ANNULUS AND BILLIARD MAPS WITH ASYMPTOTIC ENTROPY

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ABSTRACT

In this paper we derive some properties of a new geometric-topological invariant defined in Swanson[14] and Kwapisz-Swanson[11] that measures the rotational complexity of annulus maps, called asymptotic or rotational entropy. We completely characterize this concept for circle maps of degree one as a linear function of the diameter of the rotation interval. For annulus maps, the authors have proved in prior work[11] that Hölder C^1 diffeomorphisms with nonvanishing asymptotic entropy are isotopic rel a finite set to pseudo-Anosov maps. We provide a brief description of the proof of that result. Finally, in this paper, we prove that nonvanishing asymptotic entropy implies the existence of infinitely many periodic orbits and a nontrivial rotation set. We apply the concept to billiard maps to prove integrable billiard problems have continuous rotation numbers, and, more significantly, rotation discontinuities force generalized rotary horseshoes in billiard maps.

1. Introduction, definitions and main results

Smale horseshoes on the annulus may have trivial rotation sets, despite their complicated dynamics. Horseshoes which give rise to rotation intervals of positive length are called rotary horseshoes. When rotation numbers of periodic orbits vary discontinuously in a precise sense, one can infer the existence of pseudo-Anosov orbits: the original annulus map is pseudo-Anosov rel a finite invariant set (Handel[9]). Asymptotic entropy, defined below, measures this kind of rotational complexity very well. In the second section, devoted to circle maps, we derive a formula for asymptotic entropy in terms of the diameter of the rotation set. This is as it should be. For circle maps, the rotation interval

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completely describes the rotational behavior and is decisive for the existence of rotary horseshoes.

On the annulus, the authors have proved[11] that positive asymptotic entropy implies rotational discontinuities for sufficiently smooth diffeomorphisms. Homeomorphisms of the annulus that are pseudo-Anosov rel a finite set have a nontrivial rotation interval relative to any boundary component. That result is an important motivation for the study of this kind of entropy. It follows that smooth annulus maps with positive asymptotic entropy have infinitely many periodic orbits corresponding to an interval of rotation numbers. The assumption of smoothness may be unnecessary, but we cannot prove this yet.

However, for the homeomorphism case, we can prove in Section 3 that nonvanishing rotational entropy implies a nontrivial rotation interval and, hence, infinitely many periodic points, as in Section 3.

In the last section, we relate these results to the problem of characterizing nonintegrability in billiard maps. We show that in the integrable case, evidently, rotation numbers are well defined and vary continuously. If the rotation is ill-defined or discontinuous — under the generic expectation that periodic points are dense — then we can conclude the existence of pseudo-Anosov orbits.

1.1. The Definition of Asymptotic Entropy

Let $\mathbb{A}$ be the annulus $S \times [0,1]$ i.e. the quotient of $\widetilde{\mathbb{A}} = \mathbb{R} \times [0,1]$ under the action of $\mathbb{Z}$ by integer translation along $\mathbb{R}$. We will refer to the quotient map as $\pi : \widetilde{\mathbb{A}} \to \mathbb{A}$. On the strip $\widetilde{\mathbb{A}}$ we have the Euclidean metric $\tilde{d}$. There is a unique corresponding metric d on $\mathbb{A}$.

From now on it will be assumed that we are given a homeomorphism $f : \mathbb{A} \to \mathbb{A}$, with f isotopic to the identity transformation. We also fix a lift $F : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ of f. This lift determines the *displacement function* of F, $\phi_F : \mathbb{A} \to \mathbb{R}^2$, obtained as the factor of $(j \circ F - j \circ id) : \tilde{\mathbb{A}} \to \mathbb{R}^2$ where j is the embedding of $\tilde{\mathbb{A}}$ into $\mathbb{R}^2$.

