim_{n\to\infty} \mathrm{rank}(\Phi(n, x)).$$

Clearly, the sequence stabilizes and $\{x \in \mathcal{A}^{\mathbf{N}} : \ q(x) > 0\}$ coincides with X_Φ.

Proof of Theorem 4.1. To come up with a point $y \in X_\Phi$ which trajectory is dense in X_Φ, form a sequence including all the allowed blocks: $\omega_1, \omega_2, \dots$, $\Phi_{\omega_i} \neq 0$ for $i \in \mathbf{N}$, and then use Lemma 4.1 repeatedly to get η_i's such that $y := \omega_1\eta_1\omega_2\eta_2...$ belongs to X_Φ. (Allowed blocks exist by irreducibility, in particular $X_\Phi \neq \emptyset$.)

For density of $\mathrm{Per}(X_\Phi)$, it is enough to prove that $\mathrm{Per}(X_\Phi)$ accumulates on the point y found above. Take n arbitrary but large enough to have $\mathrm{rank}(\Phi(n, y)) = q(y)$. Because $(f^m y)_{m\in\mathbf{N}}$ fills X_Φ densely, there is m such that $\sigma := [y]_n = [f^{m+n}y]_n$ and so $[y]_{n+m+n} = \sigma\eta\sigma$ for some η. If $V_0 := \mathrm{im}(\Phi_\sigma) = V\Phi_\sigma$, then

$$\mathrm{im}(\Phi_{\sigma\eta\sigma}) = V_0\Phi_{\eta\sigma} = V_0\Phi_\eta\Phi_\sigma \subset V_0.$$

In fact, the inclusion above must be equality because all the involved spaces have dimension equal to $q(y)$. Thus $V_0\Phi_{\eta\sigma} = V_0$ and so $\Phi_{\eta\sigma}$ is not-nilpotent, which puts $(\eta\sigma)^\infty$ and $z = (\sigma\eta)^\infty$ in $\mathrm{Per}(X_\Phi)$. Since $d(z, y) \leq 2^{-n}$, we are done by arbitrariness of n. □

5 Spectral Decomposition.

Our next task is to represent the *recurrent* dynamics of a cocyclic subshift as a union of irreducible cocyclic subshifts. This is analogous to the spectral decomposition of a hyperbolic set ([17]) with an important caveat that the union need not be disjoint, and the points in its complement need not be wandering but merely *transient* in the following sense.

For a map $f : X \to X$ and $k \in \mathbf{N}$, we shall call a set $U \subset X$ k-*transient* if

$$\sup_{x \in X} \#\{n \in \mathbf{N} : f^n x \in U\} \leq k.$$

Actually, we are only interested in the case when X is a compact Hausdorff topological space, f is continuous, and U is open (so that 1-*transient* U is what normally is called *a wandering neighborhood*, [5].) By a *transient* U we mean U that is k-transient for some $k \in \mathbf{N}$, and *the transient set of* f is

$$T(f) := \bigcup\{U : U \text{ is open and transient}\}.$$

While avoiding detailed discussion, we relate $T(f)$ to the standard notions of *the non-wandering set* $\Omega(f) := (\bigcup\{U : U \text{ is open and wandering}\})^c$ and *the (positively) recurrent set* $R(f) := \text{cl}\{x \in X : x \in \omega(x)\}$ — where cl stands for the closure and $\omega(x)$ is the accumulation set of $(f^n x)_{n \in \mathbf{N}}$.

Proposition 5.1 *(i) The wandering points, $\Omega(f)^c$, are dense in $T(f)$.*
(ii) The transient points are not recurrent: $R(f) \subset T(f)^c$.

We remark that $R(f) = T(f)^c$ for cocyclic subshifts as will be apparent from Theorem 5.1.

Proof. (i) Clearly, $\Omega(f)^c \subset T(f)$. For density, we exhibit a non-empty wandering W in any non-empty transient U. As a function of $x \in X$, $k_U(x) := \#\{n \in \mathbf{N} : \ f^n x \in U\}$ is lower-semicontinuous and bounded from above. Thus, for $k_0 := \max_{x \in U} k_U(x)$, the set

$$W := \{x \in U : \ k_U(x) = k_0\}$$

is open. W is also wandering, as otherwise $x, f^n x \in W$ for some $n > 0$ so that $k_U(x) = k_U(f^n x) + 1 > k_0$ — which is a contradiction.

(ii) It suffices to verify that (with $k_U(x)$ as in the proof of (i))

$$\mathrm{cl}\left(\bigcup_{x\in X}\omega(x)\right) = \left(\bigcup\{U:\ U \text{ open and } k_U(x) < +\infty \text{ for } x\in X\}\right)^c. \tag{3}$$

If $y\in\omega(x)$ for some $x\in X$, then $k_U(x) = +\infty$ for any neighborhood U of y — the "$\subset$" inclusion follows. On the other hand, if y has an open neighborhood V disjoint with $\omega(x)$ for all $x\in X$, then $k_U(x) < +\infty$ for a neighborhood U of y that is pre-compactly contained in V. The "$\supset$" inclusion follows. □

Returning to cocyclic subshifts, transient dynamics may appear in X_Φ in the presence of nilpotent bi-ideals in $\mathcal{E}_\Phi$. If $J\subset\mathcal{E}_\Phi$ is such an ideal, that is $\mathcal{E}_\Phi J\mathcal{E}_\Phi\subset J$ and $J^t = 0$ for some $t\in\mathbf{N}$, then any block σ with $\Phi_\sigma\in J$ can repeat at most $t-1$ times in any allowed block ω. Indeed, if $\omega = \alpha_1\sigma\alpha_2\sigma...\alpha_t\sigma\alpha_{t+1}$, then $\Phi_\omega\in\Phi_{\alpha_1}J\Phi_{\alpha_2}...J\Phi_{\alpha_{t+1}}\subset J^t = 0$. We refer to such σ as *a transient block* because σ can occur at most $|\sigma|\cdot t$ times in any $x\in X_\Phi$, so that $\#\{n:\ f^n x\in U_\sigma\}\le|\sigma|\cdot t$ and U_σ is transient.

Assume that, for some non-zero linear spaces $V_1,...,V_r$, we have a homomorphism $R:\mathcal{E}_\Phi\to\prod_{i=1}^r\mathrm{End}(V_i)$ satisfying the following hypothesis

(H) the kernel J of R is nilpotent and the components $R_i:\mathcal{E}_\Phi\to\mathrm{End}(V_i)$, $i=1,...,r$, are surjective.

For $x\in\mathcal{A}^{\mathbf{N}}$ the homomorphism R determines *the partial eventual ranks*

$$q_i(x) := \lim_{n\to\infty}\mathrm{rank}(R_i(\Phi(n,x))),\quad i=1,...,r,$$

which add up to $q_+(x) := \sum_i q_i(x)$. Note that, if $q_+(x) = 0$, then there is $n\in\mathbf{N}$ such that $\Phi(x,n)\in J(\mathcal{E}_\Phi)$, which makes $[x]_n$ a transient block and any $x\in U_{[x]_n}$ a transient point.

Theorem 5.1 (spectral decomposition) *If X_Φ is a cocyclic subshift, and $R:\mathcal{E}_\Phi\to\prod_i\mathrm{End}(V_i)$ satisfies the hypothesis (H), then the sets $(X_\Phi)_i := \{x\in\mathcal{A}^{\mathbf{N}}:\ q_i(x) > 0\}$ are irreducible cocyclic subshifts for irreducible cocycles*

$$\Phi_i := (R_i(\Phi_k))_{k\in\mathcal{A}}\in\mathrm{End}(V_i)^m.$$

The union $\bigcup_{i=1}^r(X_\Phi)_i$ is a cocyclic subshift for

$$(R(\Phi_k))_{k\in\mathcal{A}}\in\mathrm{End}\left(\bigoplus_i V_i\right)$$

and equals $(X_\Phi)_+ := \{x \in \mathcal{A}^{\mathbf{N}} : q_+(x) > 0\}$*, which constitutes the set* $T(X_\Phi)^c$ *of all non-transient points of* X_Φ*.*

Proof. Checking that the cocycles determine the right subshifts is trivial. The irreducibility follows immediately from the surjectivity in (H). That all non-transient points are accounted for has already been observed. □

As noted before, ***the basic sets*** $(X_\Phi)_i$ need not be disjoint nor different, a flaw that can be remedied by passing to an appropriate cocyclic subshift that factors onto $(X_\Phi)_+$ (finite-to-one). Such is the cocyclic subshift with the alphabet $\{(i,k) : \ i = 1,...,r, \ k = 1,...,m\}$ and the cocycle given by $(R_i(\Phi_k))_{(i,k)}$, as it splits into disjoint transitive sets that are naturally conjugate to the $(X_\Phi)_i$'s. This is reminiscent of the situation for sofic systems[6] that lack spectral decomposition, but are factors of topological Markov chains that have spectral decomposition ([5]). Also, that there may be non-wandering points outside $(X_\Phi)_+$ can be seen in a sofic example[7] given by the space of sequences of 1's and 2's with at most two 1's occurring in each sequence (take Φ_1 nilpotent with $\Phi_1^2 \neq 0$ and $\Phi_1^3 = 0$, and $\Phi_2 = \mathrm{Id}$). Here $(X_\Phi)_+ = \{2^\infty\}$, yet every symbolic sequence with exactly one occurrence of 1 represents a non-wandering point (which is nevertheless 2-transient).

To supply a homomorphism R satisfying hypothesis (H) for any non-empty X_Φ, one can use the Wedderburn-Artin theory. Recall (see IX.2 in [8] or [6]) the Wedderburn (or Jacobson) radical $J(\mathcal{E}_\Phi)$ of the algebra $\mathcal{E}_\Phi$ is the union of all nilpotent two-sided ideals in $\mathcal{E}_\Phi$ and is a nilpotent two-sided ideal by itself. Thus, $J(\mathcal{E}_\Phi)^t = 0$ for some minimal $t = t_\Phi$; and $J(\mathcal{E}_\Phi) \neq \mathcal{E}_\Phi$ given that $X_\Phi \neq \emptyset$. The quotient $\mathcal{E}_\Phi / J(\mathcal{E}_\Phi)$ is then a semisimple algebra and, by the Wedderburn-Artin Theorem (Th 5.7, IX, [8]), it is isomorphic to $\prod_{i=1}^r \mathrm{End}(V_i)$ for some non-zero linear spaces V_i, $i = 1,...,r = r_\Phi$. Intrinsically, r_Φ is the number of simple ideals in $\mathcal{E}_\Phi / J(\mathcal{E}_\Phi)$ (c.f. Prop. 3.8, [8]) and $\sum_i \dim(V_i) \leq \dim(V)$ (see (5) ahead). (Irreducibility of X_Φ, which we do not assume, translates to $r_\Phi = 1$.)

