Quasi-Symmetry without Ratios

Jaroslaw Kwapisz
jarek@math.montana.edu
www.math.montana.edu/~jarek/

February 6, 2018

Abstract

We characterize quasi-symmetric maps between compact metric spaces as *homeomorphisms uniformly at all scales.*

The notion of a quasi-symmetric map is of interest in analysis (as a fruitful relaxation of that of a conformal map) and has important applications in dynamical systems and geometry. The standard definition refers to relative distances expressed by distance ratios, which suggests that quasi-symmetry is a form of uniform continuity at all scales. Our goal is to precisely articulate this intuition in a way that may appeal to those newly encountering the concept. For simplicity, all spaces considered are non-empty **compact metric spaces** (and d_X or d, in absence of ambiguity, denotes the underlying metric on X). The key benefit of compactness is that all subtleties take place at *arbitrarily small* scales, and we do not have to parallel our constructions and arguments to account for *arbitrarily large* scales.

We set $\overline{\mathbb{R}}^+ := [0, \infty]$ and refer to an increasing homeomorphism $\eta : \overline{\mathbb{R}}^+ \to \overline{\mathbb{R}}^+$ as a ***gauge***. Our ostentatious goal is to remove the quotients from the following standard definition ([7, 2, 4]).

Definition 1. *A bijection $f : X \to Y$ is quasi-symmetric (q.s.) iff there is a gauge $\eta : \overline{\mathbb{R}}^+ \to \overline{\mathbb{R}}^+$ such that, for all triples of distinct points $x, x', x'' \in X$, we have*

$$\frac{d(fx, fx')}{d(fx, fx'')} \leq \eta\left(\frac{d(x, x')}{d(x, x'')}\right). \tag{1}$$

In the traditional ϵ, δ-style, one writes:

Definition 2. *A bijection $f : X \to Y$ is quasi-symmetric (q.s.) iff*

$$\forall_{\epsilon>0}\ \exists_{\delta>0}\quad \frac{d(x, x')}{d(x, x'')} < \delta \quad \Longrightarrow \quad \frac{d(fx, fx')}{d(fx, fx'')} < \epsilon$$

and

$$\forall_{\epsilon>0}\ \exists_{\delta>0}\quad \frac{d(fx,fx')}{d(fx,fx'')}<\delta \implies \frac{d(x,x')}{d(x,x'')}<\epsilon$$

(where x, x', x'' are arbitrary triples of distinct points in X).

The two definitions are equivalent and symmetric under replacement of f by f^{-1}, whereby the gauge for f^{-1} is given by $1/\eta^{-1}(1/s)$. (We will build-in the $f \leftrightarrow f^{-1}$ symmetry into all our notions.) Also, one could allow triples of possibly non-distinct x, x', x'' with $x \neq x''$ (upon increasing η so that $1 \leq \eta(1)$).

To explain quasi-symmetry from the more familiar concept of a homeomorphism, let us define the latter by using gauges (aka *moduli of continuity*).

Definition 3. *A bijection $f : X \to Y$ is a* **homeomorphism** *iff there is a gauge $\alpha : \overline{\mathbb{R}}^+ \to \overline{\mathbb{R}}^+$ so that, for $x, x' \in X$,*

$$\alpha^{-1}\left(d(x,x')\right) \leq d(fx,fx') \leq \alpha\left(d(x,x')\right).$$

To give a rigorous meaning to a homeomorphism *uniformly at all scales*, we have to first consider *pieces* of the map f with rescaled metrics.

Definition 4. *Given a homeomorphism $f : X \to Y$ and $x_0 \in X$, a* **zooming of f at** x_0 *is a map $f' : (X', d_{X'}) \to (Y', d_{Y'})$ between metric spaces where $X' \subset X$ is closed with x_0 contained in its interior, $Y' := f(X')$, and f' is the restriction of f (so $f'(x) := f(x)$ for all $x \in X'$). Moreover, the new metrics are $d_{X'} := \lambda d_X$ and $d_{Y'} := \mu d_Y$ for some $\lambda, \mu \geq 1$.*

In our context, the scalars λ, μ are typically strictly bigger than 1 and quite large; so the original metrics d_X and d_Y are expanded, justifying the zooming nomenclature.