Definition 1.1 *For $n \in \mathbb{N}$. Define $\tilde{d}_n$ a metric on $\tilde{\mathbb{A}}$ by the formula:*

$$\tilde{d}_n(x,y) := \max\{\tilde{d}(F^i(x), F^i(y)) : 0 \leq i \leq n-1\}.$$

We will say that a set $E \subset \widetilde{\mathbb{A}}$ is $(\tilde{d}_n, R)$-*separated* if the distance between any two points in E is at least R with respect to $\tilde{d}_n$ metric.

Definition 1.2 *For a subset X of $\mathbb{A}$ and $R > 0$ we define :*

$$\tilde{s}_X(d_n, R) = max\ \#\ of\ (\tilde{d}_n, R) - separated\ subset\ of\ \pi^{-1}(X) \cap [0,1]^2;$$

and the 'R-entropy':

$$\tilde{h}_X(R) := \limsup_{n\to\infty} \frac{1}{n}\log(\tilde{s}_X(d_n, R));$$

Definition 1.3 *For a subset X of $\mathbb{A}$ topological entropy h_X^{top} is defined as*

$$h_X^{top} = \lim_{R\to 0} h_X(R)$$

and asymptotic entropy *(called also* rotational entropy*) as*

$$h_X^{as} = \limsup_{R\to\infty} R\cdot \tilde{h}_X(R).$$

Remarks 1.4 *The reader may be accustomed to defining h_X^{top} directly in the base space $\mathbb{A}$, but the two formulations are easily seen to be equivalent.*

Asymptotic entropy is independent of the choice of lift F and of fundamental domain. The reader may prefer to define either kind of entropy using minimal spanning sets, which is equivalent for topological entropy but may differ for asymptotic entropy. However, the spanning version of asymptotic entropy is positive if and only if the separated version is positive, the exact value being of less importance.

What kind of invariant is asymptotic entropy? Technically, it is a geometric invariant, as it depends on the choice of lift metric. If we fix the metric in the lift, however, then rotational entropy is a true topological invariant like topological entropy.

In the next section, we discuss some easy results about asymptotic entropy for circle maps.

1.2. Asymptotic Entropy for Circle Maps

We want to prove the following:

Proposition 1.5 *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ denote a circle map of degree 1. Fix a lift F of f. The asymptotic entropy of f is exactly*

$$h^{as}(F) = \mathit{Diam}\,\rho(F)\cdot\log 2.$$

This means that for circle maps asymptotic entropy is indistinguishable from knowledge of the rotation set diameter. Rotation sets for circle maps are always closed intervals[2] I. Thus, asymptotic entropy reduces to a well understood concept for circles. In particular, positive asymptotic entropy implies the existence of "rotary horseshoes".

We will need the additivity property, which holds for circle or annulus maps:

Lemma 1.6 $h_X^{as}(F^m) = m \cdot h_X^{as}(F)$.

Proof:

Clearly any (F^m, n, R)-separated pair x, y is (F, mn, R)-separated. So it is easily verified that the "$\leq$" part of the inequality is established. By uniform continuity, there exists a constant C_m depending only on m and the choice of metric such that for sufficiently large R,

$$\tilde{d}(x, y) \leq R \Rightarrow \tilde{d}_m(x, y) \leq R + C_m.$$

Hence, if $E(F, n, R)$ denotes a set of maximal cardinality comprised of (n, R)-separated points in the same fundamental domain, we have

$$\operatorname{card} E(F, mn, R + C_m) \leq \operatorname{card} E(F^m, n, R).$$

Therefore,

$$\tilde{h}(F, R + C_m) \leq \frac{1}{m} \tilde{h}(F^m, R),$$

which completes the proof of the lemma. □

Now we return to the proof of the proposition. Since the subject is extremely well known, we provide only a sketch of the ideas involved.