In order to obtain suitable $R : \mathcal{E}_\Phi \to \prod_i \mathrm{End}(V_i)$, precompose the isomorphism with the canonical projection $\mathcal{E}_\Phi \to \mathcal{E}_\Phi / J(\mathcal{E}_\Phi)$. The collection of cocyclic subshifts $((X_\Phi)_i)_{i=1}^{r_\Phi}$ thus provided by Theorem 1 will be called ***the***

[6]Think of the union of two full shifts: one on $\{1,2\}$ and another one on $\{2,3\}$; they share 2^∞.

[7]suggested by the referee

Wedderburn decomposition of X_Φ. At this point we can record the following converse of Theorem 1 (c.f. Corollary 7.1 and Question 4 in Section 6 of [21]).

Corollary 5.1 (irreducibility) *A topologically transitive non-empty cocyclic subshift is irreducible.*

Proof. A union of compact invariant sets is topologically transitive only if it coincides with one of the sets. Hence, for the Wedderburn spectral decomposition of X_Φ, we have $X_\Phi = \bigcup_i (X_\Phi)_i = (X_\Phi)_i$ for some i, and the subshift X_Φ coincides with one of its irreducible components. □

Before leaving this section, we digress that the Wedderburn-Artin homomorphism R is not the only R satisfying hypothesis (H), but it is the *simplest* such R. Let us illuminate this point and use the opportunity to record a few useful algebraic facts.

Consider another homomorphism that satisfies (H), $\tilde{R} : \mathcal{E}_\Phi \to \prod_{j=1}^{\tilde{r}} \mathrm{End}(\tilde{V}_j)$. Because $\tilde{R}(J(\mathcal{E}_\Phi)) \subset J(\prod_{j=1}^{\tilde{r}} \mathrm{End}(\tilde{V}_j)) = 0$ (see Prop. 3.1.3 in [6]), we have $J(\mathcal{E}_\Phi) \subset \ker(\tilde{R})$. From (H), the opposite inclusion holds so that $J(\mathcal{E}_\Phi) = \ker(\tilde{R})$. Thus $\tilde{R}$ induces a monomorphism

$$\rho : \prod_{i=1}^{r} \mathrm{End}(V_i) \to \prod_{j=1}^{\tilde{r}} \mathrm{End}(\tilde{V}_j)$$

such that $\tilde{R} = \rho \circ R$, and in this sense R is *simpler* than $\tilde{R}$.

Moreover, the structure of ρ is very transparent: The component homomorphisms $\rho_{ij} : \mathrm{End}(V_i) \to \mathrm{End}(\tilde{V}_j)$ are either zero or isomorphisms because $\mathrm{End}(V)$ is simple for any non-zero V (see Schur's lemma, [6]). Additionally, if $i_1 \neq i_2$, then $\rho_{i_1 i_2 j} : \mathrm{End}(V_{i_1}) \times \mathrm{End}(V_{i_2}) \to \mathrm{End}(\tilde{V}_j)$ has a non-zero kernel (by counting dimensions). The kernel must be equal to one of the two ideals $0 \times \mathrm{End}(V_{i_2})$ or $\mathrm{End}(V_{i_1}) \times 0$, so that $\rho_{i_1 j} = 0$ or $\rho_{i_2 j} = 0$. In this way, for each j there is a unique i with $\rho_{ij} \neq 0$.[8] One immediate corollary is that

$$\sum_i \dim(V_i) \leq \sum_j \dim(\tilde{V}_j). \tag{4}$$

[8]This essentially proves a standard fact (see [6]) that $\bigoplus_j \tilde{V}_j$ is isomorphic as a module over $\mathcal{E}_\Phi$ to $\bigoplus_i k_i V_i$; here $k_i := \#\{i :\ r_{ij} \neq 0\}$.

Finally, although optimal, R may not be best suited for practical calculations: it is more convenient to deal with $\tilde{R}$ derived directly from the given representation on V. The linear space V, as a right module over $\mathcal{E}_\Phi$, has a composition series (p. 375 in [8])

$$0 = W_{\tilde{r}} \subset W_{\tilde{r}-1} \subset .... \subset W_0 = V,$$

where the quotients W_{k-1}/W_k have no proper submodules. One can construct a (non-canonical) splitting of V into linear spaces $V = \bigoplus_{j=1}^{\tilde{r}} \tilde{V}_j$ so that $W_j = \tilde{V}_j \oplus ... \oplus \tilde{V}_{\tilde{r}}$. For any map $A \in \mathcal{E}_\Phi$, the component $A^{(ij)} : \tilde{V}_i \to \tilde{V}_j$ is defined as the composition of A with the canonical injection of $\tilde{V}_i$ and the canonical projection onto $\tilde{V}_j$. The map $\tilde{R}_j : A \mapsto A^{(jj)}$ is a homomorphism. It is either zero or it is onto $\mathrm{End}(\tilde{V}_j)$ because $\tilde{V}_j \simeq W_{j-1}/W_j$, having no proper submodules, is either zero or simple (over $\tilde{R}_j(\mathcal{E}_\Phi)$). Since also $A^{(ij)}$ vanishes for $i > j$, the homomorphism $\tilde{R} := \prod_{j \in \{j : \tilde{R}_j \neq 0\}} \tilde{R}_j$ has a nilpotent kernel; $\tilde{R}$ satisfies hypothesis (H).

Moreover, because $\sum_j \dim(\tilde{V}_j) \leq \dim(V)$, inequality (4) yields

$$\sum_i \dim(V_i) \leq \dim(V). \tag{5}$$

Also, on assumption that $\rho_{ij} :\neq 0$, $\rho_{ij} : \mathrm{End}(V_i) \to \mathrm{End}(\tilde{V}_j)$ being an isomorphism implies $\mathrm{rank}(A^{(jj)}) = \mathrm{rank}_{V_i}(R_i(A))$ for any $A \in \mathcal{E}_\Phi$ (where the subscript V_i indicates that the rank is computed in the representation on V_i). It follows that

$$\sum_i \mathrm{rank}_{V_i}(R_i(A)) \leq \sum_j \mathrm{rank}(A^{(jj)}) \leq \mathrm{rank}(A). \tag{6}$$

We shall need (5) and (6) later in Sections 7 and 8.

6 Aperiodicity and Aperiodic Decomposition.

The Wedderburn-Artin decomposition can be refined so as to distinguish within each transitive basic set finer aperiodic (primitive) components that are cyclicly permuted by the dynamics. A more structured algebra than $\mathcal{E}_\Phi$ serves this purpose.

Recall (from Section 3), that for any $l \in \mathbf{N}$, the iterate $f^l : X_\Phi \to X_\Phi$ is naturally conjugate to the power subshift $X^{(l)}$, which is the cocyclic subshift $X_{\Phi^{(l)}} \subset (\mathcal{A}^l)^{\mathbf{N}}$ supporting the cocycle

$$\Phi^{(l)} := (\Phi_\sigma)_{\sigma \in \mathcal{A}^l} \subset \mathrm{End}(V)^{m^l}.$$

The corresponding algebra, denoted by $\mathcal{E}_\Phi^{(l)}$, is generated by all Φ_σ with $|\sigma|$ divisible by l. Of particular importance is *the tail algebra of* Φ given by

$$\mathcal{E}_\Phi^{(\infty)} := \bigcap_{l \in \mathbf{N}} \mathcal{E}_\Phi^{(l)}.$$

Observe that $\mathcal{E}_\Phi^{(\infty)} := \bigcap_{k \in \mathbf{N}} \mathcal{E}_\Phi^{(l_k)}$ for $l_k := k!$, and this is an intersection of a descending sequence of linear spaces so $\mathcal{E}_\Phi^{(\infty)} = \mathcal{E}_\Phi^{(s)}$ for some $s \in \mathbf{N}$. We will write s_Φ for the minimal s with this property. Of special interest is the case when $s_\Phi = 1$, i.e. $\mathcal{E}_\Phi^{(l)} = \mathcal{E}_\Phi$ for all $l \in \mathbf{N}$.

Definition 6.1 *A cocycle $\Phi \in \mathrm{End}(V)^m$ is called* aperiodic *iff $V \neq \{0\}$ and its algebra coincides with its tail algebra, that is $\mathcal{E}_\Phi = \mathcal{E}_\Phi^{(\infty)}$. A cocycle Φ is called* primitive *iff it is irreducible and aperiodic, that is $V \neq \{0\}$ and $\mathcal{E}_\Phi = \mathcal{E}_\Phi^{(\infty)} = \mathrm{End}(V)$. A cocyclic subshift is* aperiodic *iff it can be represented as X_Φ for some aperiodic Φ, and it is* primitive *if such Φ exists that is primitive.*

Note that from $\mathcal{E}_{\Phi^{(lk)}} \subset \mathcal{E}_{\Phi^{(l)}}$ for $k, l \in \mathbf{N}$, it follows that

$$\mathcal{E}_\Phi^{(\infty)} = \mathcal{E}_{\Phi^{(l)}}^{(\infty)} \subset \mathcal{E}_{\Phi^{(l)}} \subset \mathcal{E}_\Phi, \quad l \in \mathbf{N}.$$

As an immediate consequence we note the following.

Corollary 6.1 *(i) If Φ is aperiodic (primitive), then so is $\Phi^{(l)}$, $l \in \mathbf{N}$.*
(ii) If $\Phi^{(l)}$ is primitive for some $l \in \mathbf{N}$, then so is Φ.

Note that, from (i), if a subshift is primitive cocyclic then its power subshift is primitive cocyclic. The opposite implication (stronger than (ii) above) will be proven only in the next section (see Corollary 8.2).

As in Section 5, to decompose an irreducible cocyclic subshift into aperiodic pieces, we use the (surjective) homomorphism $M : \mathcal{E}_\Phi^{(\infty)} \to \prod_j \mathrm{End}(W_j)$ that induces the isomorphism of $\mathcal{E}_\Phi^{(\infty)}/J(\mathcal{E}_\Phi^{(\infty)})$ and $\prod_j \mathrm{End}(W_j)$, for some non-zero linear spaces W_j, $j = 1, ..., r^\infty_\Phi$. Here we should note that $J(\mathcal{E}_\Phi^{(\infty)}) \neq \mathcal{E}_\Phi^{(\infty)}$ because $\mathrm{Per}(X_\Phi) \neq \emptyset$: given $\sigma^\infty \in \mathrm{Per}(X_\Phi)$, $\Phi_{\sigma^{s_\Phi}} \in \mathcal{E}_\Phi^{(s_\Phi)} = \mathcal{E}_\Phi^{(\infty)}$ is non-nilpotent. Again $J(\mathcal{E}_\Phi^{(\infty)})^t = 0$ for some $t \in \mathbf{N}$, and let t^∞_Φ be the minimal such t. M satisfies then the analogue of hypothesis (H) in Section 5,

(HH) the kernel J of M is nilpotent and the components $M_j : \mathcal{E}_\Phi^{(\infty)} \to \mathrm{End}(W_j)$, $W_j \neq \{0\}$, are surjective.

Given $x \in \mathcal{A}^{\mathbf{N}}$, the appropriate partial eventual ranks are

$$q_j^\infty(x) := \lim_{n\to\infty} \mathrm{rank}(M_j(\Phi(ns, x))),\ s = s_\Phi,$$

with $q_+^\infty(x) := \sum_j q_j^\infty(x)$.