Definition 5. *Let $\mathcal{I}$ be any set. A family of homeomorphisms $(f_i : X_i \to Y_i)_{i\in\mathcal{I}}$ is* **uniform** *iff there is a gauge α serving (as in Definition 3) all the maps f_i and the sets of numbers $\{\operatorname{diam}(X_i)\}_{i\in\mathcal{I}}$ and $\{\operatorname{diam}(Y_i)\}_{i\in\mathcal{I}}$ are precompact in $(0,\infty)$ (i.e., they are contained in the segment $[1/D, D]$ for some $D > 0$).*

It is easy to see that a common gauge α exists as soon as both the family of gauges $(\alpha_i)_{i\in\mathcal{I}}$ of individual f_i and the family of inverses $\left(\alpha_i^{-1}\right)_{i\in\mathcal{I}}$ are uniformly equicontinuous. Also, in presence of a common gauge α, $\{\operatorname{diam}(X_i)\}_{i\in\mathcal{I}}$ is precompact iff $\{\operatorname{diam}(Y_i)\}_{i\in\mathcal{I}}$ is precompact.

In what follows, we label all zoomings by an index i that is a pair consisting of the base point and a natural number (**zoom level**).

Definition 6. *A family* $(f_{x_0,k} : X_{x_0,k} \to Y_{x_0,k})_{x_0 \in X, k \in \mathbb{N}}$*, where* $f_{x_0,k}$ *is a zooming of* f *at* x_0*, is of* ***bounded type*** *iff there exists* $C > 1$ *such that, for any distinct* $x_0, x_0' \in X$ *with* $d_X(x_0, x_0') < 1/C$*, there is* $k \in \mathbb{N}$ *so that* $x_0' \in X_{x_0,k}$ *and*

$$\operatorname{diam}_X(X_{x_0,k}) \le C d_X(x_0, x_0') \quad \textit{and} \quad \operatorname{diam}_Y(Y_{x_0,k}) \le C d_Y(fx_0, fx_0').$$

The idea is that, for pairs of nearby points, there is a zooming (centered at the first point) of diameter *comparable* to the distance between the points. Incidentally, the simplest way to satisfy the inequality $\operatorname{diam}_X(X_{x_0,k}) \le C d_X(x_0, x_0')$ is by taking $X_{x_0,k} := \overline{B}_{C^{-k}}(x_0)$, the closed ball of radius C^{-k} about x_0, and picking the largest k for which x_0' belongs to $\overline{B}_{C^{-k}}(x_0)$. Note that, in absence of sufficiently many points of X around x_0, the diameters $\operatorname{diam}_X(\overline{B}_{C^{-k}}(x_0))$ could be much smaller than C^{-k} for many $k \in \mathbb{N}$. (Although, for the k chosen to suit x_0', as above, $\operatorname{diam}_X(\overline{B}_{C^{-k}}(x_0))$ is comparable to C^{-k}; it exceeds C^{-k-1}.) This pathology is absent under the assumption that X is ***uniformly perfect***, i.e., $\operatorname{diam}_X(\overline{B}_r(x_0)) \ge C_1^{-1} r$ for all $r > 0$ and some fixed $C_1 > 1$. For such X, the bounded type stipulation simply amounts to boundedness of the ratios $\operatorname{diam}_X(X_{x_0,k})/\operatorname{diam}_X(X_{x_0,k+1})$ as $k \to \infty$, with the analogous condition for $\operatorname{diam}_Y(Y_{x_0,k})$. Also, note that if f is q.s. then one inequality (in Definition 6) already implies the other upon adjusting the constant C, if necessary.

Theorem 1 (dynamical characterization of quasi-symmetry). *Let* X, Y *be compact metric spaces. A homeomorphism* $f : X \to Y$ *is quasi-symmetric iff one can select at each* $x_0 \in X$ *zoomings* $f_{x_0,k}$ *of* f *so that the family* $(f_{x_0,k} : X_{x_0,k} \to Y_{x_0,k})_{x_0 \in X, k \in \mathbb{N}}$ *is uniform and of bounded type.*

Before delving into proofs, consider the canonical example. (See [4], and also the elegant cataloging of all quasi-circles in [5, 3].)