Proof: Suppose $\rho(F) = [\alpha, \beta] \supset [p/q, r/q]$. There exist disjoint intervals $I_0 = [x_0, y_0]$ and $I_1 = [x_1, y_1]$ in $[0, 1]$ such that the lift $G := T^{-p} F^q$ of f^q fixes x_0 in I_0 and $F(y_1) = y_1 + (r - p)$. In fact, one can choose I_0 and I_1 so that their images under G cover $I_0 \cup I_1 + (r - p)$. Points of I_0 do not "go around the circle". Points of I_1 go around essentially $r - p$ times. This defines a rotary horseshoe $f^q|_\Lambda$.

It is not hard to compute the asymptotic entropy of such a rotary horseshoe:

$$h_\Lambda^{as} = h_\Lambda^{as}(G) = (r - p) \cdot \log 2$$

We can illustrate the former in the special case, $r = 1, p = 0, q = 1$. Points in I_0 advance 0 units in $\mathbb{R}$, while points of I_1 move 1 unit. The shift based on the Markov rectangles consisting of $\pi(I_i)$, $i = 0, 1$ is the full two-sided two shift. Fix a value of $R > 0$, and write $n = kR + m$, $0 \leq m < R$. Then a maximal (n, R) separated set E is given by lifts of words of length kR which are a union of words of length R consisting entirely of 1's or 0's. The remainder m is bounded and makes no contribution to the exponential growth rate. The cardinality of E is 2^k. The R-asymptotic entropy is $\tilde{h}(R) = \limsup_{n \to \infty} (k/n) \cdot \log 2 = (1/R) \cdot \log 2$. The additivity lemma above allows us to divide by q, if that is the common denominator of the endpoints. The reverse inequality is established below in Lemma 2.1. □

2. Prior Results: Hölder diffeomorphisms and Pseudo-Anosov Behavior

Let $F : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ be a lift of an orientation preserving and boundary preserving homeomorphism $f : \mathbb{A} \to \mathbb{A}$. Let X denote a compact subset of $\mathbb{A}$. We begin by finding an upper bound for $h^{as}_X(f)$ in terms of the rotation set $\rho_X(F)$. This bound is proved in Kwapisz-Swanson[11], but it will be convenient to give the proof here also.

Lemma 2.1 $h^{as}_X(f) \leq DIAM(\rho_X(F)) \cdot \log 2$.

Proof: Let $\{F^n(x) = x(n)\}$ denote a positive trajectory of the the lift F, and pr_1 is projection onto the first coordinate. Choose $\rho^+, \rho^- \in \mathbb{R}$ so that $\rho(f, X) \subset (\rho^-, \rho^+)$. Note that we get, for sufficiently high n :

$$(*) \qquad \rho^- \leq pr_1(x(n) - x(0))/n \leq \rho^+.$$

Therefore, there is a constant $C > 0$ such that for all $n \geq 1$

$$(**) \qquad \rho^- - C/n \leq pr_1(x(n) - x(0))/n \leq \rho^+ + C/n.$$

In fact, by the compactness of the fundamental domain and periodicity of $F^n - Id$, C can be chosen independently of $x(0)$.

Let $R > 0$ be as in the definition of asymptotic entropy.

Define

$$m = R/(\rho^+ - \rho^-).$$

If m is not already an integer, increase ρ^- and decrease ρ^+ so that m is now an integer and $(*)$ is true for $x(m)$. We may have to increase R, but R is independent of $\rho^\pm$ except for having to be sufficiently large. This may also increase C in $(**)$. Now fix those values of m, R, and C.

Let D be an R-box; i.e., $D := pr_1^{-1}(I)$ where I is an arbitrary interval of length R. Let $x(i), y(i)$ be two trajectories in the universal cover both originating in D.

Claim 2.2 *For the above choice of m, the following holds:*

(1) both $x(m)$ and $y(m)$ are in one of the two boxes $D + m\rho^+$ or $D + m\rho^-$;

(2) For $n = 0, 1, 2, ..., m$, $d(x(n), y(n)) \leq R + 2C + 2$.