Any irreducible X_Φ is *made of* a cyclicly permuted aperiodic cocyclic subshift, as described by the following result.

Theorem 6.1 (aperiodic decomposition) *If $\Phi \subset \mathrm{End}(V)^m$ is irreducible, then there exists $q \in \mathbf{N}$, $q \leq d := \dim(V)$, such that $X_\Phi = X_0 \cup ... \cup f^{q-1}X_0$ for some $X_0 \subset X_\Phi$ that is invariant under f^q, and $f^q : X_0 \to X_0$ is naturally conjugate to a cocyclic subshift with a primitive power. In fact, if $s = s_\Phi$, so that $\mathcal{E}_\Phi^{(\infty)} = \mathcal{E}_\Phi^{(s)}$, and $M : \mathcal{E}_\Phi^{(\infty)} \to \prod_{j=1}^r \mathrm{End}(W_j)$ satisfies hypothesis (HH), then X_Φ is the union of*

$$(X_\Phi)_j^{(\infty)} := \{x \in \mathcal{A}^{\mathbf{N}} :\ q_j^\infty(x) > 0\},\ j = 1, ..., r,$$

which (acted upon by f^s) are naturally conjugate to the primitive cocyclic subshifts of $(\mathcal{A}^s)^{\mathbf{N}}$ that are given by the primitive cocycles

$$\Phi_j^{(\infty)} := M_j(\Phi^{(s)}) = (M_j(\Phi_\sigma))_{\sigma\in\mathcal{A}^s} \subset \mathrm{End}(W_j)^{m^s}.$$

The set X_0, as well as each of its iterates $fX_0, ..., f^{q-1}X_0$, can be found as one of the $(X_\Phi)_j^{(\infty)}$'s; moreover, $q \leq r \leq d$ and q divides s.

Remark 6.1 *As it will become clear later (Corollary 8.2), $f^q : X_0 \to X_0$ in the theorem is in fact a primitive cocyclic subshift, even though we show now only that it has a primitive power. To exemplify the difficulty consider $\Phi = (\Phi_1)$ with $\Phi_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ so that $r^\infty_\Phi = s_\Phi = 2$ and $X_\Phi = \{1^\infty\} = (X_\Phi)_1^{(\infty)} = (X_\Phi)_2^{(\infty)} = X_0$; primitivity of $\Phi_1^{(\infty)}$ assures only that $f^2 : X_0 \to X_0$ is primitive, not $f : X_0 \to X_0$.*

Proof of Theorem 6.1. That $(X_\Phi)_j^{(\infty)}$ arises from $\Phi_j^{(\infty)}$ is a tautology. We first show that $\Phi_j^{(\infty)}$ is primitive. Since M_j is a homomorphism, we have $\mathcal{E}_{\Phi_j^{(\infty)}} = M_j(\mathcal{E}_\Phi^{(s)})$. But $M_j(\mathcal{E}_\Phi^{(s)}) = M_j(\mathcal{E}_\Phi^{(\infty)}) = \mathrm{End}(W_j)$, which makes $\Phi_j^{(\infty)}$ irreducible. Moreover, not only $\mathcal{E}_{\Phi_j^{(\infty)}} = M_j(\mathcal{E}_\Phi^{(s)})$, but for the same reason $\mathcal{E}^{(l)}_{\Phi_j^{(\infty)}} = M_j(\mathcal{E}_\Phi^{(sl)})$, for all $l \in \mathbf{N}$. By the definition of s, the right-hand sides of the two last equalities coincide so $\mathcal{E}^{(l)}_{\Phi_j^{(\infty)}} = \mathcal{E}_{\Phi_j^{(\infty)}}$; consequently, $\mathcal{E}^{(\infty)}_{\Phi_j^{(\infty)}} = \mathcal{E}_{\Phi_j^{(\infty)}} = \mathrm{End}(W_j)$ and $\Phi_j^{(\infty)}$ is primitive.

Next, we argue that $X_\Phi = \bigcup_{j=1}^r (X_\Phi)_j^{(\infty)}$. (That $r \le d$ follows from (5) in Section 5.) Periodic orbits are dense in X_Φ (Theorem 1), so it suffices to show that $x \in \mathrm{Per}(X_\Phi)$ and $q_+^\infty(x) = 0$ implies a contradiction. Represent then x as $x = \sigma^\infty$ with $|\sigma|$ divisible by s. Since $q_+^\infty(x) = 0$, we have $M_j(\Phi_{\sigma^N}) = 0$ for some N and all j. Thus $\Phi_{\sigma^N} \in \ker(M) = J(\mathcal{E}_\Phi^{(\infty)})$ and $\Phi_{\sigma^{Nt}} \in J(\mathcal{E}_\Phi^{(\infty)})^t = 0$, $t = t_\Phi^\infty$, which contradicts $x \in X_\Phi$.

To finish the argument we will show that, upon reordering of the $(X_\Phi)_j^{(\infty)}$'s, we have $X_\Phi = \bigcup_{1\le j\le q} (X_\Phi)_j^{(\infty)}$, where $(X_\Phi)^{(\infty)}_{j+1 \bmod q} = f(X_\Phi)^{(\infty)}_{j \bmod q}$ for $j = 1, ..., q$ and some $q \le r$. Note that, for any j, there is i such that $f(X_\Phi)_j^{(\infty)} \subset (X_\Phi)_i^{(\infty)}$; indeed, take z with its orbit under f^s dense in $(X_\Phi)_j^{(\infty)}$, $fz \in (X_\Phi)_i^{(\infty)}$ determines the suitable i. Of all the $(X_\Phi)_j^{(\infty)}$'s, let $(X_\Phi)_1^{(\infty)}$, $\dots$, $(X_\Phi)_{\tilde r}^{(\infty)}$ be these maximal with respect to inclusion (after renumbering perhaps) so that still $X_\Phi = \bigcup_{1\le j\le \tilde r} (X_\Phi)_j^{(\infty)}$. These $\tilde r$ sets are permuted by f (because f^s fixes them) and the permutation decomposes into cycles of the form $(X_\Phi)_{j_1}^{(\infty)} \to (X_\Phi)_{j_2}^{(\infty)} \to \dots \to (X_\Phi)_{j_q}^{(\infty)} \to (X_\Phi)_{j_1}^{(\infty)}$, where $q \le r$, q divides s, and all the maps are onto (by the maximality). The union of the $(X_\Phi)_j^{(\infty)}$'s along such a cycle is a compact invariant subset of X_Φ. Being transitive, X_Φ must coincide with one such union, and $X_0 = (X_\Phi)_{j_1}^{(\infty)}$ satisfies then the conditions of the theorem. □

In a similar fashion to the situation in Theorem 5.1, the family of primitive pieces $(X_\Phi)_j^{(\infty)}$ may be very redundant, with some of them intersecting or even coinciding. Partly to blame is the fact that we do not *optimize* Φ for the given cocyclic subshift; however, disjointness of the primitive pieces is precluded by the very nature of the dynamics on X_Φ — it breaks down already for sofic systems[9]. Of course all these problems vanish if one is willing to take finite-to-one factors.

7 Specification and Intrinsic Ergodicity.

Our goal now is to see that primitivity of a cocyclic subshift is equivalent to its topological mixing, or to a stronger property of specification. Intrinsic ergodicity of topologically transitive cocyclic subshifts is one notable corollary.

Recall that a subshift X is *topologically mixing* iff, given two blocks σ_1 and σ_2 that occur in X, there is n_0 so that $n \geq n_0$ implies that $\sigma_1\eta\sigma_2$ occurs in X for some η with $|\eta| = n$. The specification property requires furthermore that *the gap length* n is uniform: X has *specification* if there is n_0 such that given two occurring blocks σ_1 and σ_2 and $n \geq n_0$, $\sigma_1\eta\sigma_2$ occurs in X for some η with $|\eta| = n$. This can be seen ([2]) as an equivalent formulation of the following Bowen's condition on existence (*specification*) of periodic orbits (c.f. Def. 21.1 in [5]):

(S) for some $n_0 \in \mathbf{N}$, given a finite sequence of occurring blocks $\sigma_1, ..., \sigma_k$ and numbers $l_i \geq n_0$, $i = 1, ..., k$, there are connecting blocks η_i, $|\eta_i| = l_i$, such that $(\sigma_1\eta_1\sigma_2\eta_2...\sigma_k\eta_k)^\infty \in \mathrm{Per}(X_\Phi)$

We postpone the proof of the following well known fact.

Fact 7.1 *For a subshift $X \subset \mathcal{A}^{\mathbf{N}}$, if its power subshift $X^{(c)} \subset (\mathcal{A}^c)^N$ has specification for some $c \in \mathbf{N}$, then X has specification.*

Theorem 7.1 (specification) *A primitive cocyclic subshift has specification.*

Specification guarantees for a subshift good statistical properties, particularly *intrinsic ergodicity*: by the theorem due to Bowen (Th. 22.15 in [5]), (S) implies existence of a unique probability measure μ of maximal entropy. If X_Φ is not primitive but merely transitive the maximal entropy measure still exists; it is the average of the measures on the primitive components $X_0, ..., f^{q-1}X_0$ provided by Theorem 6.1. Thus we can note the following important corollary.

[9] Consider a graph with two vertices a, b and edges ab labeled 0, ab labeled 1 and ba labeled 1. The sequences starting with even number of 1's form a primitive piece which shares 1^∞ with its image under f.

Corollary 7.1 *A transitive cocyclic subshift is intrinsically ergodic.*

A construction of the maximal measure via an appropriate *transfer operator* can be found in [11].

Proof of Theorem 7.1. Let $X = X_\Phi \subset \mathcal{A}^{\mathbf{N}}$ for a primitive cocycle $\Phi \in \mathrm{End}(V)^m$. The argument is similar to that for density of periodic points in the proof of Theorem 4.1. Let

$$q_0 := \min\{\mathrm{rank}(\Phi_\sigma) : \ \sigma \text{ is an allowed block, i.e. } \Phi_\sigma \neq 0\},$$

and let η be a fixed block with $\mathrm{rank}(\Phi_\eta) = q_0$. Set $V_0 := \mathrm{im}(\Phi_\eta)$. The role of minimality of q_0 is embodied by the following implication: if ν is a block with $\Phi_{\eta\nu\eta} \neq 0$, then $\mathrm{rank}(\Phi_{\eta\nu\eta}) = q_0$ and $V_0\Phi_{\nu\eta} = V_0$. By irreducibility such ν exists (Lemma 4.1); choose one and set $\gamma := \nu\eta$, $c := |\gamma|$.