Koch Example: Let $X := [0,1]$ and let $Y := K$ be the classical Koch curve obtained from a finite segment in $\mathbb{R}^2$ by recursive replacement of the middle third subsegment with two segments of the same length (meeting at 60°). Y is homeomorphic to X via a standard map $f : X \to Y$. This map is verified to be quasi-symmetric by taking $X_{x_0,k} := \overline{B}_{1/4^{k+1}}(x_0)$ with zoom factors $\lambda_{x_0,k} := 4^k$ and $\mu_{x_0,k} := 3^k$. To see this, use the quintessential *self-similarity*: f restricted to any 4-*adic segment*, $I' := [j/4^k, (j+1)/4^k]$ (with $k \in \mathbb{N}$, $0 \le j < 4^k$), maps onto a portion K' of K that coincides with all of K upon translating and scaling by 3^k. Moreover, the restriction $f|_{I'}$ becomes equal to f once pre- and post- composed with the obvious maps $I' \to I$ and $K \to K'$ (obtained by translating and scaling). This implies that, up to pre- and post- composition with an isometry, the zooming $f_{x_0,k}$ coincides with f restricted to a subset. Therefore, uniformity of the family of zoomings is immediate from the uniform continuity of f and f^{-1} (as afforded by compactness of X).

By way of historical perspective, a mechanism similar to that in Koch example is responsible for quasi-symmetry of many *self-similar* homeomorphisms f, where the zoomings are constructed by using dynamical systems, one on X and one on Y, that are topologically conjugated by f. The uniformity of zoomings is automatic if the iterated dynamics expand small distances linearly until they become big. Generally, this dynamical *passage to the big scale* is non-linear and one has to obtain some uniform control of the non-linearity (as needed to secure uniform quasi-symmetry of the *passage*, cf. Remark 1; see [6, 1, 8]). The initial thrust behind our theorem was the sentiment that all quasi-symmetry is of dynamical origin, with the act of zooming supplanting[1] the dynamics.

We turn to proving Theorem 1. The points in all triples considered are assumed to be distinct. A triple $\gamma = (x, x', x'')$ is called δ-***big*** iff $\operatorname{diam}(\gamma) := \max\{d(x,x'), d(x',x''), d(x'',x)\} \geq \delta$. Our departure point is a natural observation that all homeomorphisms are quasi-symmetric on big triples.

Lemma 1 (big triple lemma). *If $f : X \to Y$ is a homeomorphism (with gauge α) and $\delta > 0$, then there is a gauge $\eta : \overline{\mathbb{R}}^+ \to \overline{\mathbb{R}}^+$ so that, for all δ-big triples $\gamma = (x, x', x'')$ in X,*

$$\frac{d(fx, fx')}{d(fx, fx'')} \leq \eta\left(\frac{d(x,x')}{d(x,x'')}\right). \tag{2}$$

Moreover, as long as $\operatorname{diam}(X), \operatorname{diam}(Y) \leq D$ (for some $D > 0$), then the gauge η can be chosen to depend only on δ, D, and the gauge α.

Proof of Lemma 1: Consider a δ-big triple $\gamma = (x, x', x'')$ in X. Clearly $\operatorname{diam}(X) \geq \delta$. Setting

$$\epsilon := \alpha^{-1}(\delta),$$

we see that $\sigma := (fx, fx', fx'')$ is an ϵ-big triple in Y. Also, take

$$\delta' := \alpha^{-1}(\epsilon/2) > 0$$

and define a gauge $\eta : \overline{\mathbb{R}}^+ \to \overline{\mathbb{R}}^+$ by

$$\eta(s) := \max\left\{\frac{\alpha\left(s \operatorname{diam}(X)\right)}{\epsilon/2},\ \frac{\operatorname{diam}(Y)}{\alpha^{-1}\left(\delta'/s\right)}\right\} \qquad (s > 0). \tag{3}$$

First, suppose that $d(fx, fx'') \geq \epsilon/2$. Then

$$\frac{d(fx, fx')}{d(fx, fx'')} \leq \frac{d(fx, fx')}{\epsilon/2} \leq \frac{\alpha(d(x,x'))}{\epsilon/2} \leq (\epsilon/2)^{-1}\alpha\left(\frac{d(x,x')}{d(x,x'')}\operatorname{diam}(X)\right).$$

[1] In a pinch, *zooming* is *dynamical*: it comes from the scaling $\mathbb{R}$-action on the space of metric spaces.