Proof of Claim. (Part 1) From $(*)$ we get $\inf I + m\rho^- \leq pr_1(x(m)) \leq \sup I + m\rho^+$. That is, $x(m)$ lies in a region swept out as we slide $D + m\rho^-$ to $D + m\rho^+$. But by the choice of m there is no gap between the two. □

(Part 2) We consider the case when $x(m) \in D+m\rho^+$. The other case is similar. Apply $(**)$ to the trajectory $x(i)$ twice — once starting from $x(0)$ and the second time starting from $x(m)$ and running back in time. We get:

$$pr_1(x(n)) \leq pr_1(x(0)) + n\rho^+ + C;$$

$$pr_1(x(n)) \geq pr_1(x(m)) - (m-n)\rho^+ - C.$$

Since $x(0) \in D$ and $x(m) \in D + m\rho^+$ we see that:

$$\inf I + n\rho^+ - C \leq pr_1(x(n)) \leq \sup I + n\rho^+ + C.$$

Of course we have an analogous inequality for $y(n)$. "Subtracting" the two, yields $pr_1(x(n) - y(n)) \leq R + 2C$. Our claim follows. □

The claim insures that we can match to each trajectory $x(n)$ a sequence of signs $\sigma_i = \pm$ so that $x(i \cdot m) \in D + m\rho^{\sigma_1} + ... + m\rho^{\sigma_i}$. Also this correspondence is injective on $(R+2C+2)$-separated trajectories by (2). Trivial counting implies that $\tilde{h}(R + 2C + 2) \leq \log 2/m$. The Lemma follows.

□

Let $cl(Per(f))$ denote the closure in $\mathbb{A}$ of the set of periodic points of f. Handel has provided a somewhat technical criterion[9], called the *pA-Hypothesis*, for an annulus homeomorphism to be pseudo-Anosov rel a finite set. We have proved elsewhere[11] that an equivalent criterion is for the rotation number assignment $\pi(x) \mapsto \rho(F, x)$ to be either not well-defined or not continuous on the set $cl(Per(f))$.

Theorem 2.3 (Handel's Lemma[9]) *Suppose that $f : \mathbb{A} \to \mathbb{A}$ satisfies the pA-hypothesis. The f is isotopic rel a finite invariant set to a pseudo-Anosov map ϕ. Moreover, the rotation set of f contains the rotation set of ϕ, with respect to the original annulus covering space.*

Theorem 2.4 (Theorem A.) *Suppose that f is an annulus homeomorphism. If we have $h^{as}_{cl(Per)} > 0$ then f satisfies the pA-hypothesis.*

Theorem 2.5 (Theorem B.) *If f is a $C^{1+\epsilon}$-smooth annulus diffeomorphism then*

$$h^{as}_{cl(Per)} \geq \frac{1}{2} \cdot h^{as}.$$

If we assemble these latter three results, we get

Theorem 2.6 (Kwapisz-Swanson[11]) *If f is a Hölder smooth annulus diffeomorphism isotopic to the identity and the asymptotic entropy of f is nonvanishing, then f is isotopic rel a finite invariant set to a pseudo-Anosov map.*

Remarks 2.7 *The proof is too long and technical to be included here. The proof proceeds by establishing that at least half the asymptotic entropy can be realized on the supports of invariant measures with nonzero Lyapunov exponents. Such measures are called hyperbolic. When f is a Hölder diffeomorphism hyperbolic measures are supported in the closure of the periodic points by a result due to Katok*[10].

3. The C^0 case: asymptotic entropy implies periodic orbits

In this section we will prove that positive asymptotic entropy implies the existence of a chain transitive set having a nontrivial rotation set. This, in turn, by a result of J. Franks[7], implies the existence of an interval of rotation numbers and infinitely many periodic orbits in the annulus. This correlates well with the Hölder smooth case and suggests that one might be able to remove the smoothness requirement in Theorem A and obtain the pA hypothesis simply given nonvanishing rotational entropy .