In view of Fact 7.1, it suffices to prove specification for $X^{(c)}$. List all blocks occurring in X with length divisible by c: $\sigma_1, \sigma_2, \ldots$. For each $k \in \mathbf{N}$, due to irreducibility of $X_{\Phi^{(c)}}$ (from Corollary 5.1), one can use Lemma 4.1 to find blocks α_k and β_k such that $\Phi_{\gamma\alpha_k\sigma_k\beta_k\gamma} \neq 0$ and $dc \geq |\alpha_k|, |\beta_k| \in c\mathbf{N}$. As anticipated, the minimality of q_0 assures that $V_0\Phi_{\mu_k} = V_0$ for $\mu_k := \alpha_k\sigma_k\beta_k\gamma$, as well as $V_0\Phi_\gamma = V_0$. For any two blocks σ_k and σ_j, and for $l \geq 0$, we have $V_0\Phi_{\mu_k\gamma^l\mu_j} = V_0$ so that $\sigma_k\beta_k\gamma^{l+1}\alpha_j\sigma_j$ occurs in X. In this way, we can connect σ_k with σ_j with any *gap length* n exceeding $2dc + c$. Hence, $X^{(c)}$ has specification and so does X by Fact 7.1. □

Even though cocyclic subshifts are generally not uniformly hyperbolic, let us observe that *horseshoes* are still the mechanism responsible for generating all of the topological entropy (which is reminiscent of Katok's theorem for $C^{1+\epsilon}$-diffeomorphisms of surfaces [9]). This hinges on the existence of a *synchronizing* word, as the γ in the proof of Theorem 7.1 above.

Theorem 7.2 (horseshoes) *Suppose that X_Φ is a cocyclic subshift. For any $\epsilon > 0$, there exist $N, n \in \mathbf{N}$ such that $N \geq \exp(n(h(f) - \epsilon))$ and $f^n : X_\Phi \to X_\Phi$ has an embedded full N-shift; namely, $\{\mu_{i_1}\mu_{i_2}\ldots : \ i_j \in \{1, \ldots, N\}, \ j \in \mathbf{N}\} \subset X_\Phi$ for some pairwise different blocks $\{\mu_i\}_{i=1}^N$ of length n.*

Proof. It suffices to argue in the primitive case since the full entropy must be carried on one of the mixing pieces $(X_\Phi)_j^{(\infty)}$ provided by Theorem 6.1.

Assume then that γ, $c = |\gamma|$, is a (synchronizing) block as in the beginning of proof of Theorem 7.1. From the definition of topological entropy (via separated sets, see [5]) conclude that, for arbitrarily large $n_0 \in c\mathbf{N}$, there are $N \geq \exp(n_0(h(f) - \epsilon/2))$ different blocks $\sigma_1, ..., \sigma_N$ of length n_0 with $\Phi_{\sigma_i} \neq 0$, $i = 1, ..., N$. Set $l = 2dc + c$. As before we can get blocks μ_i of the form $\mu_i := \alpha_i \sigma_i \beta_i \gamma^{d_i}$, $d_i \in \mathbf{N}$, such that $V_0 \Phi_{\mu_i} = V_0$ and $|\mu_i|$ are all equal to $n := n_0 + l$. These blocks can be freely concatenated: if $x \in \mathcal{A}^{\mathcal{V}}$ is concatenated from elements of the set $\{\mu_i\}_{i \in \mathbf{N}}$, then $V_0 \Phi_x \neq 0$ so that $x \in X_\Phi$. Since l is independent of n_0, we have $N \geq \exp(n(h(f) - \epsilon))$ provided n_0 is large enough. □

We append the proof of Fact 7.1 for completeness.

Proof of Fact 7.1. The blocks occurring in $X^{(c)}$ correspond to the blocks of X with length divisible by c. Thus specification for $X^{(c)}$ means that there is $l_0 \in \mathbf{N}$ such that if σ_1 and σ_2 with c dividing $|\sigma_i|$, $i = 1, 2$ occur in X and $l \geq l_0$, then $\sigma_1 \eta \sigma_2$ occurs in X for some η with $|\eta| = lc$.

Suppose that $n \geq l_0 c$ and μ_1 and μ_2 occur in X. Write $n = lc + r$ with $l \geq 0$ and $0 \leq r < c$. There are blocks ϵ_1, ϵ_2, and δ with $|\delta| = r$, such that $\sigma_1 := \epsilon_1 \mu_1 \delta$ and $\sigma_2 := \mu_2 \epsilon_2$ occur in X and c divides $|\sigma_i|$, $i = 1, 2$. (To find ϵ_1, use $f(X) = X$ — which follows from transitivity.)

Now, $\sigma_1 \eta \sigma_2 = \epsilon_1 \mu_1 \delta \eta \mu_2 \epsilon_2$ occurs in X for some η with $|\eta| = lc$ by specification for $X^{(c)}$. Thus $\mu_1 \gamma \mu_2$ occurs in X, for $\gamma = \delta \eta$ and $|\gamma| = n$. □

8 Primitivity from Mixing, and Irreducibility from Transitivity.

A primitive cocyclic subshift is mixing by Theorem 7.1. We set out to show the opposite implication, which complements the already proven fact that a transitive cocyclic subshift is irreducible (Corollary 5.1). In fact, we shall see that, under suitable assumptions on a cocycle, transitivity and mixing of a cocyclic subshift force, correspondingly, irreducibility and primitivity of the cocycle.

It is instrumental to consider together with a cocycle $\Phi \in \mathrm{End}(V)^m$ its exterior powers $\Phi^{\wedge r} = (\Phi_i^{\wedge r})_{i=1}^m \in \mathrm{End}(V^{\wedge r})^m$, where $V^{\wedge r}$ is the linear space of antisymmetric tensors of degree r on V and $\Phi_i^{\wedge r}$ is the map induced on tensors by Φ_i, $i = 1, ..., m$. Since, for $A \in \mathrm{End}(V)$, $\mathrm{rank}(A) \geq r$ iff $\mathrm{rank}(A^{\wedge r}) \geq 1$,

X_Φ stratifies into

$$\mathrm{X}_{\Phi^{\wedge r}} = \{x \in X_\Phi : \ \mathrm{rank}(\Phi_{[x]_n}) \geq r, \ \text{for all} \ \ n \in \mathbf{N}\}, \quad r = 1, \ldots, \dim(V).$$

In particular, if r_0 is *the minimal rank of* Φ, by definition equal to

$$r_0 = \min \mathrm{rank}(\Phi) := \min\{\mathrm{rank}(\Phi_\sigma) : \ \Phi_\sigma \neq 0\},$$

then

$$X_\Phi = \mathrm{X}_{\Phi^{\wedge r_0}}$$

and the minimal rank of $\Phi^{\wedge r_0}$ equals 1.

Proposition 8.1 (rank reduction) *If X is a cocyclic subshift, then $X = X_\Psi$ for some Ψ of minimal rank 1. Moreover, if X is irreducible, then such Ψ exists that is irreducible.*

Proof. The first assertion follows by passing to the r_0^{th} exterior power, as explained above. For the moreover part, we may already assume then that $X = X_\Phi$ for Φ with minimal rank 1. By Theorem 4.1, X_Φ is topologically transitive. In the Wedderburn-Artin decomposition of X_Φ given by Theorem 5.1, X_Φ is equal then to some (every) basic set $(X_\Phi)_i$ (c.f. the proof of Corollary 5.1). Since, $\mathrm{rank}_{V_i}(R_i(A)) \leq \mathrm{rank}_V(A)$ for any $A \in \mathcal{E}_\Phi$ (by (6) in Section 5), the minimal rank of $R_i(\Phi)$ does not exceed that of Φ — so it equals 1, and $\Psi = R_i(\Phi)$ is the desired cocycle. □

Here is one advantage of reducing the minimal rank to one:

Theorem 8.1 *If $\Phi \in \mathrm{End}(V)^m$ is such that $\mathcal{E}_\Phi$ has no radical, i.e $J(\mathcal{E}_\Phi) = \{0\}$, and Φ has minimal rank 1, then*
(i) if X_Φ is transitive, then Φ is irreducible;
(ii) if X_Φ is mixing, then Φ is primitive.

We should note that $J(\mathcal{E}_\Phi) = \{0\}$ for any irreducible Φ. In fact, $J(\mathcal{E}_\Phi) = \{0\}$ means that $\mathcal{E}_\Phi$ is semisimple so that Φ is a direct sum of irreducible cocycles.

Corollary 8.1 (primitivity) *A non-empty mixing cocyclic subshift is primitive.*

Proof of Corollary 8.1. If X is a mixing cocyclic subshift, then it is transitive and so it is irreducible by Corollary 5.1. The cocycle Ψ provided by Proposition 8.1 satisfies then the hypothesis of Theorem 8.1 and so Ψ is the desired primitive cocycle with $X = X_\Psi$. □

Since mixing is preserved under taking roots, Proposition 3.1 and Corollary 8.1 yield the following corollary, which shows that ultimately the X_0 in Theorem 6.1 is primitive (c.f. Remark 6.1).

Corollary 8.2 *If X is a subshift and its power $X^{(l)}$ is a primitive cocyclic subshift for some $l \in \mathbf{N}$, then X is also a primitive cocyclic subshift.*

Proof of Theorem 8.1.
(i): As in the proof of Corollary 5.1 or Proposition 8.1, X_Φ coincides with its every irreducible component, $X_\Phi = (X_\Phi)_i$, $i = 1, \ldots, r_\Phi$. Since the inequality (6) in Section 5 implies that

$$\sum_{i=1}^{r_\Phi} \min \operatorname{rank}(R_i(\Phi)) \leq \min \operatorname{rank}(\Phi),$$

we must have $r_\Phi = 1$, which means that $\mathcal{E}_\Phi = \mathcal{E}_\Phi / J(\mathcal{E}_\Phi) = \operatorname{End}(V_i)$ for $i = 1 = r_\Phi$, i.e. Φ is irreducible.
(ii): First note that if X_Φ is mixing then it is topologically transitive under any power of f, which makes X_Φ equal to X_0 in the aperiodic decomposition given by Theorem 6.1. Thus X_Φ has a primitive power, and so X_Φ has specification by Theorem 7.1 and Fact 7.1.

By the already proven (i), Φ is irreducible. We have to show that Φ is primitive, i.e. that $\mathcal{E}_\Phi^{(l)} = \operatorname{End}(V)$ for all $l \in \mathbf{N}$, which is equivalent to $v\mathcal{E}_\Phi^{(l)} = V$ for any non-zero $v \in V$ (c.f. the beginning of Section 4). Fix then $v \in V \setminus \{0\}$ and consider the subspaces

$$W^{(l)} := \operatorname{lin}\{v\Phi_\sigma : \ \operatorname{rank}(\Phi_\sigma) = 1, \ l \text{ divides } |\sigma|\} \subset v\mathcal{E}_\Phi^{(l)}, \quad l \in \mathbf{N}.$$

We note that $W^{(l)}$'s are invariant, $W^{(l)}\mathcal{E}_\Phi^{(l)} \subset W^{(l)}$; and we claim that $W^{(l)} = V$, $l \in \mathbf{N}$. For $l = 1$, $\mathcal{E}_\Phi = \operatorname{End}(V)$ from irreducibility, and $W^{(1)} = V$ by the invariance because $W^{(1)} \neq \{0\}$. For $l > 1$ we show that $W^{(l)} = W^{(1)}$. Fix a block σ with $\operatorname{rank}(\Phi_\sigma) = 1$ and suppose that $u := v\Phi_\sigma \neq 0$. Specification supplies a block η such that $\Phi_{\sigma\eta\sigma} \neq 0$ and l divides $|\sigma\eta\sigma|$. Hence $v\Phi_{\sigma\eta\sigma} = u\Phi_{\eta\sigma} = c \cdot u$ for some non-zero scalar c, which proves that $u \in W^{(l)}$. By arbitrariness of σ, $W^{(l)} = W^{(1)} = V$; and $v\mathcal{E}_\Phi^{(l)} = V$ follows. □

9 Zeta Function and the Zero Entropy Case.

Sections 5 and 6 give the following picture of a general cocyclic subshift.