Second, suppose that $d(fx, fx'') < \epsilon/2$. Then $d(fx, fx') > \epsilon/2$ (as otherwise σ would not be ϵ-big). By the choice of δ', $d(x, x') \geq \delta'$, yielding

$$\frac{d(fx, fx')}{d(fx, fx'')} \leq \frac{\mathrm{diam}(Y)}{d(fx, fx'')} \leq \frac{\mathrm{diam}(Y)}{\alpha^{-1}(d(x, x''))} \leq \mathrm{diam}(Y) \left(\alpha^{-1} \left(\frac{d(x, x'')}{d(x, x')} \delta' \right) \right)^{-1}.$$

The two displayed estimates above combine to establish inequality (2) in the lemma. Moreover, from monotonicity of α, it is clear that the diameters in (3) can be replaced by their upper bound D. □

The assertion of quasi-symmetry in the theorem can now be shown based on the simple idea that any triple is big in an appropriate zooming.

Proof of $\Leftarrow$ implication of Theorem 1: First let $D > 0$ be such that $1/D \leq \mathrm{diam}(X_{x_0,k}), \mathrm{diam}(Y_{x_0,k}) \leq D$ for all k, x_0. Proceeding by contradiction, we assume that Definition 2 fails. All things being symmetric with respect to $f \leftrightarrow f^{-1}$, we may suppose that there is a sequence of triples $\gamma_n := (x_n, x'_n, x''_n)$ for which $\frac{d(x_n, x'_n)}{d(x_n, x''_n)} \to 0$ but $\frac{d(fx_n, fx'_n)}{d(fx_n, fx''_n)} \geq \kappa$ for some fixed $\kappa > 0$. By Lemma 1, it must be that $\mathrm{diam}_X(\gamma_n) \to 0$.

For each $n \in \mathbb{N}$, pick $r_n > 0$ minimal such that $\gamma_n \subset \overline{B_{r_n}(x_n)}$. Thus one of x'_n or x''_n is r_n distance away from x_n, and the bounded type property yields, for all large enough n, a zooming $f_{x_n,k_n} : X_{x_n,k_n} \to Y_{x_n,k_n}$ so that $\mathrm{diam}_X(X_{x_n,k_n}) \leq C r_n$. Therefore,

$$\frac{\mathrm{diam}_{X_{x_n,k_n}}(\gamma_n)}{\mathrm{diam}(X_{x_n,k_n})} = \frac{\lambda_{x_n,k_n} \mathrm{diam}_X(\gamma_n)}{\lambda_{x_n,k_n} \mathrm{diam}_X(X_{x_n,k_n})} \geq \frac{r_n}{C r_n} \geq 1/C. \tag{4}$$

Since also $\mathrm{diam}(X_{x_n,k_n}) \geq 1/D$ (by uniformity), the triple γ_n is $\frac{1}{DC}$-big when viewed in X_{x_n,k_n} and we can apply Lemma 1 to maps f_{x_n,k_n}. By uniformity, the lemma yields a common gauge η so that, for all $n \in \mathbb{N}$,

$$\frac{d_{Y_{x_n,k_n}}(fx_n, fx'_n)}{d_{Y_{x_n,k_n}}(fx_n, fx''_n)} \leq \eta \left(\frac{d_{X_{x_n,k_n}}(x_n, x'_n)}{d_{X_{x_n,k_n}}(x_n, x''_n)} \right). \tag{5}$$

For a contradiction, note that

$$\frac{d_{X_{x_n,k_n}}(x_n, x'_n)}{d_{X_{x_n,k_n}}(x_n, x''_n)} = \frac{d(x_n, x'_n)}{d(x_n, x''_n)} \to 0 \tag{6}$$

and

$$\frac{d_{Y_{x_n,k_n}}(fx_n, fx'_n)}{d_{Y_{x_n,k_n}}(fx_n, fx''_n)} = \frac{d(fx_n, fx'_n)}{d(fx_n, fx''_n)} \geq \kappa > 0. \tag{7}$$