3.1. Asymptotic entropy and chain transitive components

There are many good references for chain recurrence, and we refer the reader to any of those (e.g. Conley[4]) for detailed background information.

Definition 3.1 *An f-periodic ϵ-chain is a sequence $\{x_0, \dots, x_n = x_0\}$ with $d(f(x_i), x_{i+1}) < \epsilon$ for $0 \le i < n$ and $n > 2$. The point x is* chain recurrent *if for every $\epsilon > 0$, there exists a periodic ϵ chain containing x. Two points x and y lie in the same* chain transitive component *$\mathcal{C}$, if for all $\epsilon > 0$, there exists a periodic ϵ chain containing x and y.*

As in Remark 2.7, at least one half of the total asymptotic entropy is attained on the closure of the union of supports of the invariant measures of f. But every invariant measure is supported in the closure of the set of f-recurrent points by the Poicaré recurrence theorem (e.g. Walters[18]).

Therefore, we can conclude

Proposition 3.2 *If asymptotic entropy is positive, then the asymptotic entropy is positive on the chain recurrent set CR.*

However, the following conjecture remains open:

Conjecture 3.3 *Let f denote an annulus homeomorphism isotopic to the identity with chain recurrent set CR. If $h^{as}_{CR}(f) > 0$, then there exists a chain transitive component C of CR such that $h^{as}_{C}(f) \ge (1/2) h^{as}_{CR}(f)$.*

Such a result is not difficult for topological entropy , but scaling differences between topological and asymptotic entropy require a new argument in the case of asymptotic entropy. We will prove, however, that for arbitrary homeomorphisms isotopic to the identity, positive asymptotic entropy forces infinitely many periodic orbits. To do this, we will need the following lemma, due to J. Franks[7]:

Lemma 3.4 *If a chain transitive set contains points x and y with rotation numbers $\rho(f,x) = r$ and $\rho(f,y) = s$, $r < s$, then*
a) the full rotation set of f contains the interval $[r,s]$, and
b) for each reduced rational $p/q \in [r,s]$ there exists a periodic point of period q and rotation number p/q.

We need two additional lemmas:

Lemma 3.5 *Let X denote a chain transitive subset of $f : \mathbb{A} \to \mathbb{A}$ such that $\rho(f,X) \subset (a,b)$. then there exists a number $\delta > 0$ such that if $d(x,X) < \delta$, then $\rho(f,x) \subset (a,b)$.*

Lemma 3.6 *If every chain transitive set admits at most a single rotation number, then the rotation number mapping is continuous on the set of chain recurrent points.*

For a proof of the first lemma, see Swanson[15]. where a slightly stronger result is proved. The second lemma is a corollary of the first.

Theorem 3.7 *If asymptotic entropy is nonvanishing, then there are infinitely many periodic orbits corresponding to a nontrivial rotation interval.*

Proof: If some chain component admits more than one rotation number then the theorem follows from the result of Franks cited above. Otherwise, the rotation number mapping is continuous on the chain recurrent set by the last lemma. If so, for each $\delta > 0$ there is finite cover of the chain recurrent set by compact subsets X_i, $i = 1, \ldots, N$ whose rotation sets each have diameters less than δ. However, one of X_i must support full asymptotic entropy $h > 0$. Since by Lemma 2.1, $h \leq (\log 2)\delta$, we arrive at a contradiction, since δ was arbitrary, unless $h^{as} = 0$. □

4. Area Preserving and Billiard Maps of the Annulus

Consider an area preserving diffeomorphism $f : \mathbb{A} \to \mathbb{A}$. Such maps arise naturally for billiard problems[16] and in celestial mechanics. As J. Franks[7] has noted elsewhere, the area preserving case enjoys many advantages in rotation number theory. The recurrent set is now the entire annulus. If the annulus map admits two distinct rotation numbers, then the map realizes the full interval of rotation numbers between those extremes, including infinitely many reduced rationals corresponding to periodic orbits[7].