Corollary 9.1 *For* $\Phi \in \mathrm{End}(V)^m$*, the non-transient set* $(X_\Phi)_+$ *of the cocyclic subshift* X_Φ *is a union of at most* $\dim(V)$ *sets, each invariant under some positive iterate of the shift map and conjugate to a primitive cocyclic subshift.*

The proof amounts to superimposing Theorem 6.1 onto Theorem 5.1: by recognizing the simple components of the algebra $\mathcal{E}_\Phi$, pass to a number of irreducible cocycles, and then further split each of these into primitive cocycles according to the simple components of its tail algebra. The total dimension of the representation spaces for all the cocycles involved in each step does not exceed $d = \dim(V)$ (c.f. (5) in Section 5) — thus the estimate.

A primitive cocyclic subshift satisfies specification (Theorem 7.1) and so it has positive topological entropy unless it is just one point (Prop. 21.6 in [5]). This yields the following complement to Theorem 7.2.

Corollary 9.2 (zero entropy) *A non-empty cocyclic subshift* X_Φ *has zero topological entropy iff its primitive pieces are single points; that is when the non-transient set consists of at most* $\dim(V)$ *periodic points.*

Sharpness of the estimate is confirmed by a trivial example.

Example. Take V with a basis $(e_1, ..., e_d)$ and rank-one $\Phi_i : V \to V$ with $e_i \mapsto e_{i+1}$ (mod d), $i = 1, ..., d$. Then $\mathrm{Per}(X_\Phi)$ is readily seen to be a single periodic orbit of period d. (Also $\mathcal{E}_\Phi = \mathrm{End}(V)$, while $\mathcal{E}_\Phi^{(\infty)} \simeq \mathbf{C}^d$.)

In applying the Conley index methods to proving chaos ([15, 20, 18, 3]), the issue of recognizing whether X_Φ has positive entropy becomes particularly important because then a power of X_Φ (by Theorem 7.2) factors onto the full two-shift[10], and so does the original dynamical system by the algebraic topology of the Conley index (see [3]). In view of our structure theory, the problem is completely resolved through inspection of the semisimple algebras $\mathcal{E}_\Phi/J(\mathcal{E}_\Phi)$ and $\mathcal{E}_\Phi^{(\infty)}/J(\mathcal{E}_\Phi^{(\infty)})$.

[10] i.e. a continuous $h : X_\Phi \to \{1,2\}^{\mathbf{N}}$ exists such that $h \circ f^k = f_2 \circ h$ for some $k \in \mathbf{N}$, where f_2 is the shift map on $\{1,2\}^{\mathbf{N}}$

Corollary 9.3 *A cocyclic subshift $f : X_\Phi \to X_\Phi$ is chaotic, i.e. f has positive entropy and f^k continuously factors for some k onto the full two-shift iff X_Φ has a primitive component that is not a point.*

Proof. If all primitive components are points then we are in the situation of Corollary 9.2 and the entropy is zero which precludes existence of the factor map. If one of the primitive components is not a point then specification implies positive entropy, and the factor map exists via Theorem 7.2. □

With some additional work, Corollary 9.3 leads to efficient numerical algorithms. Without slowing down to discuss the details (relegated to the appendix), we turn to a sufficient condition for chaos in X_Φ readily verifiable by inspecting Φ. The way leads through a certain *zeta function*, an approach that we developed for the proof of a conjecture due to K. Mischaikow and M. Mrozek. In our language, the conjecture reads:

For some power of a cocyclic subshift X_Φ to factor onto the full shift $\{0,1\}^{\mathbf{N}}$, it suffices that either of the two hypotheses below is satisfied

$$\sum_{i=1}^{m} \operatorname{rank}^\infty(\Phi_i) > \operatorname{rank}^\infty\left(\sum_{i=1}^{m} \Phi_i\right),$$

$$\sum_{i=1}^{m} \operatorname{rank}^\infty(\Phi_i) = \operatorname{rank}^\infty\left(\sum_{i=1}^{m} \Phi_i\right) - 1,$$

where $\operatorname{rank}^\infty(A) := \lim_{n\to\infty} \operatorname{rank}(A^n)$.

Based on a different approach, special cases were established by A. Szymczak who, arguing under the first hypothesis only, required that $m = 2$, $\operatorname{rank}^\infty(\Phi_0) = 1$, and $\operatorname{rank}^\infty(\Phi_0 + \Phi_1) = 0$ ([20]). In a subsequent refinement, M. Carbinatto allowed for $\operatorname{rank}^\infty(\Phi_0) > 1$ (private communication). Observe that, in view of Theorem 7.2, the conjecture addresses exactly the problem of verifying positive topological entropy on X_Φ.

With a periodic orbit $P \subset \operatorname{Per}(X_\Phi)$ associate a rational function

$$\zeta_P(z) := \det(I - z^{p(x)}\Phi_{[x]_{p(x)}})^{-1},\ z \in \overline{\mathbf{C}},$$

where $x \in P$ and $p(x)$ is the period (and recall that $[x]_{p(x)} = (x_1, ..., x_{p(x)})$). This is the restriction to the diagonal in $\overline{\mathbf{C}}^m$ of a more natural[11] function

$$\zeta_P(z_1, ..., z_m) := \det(I - z_1^{p_1(x)} \cdot ... \cdot z_m^{p_m(x)} \Phi_{[x]_{p(x)}})^{-1}, \quad z_1, ..., z_m \in \overline{\mathbf{C}},$$

where $x \in P$ and $p_i(x)$ is the number of times i occurs in the block $[x]_{p(x)}$ so that $p_1(x) + ... + p_m(x) = p(x)$. Note that, the definitions do not depend on $x \in P$ because $\det(I - AB) = \det(I - BA)$ for any matrices A, B. Also, we include the exponent -1 to stress the analogy with the classical zeta function — although working with polynomials, not their reciprocals, is usually more convenient.

The arrangement of the periodic orbits in X_Φ is to some extent governed by an explicit function

$$\zeta_\Phi(z_1, ..., z_m) := \det(I - z_1\Phi_1 - ... - z_m\Phi_m)^{-1}.$$

Theorem 9.1 (zeta function) *For a cocyclic subshift $X_\Phi \subset \{1, ..., m\}^{\mathbf{N}}$,*

$$\zeta_\Phi(z_1, ..., z_m) = \prod_{P \subset \mathrm{Per}(X_\Phi)} \zeta_P(z_1, ..., z_m) \tag{7}$$

where the product is taken over all periodic orbits P and converges absolutely for $(z_1, ..., z_m)$ in a neighborhood of the origin in $\overline{\mathbf{C}}^m$.

Proof of Theorem 9.1. This is a version of the standard zeta function trick. We carry out only the formal calculation leaving the convergence as a simple exercise. Also, no generality is lost in assuming that $z_i = 1$. For any $A \in \mathrm{End}(V)$,

$$\sum_{k=1}^{\infty} \mathrm{trace}(A^k) z^k / k = -\ln \det(I - zA). \tag{8}$$

Hence,

$$\sum_{n=1}^{\infty} \sum_{\sigma \in \mathcal{A}^n} \mathrm{trace}(\Phi_{\sigma_1} ... \Phi_{\sigma_n})/n = \sum_{n=1}^{\infty} \mathrm{trace}((\Phi_1 + ... + \Phi_m)^n)/n,$$

[11] reflecting the fact that the projective action of the cocycle solely determines its supporting subshift.

which is ln(the left side of (7)). On the other hand, the above sum can be calculated over periodic points to give

$$\sum_{x\in\mathrm{Per}(X_\Phi)}\sum_{k=1}^{\infty}\mathrm{trace}\left(\Phi^k_{[x]_{p(x)}}\right)/(kp(x)) = \sum_{x\in\mathrm{Per}(X_\Phi)}\frac{1}{p(x)}\sum_{k=1}^{\infty}\mathrm{trace}\left(\Phi^k_{[x]_{p(x)}}\right)/k,$$

which is ln(the right side of (7)). □

Proof of the conjecture. Observe that $\mathrm{rank}^\infty(A)$ is the degree of $\det(I - zA)$ as a polynomial in z for any $A \in \mathrm{End}(V)$. Due to Theorem 7.2, it suffices to prove non-vanishing of topological entropy on X_Φ. Suppose the entropy is zero. By Corollary 9.2, $\mathrm{Per}(X_\Phi)$ is finite and the reciprocals of both sides in the formula (7) are polynomials in $z = z_1 = ... = z_m$. Each fixed point $x = i^\infty$, $i = 1, ..., m$, contributes to the product the characteristic polynomial $\det(1 - z\Phi_i)$, which implies that $\sum_i \mathrm{rank}^\infty(\Phi_i) \le \mathrm{rank}^\infty(\Phi_1 + ... + \Phi_m)$ and contradicts the first hypothesis. If the inequality above is strict, this is due to some $x \in \mathrm{Per}(X_\Phi)$ with the period $p(x) > 1$. The periodic orbit P of x contributes a polynomial factor ζ_P^{-1} of degree at least $p(x)$. Thus the discrepancy between the two sides must be at least two; the second hypothesis is contradicted. □

10 Sofic and Non-Sofic Cocyclic Subshifts.

As indicated in the introduction, cocyclic subshifts include sofic systems. This can be seen in at least two ways: algebraic (Theorem 10.1) and graph theoretic (Theorem 10.2). The main purpose of this section is to point out that the inclusion is proper and to give a concrete example of an interesting cocyclic subshift.

Recall that a *sofic system*, as introduced to ergodic theory by [21], is a subshift $X_{\mathcal{G}}$ of the full shift on $\mathcal{A}^{\mathbf{N}}$, $\mathcal{A} = \{1, ..., m\}$, where $\mathcal{G}$ is a finite semigroup with a fixed set of generators $\{g_1, ..., g_m\}$ and $(x_i)_{i=1}^{\infty} \in X_{\mathcal{G}}$ iff $g_{x_1} \cdots g_{x_n} \neq 0$ for all $n \in \mathbf{N}$. Sofic systems and their applications have a considerable amount of literature devoted to them — consult [4, 10, 12] and the references therein (see also Section 12).

Theorem 10.1 *(i) Every sofic system is a cocyclic subshift.*
(ii) There exists a cocyclic subshift that is not sofic.

A simple sufficient condition for X_Φ to be sofic is positivity of the cocycle.