□

Remark 1. *Looking back at the proof, in the steps where the original and the zoomed metrics had to be related (i.e., in (4), (6), and (7)), the comparison was that of ratios of distances. The argument then readily generalizes to the situation when* $d_{X_{x_n,k_n}}$ *and* $d_{Y_{x_n,k_n}}$ *are merely uniformly quasi-symmetrically equivalent to the restrictions of* d_X *and* d_Y*, respectively. That is, it suffices that the identity maps* $(X_{x_0,k}, d_{X_{x_0,k}}) \to (X_{x_0,k}, d_X)$ *are q.s. with a common gauge for all* $x_0 \in X$ *and* $k \in \mathbb{N}$*, and that the analogous condition holds for* $Y_{x_0,k}$*.*

The other implication of the theorem follows the natural idea — already broached after Definition 6 — of zooming to geometrically decreasing balls; although, some extra care has to be exercised due to the possibility of isolated points in X.

Lemma 2. *Suppose that* $f : X \to Y$ *is quasi-symmetric with gauge* $\eta : \overline{\mathbb{R}}^+ \to \overline{\mathbb{R}}^+$ *and* $A := \mathrm{diam}(X)$ *and* $B := \mathrm{diam}(Y)$*. If* $X' \subset X$ *and* $Y' := f(X')$ *are not single points so that* $A' := \mathrm{diam}_X(X') > 0$ *and* $B' := \mathrm{diam}_Y(Y') > 0$*, then taking* $\lambda := A/A' \geq 1$ *and* $\mu := B/B' \geq 1$ *and defining a gauge via*

$$\beta(s) := \max\left\{ B\eta\left(\frac{4s}{A}\right), \frac{4Bs}{A}, \frac{4As}{B}, A\frac{1}{\eta^{-1}\left(\frac{B}{4s}\right)} \right\}$$

secures

$$\beta^{-1}(d_{X'}(x,x')) \leq d_{Y'}(fx, fx') \leq \beta(d_{X'}(x,x')) \quad (x, x' \in X').$$

Note that, the gauge β and the diameters of X' and Y' with respect to the rescaled metrics $d_{X'} := \lambda d_X$ and $d_{Y'} := \mu d_Y$ used above do not depend on X'. (Indeed, $\mathrm{diam}(X') = \mathrm{diam}(X)$ and $\mathrm{diam}(Y') = \mathrm{diam}(Y)$ by design.) Therefore, the lemma immediately gives:

Corollary 1. *If* f *is q.s. then the family of all restrictions* $(f|_{X'} : X' \to Y')_{X'}$*, where* X' *ranges over subsets of* X *that are not a single point, is a uniform family (when taken with* $d_{X'}$ *and* $d_{Y'}$ *and zoom factors* λ, μ *as in Lemma 2).*

Proof of Lemma 2: Consider any $x, x' \in X'$. First, assume that $d(x,x') \geq A'/4$. Then

$$\mu d(fx, fx') = B\frac{d(fx,fx')}{B'} \leq B \leq B\frac{d(x,x')}{A'/4} = \frac{4B}{A}\lambda d(x,x').$$

Second, assume that $d(x,x') < A'/4$. Then there is $x'' \in X'$ such that $x'' \neq x$ and $d(x,x'') > A'/4$ (as otherwise $\mathrm{diam}_X(X') \leq 2A'/4 < A'$), and we can write

$$\begin{aligned} \mu d(fx, fx') &= B\frac{d(fx,fx')}{B'} \leq B\frac{d(fx,fx')}{d(fx,fx'')} \\ &\leq B\eta\left(\frac{d(x,x')}{d(x,x'')}\right) < B\eta\left(\frac{d(x,x')}{A'/4}\right) = B\eta\left(\frac{4}{A}\lambda d(x,x')\right). \end{aligned}$$

The last two displayed inequalities combine to give $\mu d(fx, fx') \le \beta(\lambda d(x, x'))$ for any $x, x' \in X'$. By switching the roles of f and f^{-1} (and the associated swapping $A \leftrightarrow B$ and $\eta(s) \leftrightarrow 1/\eta^{-1}(1/s)$) we get the other inequality. □