From Section 2, we can easily infer the following theorem.

Theorem 4.1 *Suppose that* $f : \mathbb{A} \to \mathbb{A}$ *is a* C^2 *area preserving annulus diffeomorphism, whose periodic points are dense. If this map is integrable, then the rotation number map* $\rho : \mathbb{A} \to \mathbb{R}$ *is well-defined and continuous.*

Proof: If the map ρ is not well-defined or is discontinuous, Handel's criterion is met for the pA-hypothesis (as in Theorem 2.1). As a consequence, there exist finite pseudo-Anosov orbits and a compact set K such that $f|_K$ is semiconjugate to a rotary horseshoe. No such map can be integrable. □

If the conclusions of the last theorem hold, then we shall say that f admits a *generalized rotary horseshoe.*

4.1. Billiard Maps

I would like to extend my thanks to the references[8,19] for helpful conversations concerning the material of this section. We remind the reader of the reduction of billiard orbits to the study of area preserving annulus maps. Parameterize the convex table T by a circle coordinate $p \in \mathbb{S}^1$. Trajectories are geodesics or lightrays which meet the table and exit at a common angle α. Consider the space $K := (0, \pi) \times \mathbb{S}^1$. The map $f : K \to K$ is such that $f(\alpha, p) = (\beta, q)$, where β is the incident angle at point q in the elastic collision immediately following the one at p. Now K is an open annulus. However, for *convex* billiards we may compactify to a compact annulus and we shall assume that this has been carried out, replacing K with $\mathbb{A}$, the usual annulus.

The Aubry-Mather theorem asserts that the set of rotation numbers is precisely the interval whose endpoints are attained on the boundary circles. More significantly, for each number in this interval there corresponds a periodic orbit or, in the irrational case, a Denjoy minimal set or (minimal) invariant circle.

Sometimes called *the Birkhoff conjecture*, the assertion that a billiard map is integrable iff the billiard table is an ellipse has long defied the most powerful methods of analysis and topology.

For our purposes we will use the version due to S. Tabachnikov[16]:

Conjecture 4.2 (Birkhoff's Billiards' Conjecture) *If a neighborhood of a strictly convex smooth billiard curve is foliated by caustics, then the curve is an ellipse.*

The result then implies that the entire billiard disc is foliated by confocal ellipses[16].

We certainly will not resolve this problem in the present discussion! However, one can at least sharpen the problem a little by looking for pseudo-Anosov orbits as in the previous theorem.

We make the following standing assumption:

(*) Suppose every nonempty open set of nonperiodic points of the billiard transformation has a constant (irrational) rotation number.

In fact, it seems to be open whether there *can* be an open set of nonperiodic points.

Theorem 4.3 *For integrable billiard problems the rotation number is continuous in a neighborhood of the annulus boundary curve corresponding to the billiard table curve. If the rotation mapping is ill defined or discontinuous at some phase cylinder point then, in fact, there is a generalized rotary horseshoe in the phase space cylinder.*

Proof: In the integrable case the phase cylinder is foliated into invariant curves. The curves correspond to caustics in the billiard table. Such an invariant curve has a constant rotation number, since that is true of homeomorphisms on circles. The rotation number varies continuously, for that is true of one-parameter families of circle maps.

For area preserving twist maps an open set of nonperiodic points must have a common irrational rotation number ω. It follows that if ρ is ill-defined or discontinuous on the annulus, then the rotation map is ill-defined or discontinuous on the closure of the periodic orbits. □

Conjecture 4.4 *Suppose the billiard curve*

is smooth and strictly convex. The asymptotic entropy of the billiard map is positive iff the billiard table is not an ellipse. In other words either the system is completely integrable or it admits a generalized rotary horseshoe.

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