Theorem 10.2 *If $\Phi = (\Phi_i)_{i\in\mathcal{A}}$ where Φ_i's are represented by matrices with non-negative entries, then X_Φ is a sofic system. Any sofic system arises in this way.*

We will prove Theorem 10.1 now and Theorem 10.2 in the next section.

Proof of Theorem 10.1, part (i). This amounts to the standard task of representing $\mathcal{G}$ by linear transformations. Append the unity to $\mathcal{G}$ if necessary to get a semigroup with unity $\tilde{\mathcal{G}}$. Take for the linear space V the semigroup algebra of $\tilde{\mathcal{G}}$, $V := \bigoplus_{g\in\tilde{\mathcal{G}}}\mathbf{C}$, and associate to each $i \in \mathcal{A}$ the linear transformation Φ_i induced on V by the right multiplication by g_i. If $\Phi = (\Phi_i)_{i\in\mathcal{A}}$, then $\Phi_{(x_1,\ldots,x_n)} = 0$ iff $g_{x_1}...g_{x_n} = 0$ — as a result $X_\Phi = X_{\mathcal{G}}$. □

For a proof of (ii) consider the following example.

Example (of a cocyclic subshift that is not sofic).
Take two copies V_1 and V_2 of $\mathbf{R}^2$ and linear maps $\Phi_{ij} : V_i \to V_j$ given by the matrices (acting on the right)

$$\Phi_{11} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\ \Phi_{12} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\ \Phi_{21} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ \Phi_{22} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}.$$

Let $V := V_1 \oplus V_2$. Set $\Phi_1 : v_1 \oplus v_2 \mapsto v_1\Phi_{11} \oplus v_1\Phi_{12}$ and $\Phi_2 : v_1 \oplus v_2 \mapsto v_2\Phi_{21} \oplus v_2\Phi_{22}$. Take $\Phi := (\Phi_1, \Phi_2)$. By definition, $(x_1, x_2, ...) \in X_\Phi$ iff $\Phi_{x_1}...\Phi_{x_n} : V \to V$ is not vanishing for all $n \in \mathbf{N}$. Observe that this is equivalent to $\Phi_{x_1x_2}...\Phi_{x_{n-1}x_n} : V_{x_1} \to V_{x_n}$ being nonzero for all $n \in \mathbf{N}$, which is why the block representation of Φ_1 and Φ_2 is so convenient, and why we can abuse notation by writing $\Phi_{x_1...x_n}$ for $\Phi_{x_1x_2}...\Phi_{x_{n-1}x_n}$.

To determine the sequences of 1's and 2's forming X_Φ, we will look then at the projective action in V_i, $i = 1, 2$. The diagram on Figure 1 conveniently encodes all the relevant data (c.f. Section 12).

Note that all the Φ_{ij}'s are nondegenerate matrices with the exception of Φ_{12}, which has the line W of slope $s = 1$, $W := \{(x, y) \in \mathbf{R}^2 : \ x = y\}$, for both its kernel and its image. The action of Φ_{11} and Φ_{22} on the slope $s := y/x$ is given by $\phi_{11}(s) = s/2$ and $\phi_{22}(s) = s + 3$ respectively. For $\Phi_{2^n1^m}$, it is $\phi_{2^n1^m}(s) = (s+3(n-1))/2^{(m-1)}$, $m, n \in \mathbf{N}$. Hence, we have $\Phi_{12^n1^m2} = 0$, if

$$1 + 3(n - 1) = 2^{m-1}, \tag{9}$$

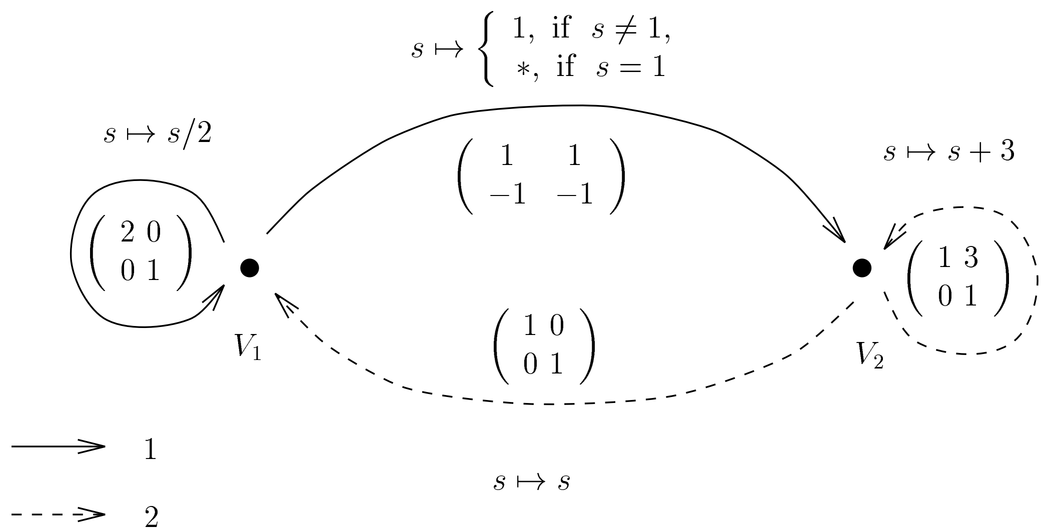

Figure 1: Graph with propagation of a nonsofic cocyclic subshift. A path in the graph determines a sequence of 1's and 2's, which is in X_Φ iff the corresponding product of matrices over the edges is non-zero (equivalently, s does not get mapped to $*$).

and otherwise $\phi_{12^n1^m2}(s) = 1$ for all slopes s. It follows that, for $\alpha = 2^{n_1}1^{m_1}2^{n_2}1^{m_2}\ldots$, Φ_α does not vanish iff $1 + 3(n_i - 1) \neq 2^{m_i - 1}$, $i = 2, 3, 4\ldots$. Note that no restrictions are put on n_1 and m_1, as the slope did not stabilize to $s = 1$ at the outset. Accounting for the sequences terminating with 1^∞ or 2^∞ and those starting with 1 leads to the following formula for X_Φ:

$$X_\Phi = \{2^{n_1}1^{m_1}2^{n_2}1^{m_2}\ldots :\ m_1, n_1 \in \mathbf{N} \cup \{0, \infty\},\ \text{and } m_i, n_i \in \mathbf{N} \cup \{\infty\},\ \text{with } 1 + 3(n_i - 1) \neq 2^{m_i - 1} \text{ for } i = 2, 3, 4, \ldots\}.$$

As a side remark, let us indicate that X_Φ is primitive. A simple calculation with *Mathematica* confirmed that the linear span of $\{\Phi_\sigma :\ |\sigma| = 4\}$ is the whole $\mathrm{End}(V)$; in particular, it contains the identity so that $\mathrm{End}(V) = \mathcal{E}_\Phi^{(4)} = \mathrm{Id}^{l/4-1}\mathcal{E}_\Phi^{(4)} \subset \mathcal{E}_\Phi^{(l)}$ for all $l \in 4\mathbf{N}$. Hence, $\mathcal{E}_\Phi = \mathcal{E}_\Phi^{(\infty)} = \mathrm{End}(V)$.

Proof of Theorem 10.1, (ii). For a block σ, the set of ω for which $\sigma\omega$ occurs in some $x \in X_\Phi$ is called *the follower set* of σ. To see that X_Φ (from the example) is not a sofic system, it is enough to establish that there are infinitely many different follower sets (see [21] or page 252 in [5]). To this end, let (m_k, n_k) for $k \in \mathbf{N}$ be different solutions to (9), say $m_k := 2k+1$ and $n_k := (2^{2k}+2)/3$. The block $1^{m_l}2^\infty$ is a follower of 12^{n_k} iff $k \neq l$. Thus the follower sets of $12^{n_1}, 12^{n_2}, \ldots$ are different from each other. □

11 Factors of Cocyclic Subshifts and Beyond.

We turn our attention briefly to factor subshifts of cocyclic subshifts and show that they do not exhaust the whole class of subshifts with specification. (In particular, a subshifts with specification need not be cocyclic.)

Recall that, given a cocyclic subshift $X_\Phi \subset \tilde{\mathcal{A}}^{\mathbf{N}}$, a map $h : X_\Phi \to \mathcal{A}^{\mathbf{N}}$ is a *factor map* if it is continuous and $h \circ \tilde{f} = f \circ h$ with $\tilde{f}$ equal to the shift on $\tilde{\mathcal{A}}^{\mathbf{N}}$. The subshift $Y := h(X_\Phi)$ is referred to as *the factor of* X_Φ (*via* h). For example, if one identifies symbols via a surjective map $\lambda : \tilde{\mathcal{A}} \to \mathcal{A}$, then $h : (x_i) \mapsto (\lambda(x_i))$ is a factor map. Actually, as observed already by Hedlund (see [5]), any factor map h has this form provided one is willing to replace X with its (conjugate) r-block presentation ${X_\Phi}^{[r]}$ for some $r \in \mathbf{N}$ (see Section 3 for definitions).

For convenient algebraization of factors of cocyclic subshifts, we abandon $\mathrm{End}(V)$ in favor of a new larger semigroup made of all linear subspaces in $\mathrm{End}(V)$.

Definition 11.1 *For a linear space V, the semigroup of linear subspaces of $\mathrm{End}(V)$, which we also call **the subspace semigroup**[12] of $\mathrm{End}(V)$, is*

$$\mathbf{End}(V) := \{W \subset \mathrm{End}(V) : \ W \text{ is a linear subspace}\}$$

with the product of W and $\tilde{W} \in \mathbf{End}(V)$ defined as

$$W \cdot \tilde{W} := \mathrm{lin}\{A\tilde{A} : \ A \in W, \ \tilde{A} \in \tilde{W}\}.$$

It is easy to see that $\mathbf{End}(V)$ is indeed a semigroup with the zero subspace $\{0\}$ serving as the zero element denoted by 0. Thus given $\mathcal{V} \in \mathbf{End}(V)^m$ we have the corresponding cocycle and the supporting it subshift is

$$X_{\mathcal{V}} := \{x \in \mathcal{A}^{\mathbf{N}} : \ \mathcal{V}_{x_1} \cdot \ldots \cdot \mathcal{V}_{x_n} \neq 0, \ \forall n \in \mathbf{N}\} \subset \mathcal{A}^{\mathbf{N}}.$$

[12] This name has been coined in [16].

Proposition 11.1 (factor) *A subshift $Y \subset \mathcal{A}^{\mathbf{N}}$ is a factor of a cocyclic subshift iff there is a finite dimensional linear space V and $\mathcal{V} \in \mathbf{End}(V)^m$ such that $Y = X_{\mathcal{V}}$.*

Proof. Suppose that $X = X_\Phi \subset \tilde{\mathcal{A}}^{\mathbf{N}}$, $\Phi \in \mathrm{End}(V)^{\tilde{m}}$, is a cocyclic subshift and $Y = h(X) \subset \mathcal{A}^{\mathbf{N}}$ is its factor via h. We may assume that h is given by a symbol identification $\lambda : \tilde{\mathcal{A}} \to \mathcal{A}$ since we can always replace X with its r-block presentation for some $r \in \mathbf{N}$, which is also cocyclic by Theorem 3.1. Setting $\mathcal{V}_j := \mathrm{lin}\{\Phi_i : \ i \in \lambda^{-1}(j)\}$, $j \in \mathcal{A}$, easily yields $Y = X_{\mathcal{V}}$.