Proof of $\implies$ implication of Theorem: Assume that f is q.s. and set $r_k := \epsilon^k$ with a fixed $\epsilon \in (0,1)$ selected at will. Consider $x_0 \in X$ and $k \in \mathbb{N}$. If $B_{r_{k-1}}(x_0) \setminus B_{r_k}(x_0)$ is empty we default to $X_{x_0,k} := X$ and $Y_{x_0,k} := Y$ (with $\lambda_{x_0,k} = \mu_{x_0,k} = 1$). Suppose then that there is $x_0' \in B_{r_{k-1}}(x_0) \setminus B_{r_k}(x_0)$. In such case we let $X_{x_0,k} := \overline{B_{r_{k-1}}(x_0)}$ and $Y_{x_0,k} := f(X_{x_0,k})$, and we take $\lambda_{x_0,k}, \mu_{x_0,k}$ and $\beta_{x_0,k} = \beta$ as in Lemma 2 with $X' := X_{x_0,k}$. By Corollary 1, the family of thus obtained zoomings $(f_{x_0,k} : X_{x_0,k} \to Y_{x_0,k})_{x_0 \in X, k \in \mathbb{N}}$ is uniform (as the gauges $\beta_{x_0,k}$ and the rescaled diameters $\operatorname{diam}(X_{x_0,k})$ and $\operatorname{diam}(Y_{x_0,k})$ do not depend on x_0 and k).

It remains to verify the bounded type property. Consider distinct $x_0, x_0' \in X$. We may well require that $r := d(x_0, x_0') < 1$, in which case we can pick $k \in \mathbb{N}$ so that $r_k < r \le r_{k-1}$. Then $x_0' \in \overline{B}_{r_{k-1}}(x_0) \setminus B_{r_k}(x_0)$, so $X_{x_0,k} = \overline{B}_{r_{k-1}}(x_0)$ and $\operatorname{diam}_X(X_{x_0,k}) \le 2r_{k-1} = 2\epsilon^{-1} r_k < 2\epsilon^{-1} r$. In particular, we verified $\operatorname{diam}_X(X_{x_0,k}) \le 2\epsilon^{-1} d(x_0, x_0')$.

On the other side of f, one has $\operatorname{diam}_Y(Y_{x_0,k}) \le 2\eta(2\epsilon^{-1}) d(fx_0, fx_0')$ because, for any $x_0'' \in X_{x_0,k}$, quasi-symmetry of f gives $d(fx_0, fx_0'') \le \eta\left(\frac{d(x_0,x_0'')}{d(x_0,x_0')}\right) d(fx_0, fx_0') \le \eta(2\epsilon^{-1}) d(fx_0, fx_0')$. Therefore, the bounded type property is satisfied with $C := \max\{1, 2\epsilon^{-1}, 2\eta(2\epsilon^{-1})\}$. □

We finish with an example illustrating the theorem in the context of simple self-similar maps of the interval.

Example: Let $m \ge 2$ and $a_i > 0$ satisfy $\sum_{i=1}^m a_i = 1$. Going from left to right, cut $X := [0,1]$ into m subsegments of lengths a_i. These are what we call 1*st generation segments*. Further cutting each such segment (in the same proportions a_i) yields 2*nd generation segments*, etc. There is an obvious way to index the m^n segments of generation n by sequences $\sigma \in \{1, \ldots, m\}^n$ so that, if I_σ is the segment corresponding to σ, then its length is $|I_\sigma| := \prod_{i=1}^n a_{\sigma(i)}$. Now, repeat the same process for $a_i := 1/m$ and $Y := [0,1]$ to get subsegments $I'_\sigma \subset Y$. (Specifically, for $\sigma \in \{1, \ldots, m\}^n$, I'_σ consists of $y \in [0,1]$ whose m-ary expansion starts with $0.\sigma_1\sigma_2\ldots\sigma_n$.) Let $f_n : X \to Y$ be the unique increasing homeomorphism sending I_σ linearly onto I'_σ (for all $\sigma \in \{1, \ldots, m\}^n$). It is easy to see that $f := \lim_{n\to\infty} f_n$ is a homeomorphism. We leave it as an exercise for the reader to use the theorem to show that $f : X \to Y$ is quasi-symmetric iff $a_1 = a_m$.

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