For the opposite implication, given $\mathcal{V} \in \mathbf{End}(V)^m$, select a basis in each $\mathcal{V}_j$, so that $\mathcal{V}_j = \mathrm{lin}\{\Phi_i : \ i \in I_j\}$ where $\Phi_i \in \mathrm{End}(V)$ and I_j's are disjoint index sets, $j = 1, \ldots, m$. Then $X_{\mathcal{V}}$ is a factor of a cocyclic subshift X_Φ with $\Phi = (\Phi_i)_{i \in \bigcup_j I_j}$. □

Proposition 11.1 reveals little. Confronted with the exotic $\mathbf{End}(V)$, we are left eluded by the obvious problem:

Question 1: Are factors of cocyclic subshifts cocyclic?

From an algebraic standpoint, one may rather ask a weaker question.

Question 2: Can any finitely generated sub-semigroup $G \subset \mathbf{End}(V)$ be realized as a matrix semigroup? Precisely, is there a finite-dimensional linear space V' and a homomorphism $\phi : G \to \mathrm{End}(V')$ such that $\phi^{-1}(0) = 0$?

Such representation ϕ does not exist for $G = \mathbf{End}(V)$ as pointed out in [16] — a work sparked by our inquiry about the nature of $\mathbf{End}(V)$. Question 1 aside, the theory of cocyclic subshifts sheds some light on their factors. We mention only one such easy corollary without proof.

Corollary 11.1 *Suppose that Y is a factor of a cocyclic subshift.*
(i) If Y is topologically transitive, then Y is intrinsically ergodic.
(ii) If Y is mixing, then Y has specification.

To exhibit examples of subshifts that are not cocyclic, one can use the following result in the spirit of *the pumping lemma*, see [12].

Theorem 11.1 (pumping) *If a subshift X is a factor of a cocyclic subshift, then there exists $n_0 \in \mathbf{N}$ such that, for any finite blocks α, σ, and an infinite block β, $\sup\{n \in \mathbf{N} : \alpha\sigma^n\beta \in X\}$ is either infinite or less than n_0.*

The context free shift over the alphabet $\{0, 1, 2\}$ is defined by disallowing the blocks $01^m 2^n 0$ where $m \neq n \in \mathbf{N}$. This is a standard example of a subshift that is not sofic but has specification (see [12]).

Corollary 11.2 (non-cocyclic specification) *The context free shift is not a factor of a cocyclic subshift.*

Proof of Corollary 11.2. Otherwise take n_0 as in Theorem 11.1 and fix $m > n_0$. The sequence $01^n 2^m 0^\infty$ is disallowed for all $n > m$ so, by Theorem 11.1, it is disallowed for all $n > n_0$ and thus for $n = m$ — a contradiction. □

Theorem 11.1 depends on the following fact.

Fact 11.1 *Suppose that* $\mathcal{V} \in \mathbf{End}(V)$. *If* $n \geq n_0 := \dim(V)^2$, *then* $\mathcal{V}^n \subset \operatorname{lin}\{\mathcal{V}^k : k \geq n_1\}$ *for any* $n_1 \in \mathbf{N}$.

Proof. In $\operatorname{End}(V)$ we have a descending sequence of linear spaces $\mathcal{M}_n := \operatorname{lin} \bigcup\{\mathcal{V}^k : \ k \geq n\} \in \mathbf{End}(V)$, $n \in \mathbf{N}$. There is then $n_* \leq \dim(\operatorname{End}(V)) = \dim(V)^2$ such that $\mathcal{M}_{n_*} = \mathcal{M}_{n_*+1}$. Since $\mathcal{M}_{n+1} = \mathcal{M}_n \cdot \mathcal{V}$, $n \in \mathbf{N}$, we have $\mathcal{M}_{n_*+k} = \mathcal{M}_{n_*}\mathcal{V}^k = \mathcal{M}_{n_*}$ for all $k \in \mathbf{N}$, and the fact follows. □

Proof of Theorem 11.1. By Proposition 11.1, $X = X_{\mathcal{V}}$ for some $\mathcal{V} \in \mathbf{End}(V)^m$. Consider *the kernel of* $\mathcal{V}_\beta$, that is $K_\beta := \{A \in \operatorname{End}(V) : A\mathcal{V}_{[\beta]_n} = 0,\ \exists n \in \mathbf{N}\}$, (where $A\mathcal{V}_{[\beta]_n} := \{AW : \ W \in \mathcal{V}_{[\beta]_n}\}$). Clearly, $\alpha\sigma^n\beta \notin X$ if and only if $\mathcal{V}_\alpha \mathcal{V}_\sigma^n \subset K_\beta$. If the supremum in the theorem is finite and equal to n_1 then the inclusion holds for all $n > n_1$, and Fact 11.1 (with $\mathcal{V} = \mathcal{V}_\sigma$) guarantees the inclusion for all $n \geq n_0 := \dim(V)^2$. Thus $n_1 < n_0$. □

We see from the proof that, if X is presented in $\mathbf{End}(V)$, then one can take $n_0 = \dim(V)^2$ in Theorem 11.1. If X is cocyclic, already $n_0 = \dim(V)$ suffices by the following remark.

Remark 11.1 *In the cocyclic case, i.e. if* $\mathcal{V} = \operatorname{lin}(L)$ *for some* $L \in \operatorname{End}(V)$, *the assertion of Fact 11.1 holds for* $n_0 = \dim(V)$, *i.e.* $L^n \in \operatorname{lin}\{L^k : \ k \geq n_1\}$ *for* $n \geq n_0$ *and any* $n_1 \in \mathbf{N}$.

Proof. It suffices to consider $n_1 > n \geq \dim(V)$. Set $\tilde{V} := \operatorname{im}^\infty(L) := \bigcap_{k \in \mathbf{N}} \operatorname{im}(L^k)$. From the Jordan theorem, for $n \geq \dim(V)$, $\operatorname{rank}(L^n) = \operatorname{rank}^\infty(L) := \lim_{k\to\infty} \operatorname{rank}(L^k)$, and $\tilde{L} := L|_{\operatorname{im}^\infty(L)}$ is a self isomorphism of $\tilde{V}$. In order that $L^n \in \text{lin}\{L^k : k \geq n_1\}$, it is enough that $\tilde{L}^n \in \tilde{W} := \text{lin}\{\tilde{L}^k : \ k \geq n_1\} \subset \text{End}(\tilde{V})$ because all maps L^k for $k \geq n_0$ agree with $\tilde{L}^n$ precomposed with the projection along $\ker^\infty(L) := \bigcup_{k\in\mathbf{N}} \ker(L^k)$ onto $\tilde{V}$. Clearly, $\tilde{W}\tilde{L} \subset \tilde{W}$ (mind that the endomorphisms act on the right). However, since $\tilde{L}$ is an isomorphism, $\tilde{W}\tilde{L} = \tilde{W}$, and $\tilde{W} = \tilde{W}\tilde{L}^{-1}$. Thus $\tilde{L}^n = \tilde{L}^n \tilde{L}^{n_1-n}\tilde{L}^{-(n_1-n)} \in \tilde{W}$. □

12 Graphs with Propagation.

Another way to cast cocyclic subshifts and their factors is by generalizing the graph theoretic description of sofic systems. We value this approach as it makes working with concrete examples so much more pleasurable.

Think of the elements of the alphabet $\mathcal{A} = \{1, ..., m\}$ as encoding colors. Suppose $\mathbf{G}$ is a directed graph with colored edges: $\mathbf{V}$ is the set of vertices, $\mathbf{E}$ is the set of edges, and the colors are assigned to the edges by $l : \mathbf{E} \to \mathcal{A}$. A sequence of edges (e_i) is a path in $\mathbf{G}$ iff $e_i^+ = e_{i+1}^-$, where e^- and e^+ stand for the head and the tail of the edge e, respectively. Each finite path $a = (e_1, ..., e_n)$ determines a block $\sigma = (l(e_1), ..., l(e_n))$; we say that σ is *the coloring* of a. The *sofic system* of the labeled directed graph $\mathbf{G}$ is the subshift defined by allowing only the blocks that are colorings of some path, that is $X_{\mathbf{G}} := \{(l(e_i))_{i\in\mathbf{N}} : \ (e_i)_{i\in\mathbf{N}} \text{ a path in } \mathbf{G}\} \subset \mathcal{A}^{\mathbf{N}}$. All sofic systems arise in this way and this characterization was introduced in [7].

For an analogous description of cocyclic subshifts, one needs multiplicative matrix weights along the edges of $\mathbf{G}$. More precisely, by a *colored graph* $\mathbf{G}$ *with propagation* $\mathbf{\Gamma}$ we understand a colored directed graph $\mathbf{G}$ (as above) that has each vertex $v \in \mathbf{V}$ equipped with a linear space V_v and each edge $e \in \mathbf{E}$ equipped with a linear transformation $\Gamma_e : V_{e^-} \to V_{e^+}$; $\mathbf{\Gamma} = (\Gamma_e)_{e\in\mathbf{E}}$. Denote the pair $(\mathbf{G}, \mathbf{\Gamma})$ by $\mathbf{P}$. For a path $a = (e_1, ..., e_n)$, write $\Gamma_a := \Gamma_{e_1}...\Gamma_{e_n}$ and say that a *propagates* iff $\Gamma_a \neq 0$. By definition, a finite block of colors $\sigma = (\sigma_1, ..., \sigma_n)$ is *allowed* iff it is a coloring of some propagating path a; an infinite block is allowed if its every finite sub-block is allowed.

We claim that the set of all infinite allowed blocks, $X_{\mathbf{P}} := \{(l(e_i))_{i\in\mathbf{N}} : (e_i)_{i\in\mathbf{N}} \text{ allowed path in } \mathbf{P}\}$, is a factor of a cocyclic subshift. To see that, set $V = \bigoplus_{v\in\mathbf{V}} V_v$. Let $P_v : V \to V_v$ and $J_v : V_v \to V$ be the canonical projection and injection, respectively; and put $\Psi_e := P_{e^-}\Gamma_e J_{e^+}$ for each edge e (where as usual we compose linear maps on the right). The cocyclic subshift $X_\Psi \subset \mathbf{E}^{\mathbf{N}}$

for $\Psi := (\Psi_e)_{e\in\mathbf{E}}$ factors onto $X_{\mathbf{P}}$ under the symbol identification given by the coloring l of $\mathbf{G}$. In fact, every factor of a cocyclic subshift can be obtained as $X_{\mathbf{P}}$ for some $\mathbf{P}$.

It is an open problem (see Question 1 in Section 11) when $X_{\mathbf{P}}$ is actually cocyclic. We mention only a simple sufficient condition. A colored graph $\mathbf{G}$ is *right (left) resolving*, if no two edges with tails (heads) at the same vertex have the same color, i.e. if $e^- = \tilde{e}^-$ and $l(e) = l(\tilde{e})$ then $e = \tilde{e}$ for any $e, \tilde{e} \in \mathbf{E}(\mathbf{G})$. A colored graph with propagation $\mathbf{P} = (\mathbf{G}, \mathbf{\Gamma})$ is *right (left) resolving* iff $\mathbf{G}$ is right (left) resolving. The right and left resolving are dual notions, where the dual $\mathbf{P}^*$ of $\mathbf{P}$ is obtained by inverting all edges and replacing Γ_e's with their adjoints Γ_e^*'s. (Note that reading an allowed block of $\mathbf{P}^*$ in the reverse order gives an allowed block of $\mathbf{P}$, and vice versa.)

Proposition 12.1 *If $\mathbf{P} = (\mathbf{G}, \mathbf{\Gamma})$ is right (left) resolving, then $X_{\mathbf{P}}$ is a cocyclic subshift.*

Proof. Set $V := \bigoplus_{v\in\mathbf{V}} V_v$. To define a cocycle $\Phi \in \mathrm{End}(V)^m$, set $x\Phi_i = \sum_{e\in\mathbf{E}:\ e^-=v} x\Gamma_e$ for $v \in \mathbf{V}$ and $x \in V_v$ (naturally embedded in V), $i = 1, \ldots, m$. In the right resolving case, $X_{\mathbf{P}} = X_\Phi$ follows from the fact that $x\Phi_i = x\Gamma_e$ where e is the (only) vertex colored i with $e^- = v$, or $x\Phi_i = 0$ if such an edge does not exist. In the left resolving case, that $X_{\mathbf{P}} = X_\Phi$ is best seen via duality: the adjoint operator to Φ_i is given by $x^*\Phi_i^* = \sum_{e\in\mathbf{E}:\ e^+=v} x^*\Gamma_e^*$ for any $x^* \in V_v^*$. By left resolving, the sum has at most one non-zero term, and one can argue as in the right resolving case. □

We should stress that any cocyclic subshift $X_\Phi \subset \mathcal{A}^{\mathbf{N}}$ arises trivially from a graph with only one vertex and a loop for each Φ_i, $i \in \mathcal{A}$. Nevertheless, by choosing a more complicated graph one can gain better insight into the structure of the subshift. The diagram in Section 10 may serve as an example. Also, note that the sofic system $X_{\mathbf{G}}$ may be cast as a cocyclic subshift by associating with each vertex of $\mathbf{G}$ a copy of $\mathbf{R}$ and with every edge the identity $\mathbf{R} \to \mathbf{R}$. However, even for an irreducible aperiodic topological Markov chain, the resulting cocyclic subshift may fail to be irreducible. As an example one can take the Markov chain associated with the edge graph of the full graph over two vertices — the edges, all with different colors, are: $(1,1)$, $(1,2)$, $(2,3)$, $(2,4)$, $(3,3)$, $(3,4)$, $(4,1)$, $(4,2)$. The subshift is conjugate to the full two-shift, but $\mathcal{E}_\Phi \neq \mathrm{End}(V)$. In fact, a straightforward calculation (with the aid of *Mathematica*) confirmed that $\mathcal{E}_\Phi$ is of co-dimension 8 in $\text{End}(V)$.

Finally, we turn to the proof of Theorem 10.2.

Proof of Theorem 10.2. First we show that X_Φ is a sofic system for positive Φ by producing a colored graph $\mathbf{G}$ for which $X_\Phi = X_{\mathbf{G}}$. Let Φ_i be represented by a matrix $(a_{kl}^{(i)})_{k,l=1}^d$ with non-negative entries. Take $\{1, ..., d\}$ for vertices. For each *positive* $a_{kl}^{(i)}$, $k, l = 1, ..., d$, $i \in \mathcal{A}$, span an edge of color i from k to l with the weight $A_i = a_{kl}^{(i)}$ over it. The positivity of weights over all edges guarantees that if σ is a coloring of a path then σ is allowed and $\Phi_\sigma \neq 0$. Since the opposite implication always holds, it follows that indeed $X_\Phi = X_{\mathbf{G}}$.

For the second assertion of the theorem, invert the above construction to obtain from a colored graph (with weights defaulted to 1) a suitable positive cocycle. For a fixed color i, the corresponding matrix Φ_i is just the incidence matrix of the graph obtained from $\mathbf{G}$ by removing all the edges of color different than i. $\Box$

A Implementing Chaos Detection.

Section 9 spells out sharp criteria for chaos in X_Φ but ignores the issues of numerical implementation. Short of writing the actual code, we sketch here possible algorithms based on the dichotomy: X_Φ is either chaotic with positive entropy and has the full two-shift as a factor (of some power), or X_Φ has a zero entropy with all non-transient dynamics limited to at most $d := \dim(V)$ periodic orbits. The proposed algorithms can be integrated with Szymczak's Conley index methods for efficient chaos detection in the spirit of [3].

To start with the simplest case of irreducible $\Phi \in \text{End}(V)^m$, whether X_Φ is chaotic can be decided simply by testing if X_Φ is a single periodic orbit of period $p \leq d := \dim(V)$. Roughly, one can do the following:

Recursively construct sets $\mathcal{B}_k := \{\sigma : \sigma$ allowed and $|\sigma| = k\}$ starting with $k = 1$. If $\#\mathcal{B}_k > d$ for some k, then X_Φ is chaotic — stop; otherwise, continue to get $\mathcal{B}_d$. Now, set $p := \#\mathcal{B}_d$ (the potential period), and see if all initial p-segments of blocks in $\mathcal{B}_d$ coincide up to a cyclic permutation. If it is not so, then X_Φ is chaotic; otherwise, X_Φ has zero entropy (and we have found the only periodic orbit that constitutes X_Φ).

The case of a general $\Phi \in \text{End}(V)^m$, in principle, reduces to irreducible cases via the spectral decomposition. Yet this involves solving for eigenvectors of $d \times d$ matrices, which seldom can be done exactly — so we follow a more direct path.

To fix notation, for a block σ, let us call $p \in \mathbf{N}$ a *period of* σ iff, for some block α with $|\alpha| = p$, σ is an initial sub-block of α^∞, i.e $\sigma = (\alpha_1^\infty, ..., \alpha_{|\sigma|}^\infty)$. For the minimal such period we write $p(\sigma)$. Clearly, $p(\sigma) \leq |\sigma|$; and note the usual uniqueness property of $p(\sigma)$: if $\sigma = \alpha^l$ with $|\alpha| = p(\sigma)$ and $\sigma = \beta^k$, then $\beta = \alpha^m$ for some m.

Theorem A.1 (chaos detection) *For $\Phi \in \mathrm{End}(V)^m$, $d := \dim(V)$, X_Φ has zero entropy iff any non-transient allowed block σ of length $d^2 + 1$ has minimal period $p(\sigma) \leq d$. Moreover, then there are at most d such blocks.*

Recall from Section 5 that σ is called non-transient iff $\Phi_\sigma \notin J$ where J is the Jacobson radical of $\mathrm{End}(V)$. This, in fact, can be decided without determining J and at a modest cost of d multiplications in the subspace semigroup $\mathbf{End}(V)$ (c.f. Definition 11.1):

Fact A.1 *A block σ is non-transient iff $\mathcal{W}^d \neq 0$ for $\mathcal{W} := \Phi_\sigma \mathcal{E}_\Phi \in \mathbf{End}(V)$.*

Before we give proofs, let us note that Theorem A.1 (coupled with Fact A.1) can be implemented as a finite calculation:
Compute recursively $\mathcal{B}_k := \{\sigma : \ \sigma$ allowed and non-transient, $|\sigma| = k\}$ starting with $k = 1$; weed out transient blocks at each stage via Fact A.1. If $\#\mathcal{B}_k > d$ for some k, then X_Φ is chaotic — stop; otherwise, continue to get $\mathcal{B}_{d^2+1}$. Finally, check whether $p(\sigma) \leq d$ for each $\sigma \in \mathcal{B}_{d^2+1}$. If yes, X_Φ has zero entropy; if not, X_Φ is chaotic.

The algorithm would require a polynomial (in d) number of matrix multiplications; however, exact arithmetic of evaluating Φ_σ may bare exponential cost even for integer cocycles. That the algorithm is correct we again leave to the reader.

Proof of Fact A.1. If σ is transient, i.e. $\Phi_\sigma \in J$, then $\mathcal{W} \subset J\mathcal{E}_\Phi \subset J$ so that $\mathcal{W}^d \subset J^d = 0$. On the other hand, if σ is non-transient then Φ_σ has a non-zero irreducible component $R_i(\Phi_\sigma)$ in the Wedderburn-Artin spectral decomposition (Theorem 5.1) and σ can be extended to $x \in (X_\Phi)_i$ via Lemma 4.1, so that $\sigma = [x]_n$ for $n := |\sigma|$. By approximating x with a periodic point (Theorem 5.1), we get $(\sigma\alpha)^\infty \in X_\Phi$ for some α. Hence, $(\Phi_\sigma \Phi_\alpha)^d \neq 0$, and so $\mathcal{W}^d \neq 0$. □

Lemma A.1 *For $d \in \mathbf{N}$ and $x \in \mathcal{A}^{\mathbf{N}}$, if every sub-block σ of x with length $|\sigma| = d^2 + 1$ has its period $p(\sigma) \leq d$, then x is periodic (with period $p \leq d$).*

Proof of Lemma A.1. Set $\sigma_n := (x_n, ..., x_{n+d^2})$ and $p_n := p(\sigma_n)$. Let α_n, $|\alpha_n| = p(\sigma_n)$, be such that σ_n is the initial segment of α_n^∞. It suffices to see that, for $n \in \mathbf{N}$, $p_{n+1} = p_n$ and that $\alpha_{n+1} = \overline{\alpha_n}$, where $\overline{\alpha_n}$ is the cyclic shift of α_n by one place to the left. The block $\mu := (x_{n+1}, ..., x_{n+p_n p_{n+1}})$ is a sub-block of both σ_n and σ_{n+1} because $p_n, p_{n+1} \leq d$. Thus $\mu = \overline{\alpha_n}^{p_{n+1}} = \alpha_{n+1}^{p_n}$, and the uniqueness property of the minimal period implies that $p_{n+1} = p_n$ and that $\alpha_{n+1} = \overline{\alpha_n}$. □

Proof of Theorem A.1. If X_Φ has zero entropy, then by Corollary 9.2 the non-transient set $(X_\Phi)_+$ of X_Φ consists of at most d periodic points. The assertion on non-transient blocks follows as they can occur as sub-blocks of non-transient points.

In the other direction, if every non-transient block σ of length $d^2 + 1$ has $p(\sigma) \leq d$, then every non-transient point must be periodic of period not exceeding d by Lemma A.1. Hence, $(X_\Phi)_+$ is finite and thus carries no entropy. □

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