Topological friction in aperiodic minimal $\mathbb{R}^m$-actions

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Abstract

For a continuous map f preserving orbits of an aperiodic $\mathbb{R}^m$-action on a compact space, its displacement function assigns to x the "time" $t \in \mathbb{R}^m$ it takes to move x to $f(x)$. We show that this function is continuous if the action is minimal. In particular, f is homotopic to the identity along the orbits of the action.

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Let X be a compact metric space with a continuous $\mathbb{R}^m$-action $T : \mathbb{R}^m \times X \to X$. Denote $T(t, x)$ simply by $x + t$. Assume that T is *aperiodic*, i.e., $x + t = x$ implies $t = 0$. We are interested in continuous $f : X \to X$ mapping every orbit of T to itself, which is to say that there is $\phi : X \to \mathbb{R}^m$ such that

$$f(x) = x + \phi(x), \qquad x \in X. \tag{0.1}$$

Note that the aperiodicity of T makes ϕ, called the *displacement function*, uniquely determined by f. Recall that T is *minimal* iff its orbits are dense in X.

Theorem. *If the action T is minimal, then ϕ is continuous.*

The twist map $f : (x_1, x_2) \mapsto (x_1 + x_2, x_2)$ on the two torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ does not admit a continuous displacement function yet preserves the orbits of a *periodic* $\mathbb{R}$-action $(x_1, x_2) \to (x_1 + t, x_2)$. Whether the minimality hypothesis can be relaxed to transitivity for $\mathbb{R}$-actions is unknown to the author. However, the following simple example shows that some hypothesis beyond the aperiodicity is necessary. Take $X := D \times \mathbb{R}^2/\mathbb{Z}^2$ where $D := \{1/n : n \in \mathbb{N}\} \cup \{0\}$. Let $T(t,(x,y)) = (x, y + t\omega)$ with $\omega = (1, \sqrt{2})$, and set $f(0,y) := (0,y)$ and $f(1/n, y) := (1/n, y + \phi_n \omega)$ where $\phi_n \to \infty$ are chosen so that $\mathrm{dist}(\phi_n \omega, \mathbb{Z}^2) \to 0$. Since $\phi(1/n, y) = \phi_n$ for $y \in \mathbb{T}^2$, ϕ is unbounded and so discontinuous.

The subtlety underlying the continuity of ϕ surfaces already in the "algebraic case" when $m = 1$, T is a minimal translation on a compact abelian group G, and f is a homeomorphism. Any homemorphism f_0 of the real line, $f_0 : x \mapsto x + \phi_0(x)$, where ϕ_0 is not periodic but merely (Bochner-Bohr) almost periodic, generates such $f : G \to G$ with the displacement $\phi : G \to \mathbb{R}$ being the continuous extension of ϕ_0 to the translation hull of ϕ_0 (c.f. [3], and [2] for further discussion). It is easy to see that a homeomorphism $f : G \to G$ (satisfying our hypotheses) arises from an almost periodic ϕ_0 via the hull construction exactly when ϕ is continuous.

When G is a torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ $(d \geq 2)$ with a minimal Kronecker action, $T(t,x) = x + t\omega$, the theorem follows readily from the homotopy classification of maps of $\mathbb{T}^d$ according to which f lifts to $\tilde{f} : \mathbb{R}^d \to \mathbb{R}^d$ of the form $\tilde{f} = A + \Phi$ where $A \in \mathrm{GL}_d(\mathbb{Z})$ and Φ is $\mathbb{Z}^2$-periodic and continuous. Indeed, one of the lifts $\tilde{f}$ preserves the pencil of lines $\{x + \mathbb{R}\omega\}_{x \in \mathbb{R}^2}$ constituting the lifted orbits of T and the irrationality of ω forces $A = \mathrm{Id}$. Hence, ϕ is just a component of Φ, $\phi = \langle \Phi | \omega \rangle$, which makes it continuous.

A similar argument can be made for G that is a solenoid but one has to contend with a trickier homotopy classification due to disconnectedness of X in the direction transversal to the flow (see [7] and the references therein). In fact, this is the continuity of ϕ in the transverse direction that relies on the minimality whereas the continuity along the flow is rather general. For non-algebraic minimal flows that are poorly understood one might worry that f is *shearing off* and radically rearranging nearby orbit segments to render ϕ unbounded. The goal of this note is to identify a mechanism ("*friction*") that precludes this scenario in minimal flows.

The theorem is also a step in homotopy classification of homeomorphisms of X that permute the orbits of T. Observe that f is homotopic to the identity along the orbits of T exactly when ϕ is continuous; and if it is, $\{x + \lambda\phi(x)\}_{\lambda \in [0,1]}$ supplies such a homotopy.

In another connection, continuity of ϕ guarantees that ϕ is integrable with respect to any ergodic f-invariant measure μ. The Birkhoff averages $\int_X \phi \, d\mu$ are called rotation vectors and constitute an important topological invariant of the dynamics of f (cf. [6, 1]).

An orbit $\mathcal{O}_x := x + \mathbb{R}^m$ is an immersed copy of $\mathbb{R}^m$ parametrized by $\alpha_x : t \mapsto x + t$. That f maps $\mathcal{O}_x$ continuously with respect to its intrinsic topology is a corollary of the following lemma. We shall not use here the minimality (which is invoked only once in the proof of the theorem), nor the full strength of the aperiodicity hypothesis but merely that there are $\epsilon_0, L_0 > 0$ so that if $J \subset \mathbb{R}^m$ is connected and $\mathrm{diam}(J) > L_0$, then $\mathrm{diam}(x + J) > \epsilon_0$

for any $x \in X$.[1]

Lemma. *If I is a line segment and $\gamma : I \to X$ is continuous with the image $\gamma(I)$ contained in $\mathcal{O}_x$ for some $x \in X$ and* $\operatorname{diam}(\gamma(I)) < \epsilon_0/2$*, then $\alpha_x^{-1} \circ \gamma : I \to \mathbb{R}^m$ is continuous and* $\operatorname{diam}(\alpha_x^{-1} \circ \gamma(I)) \leq L_0$.

Proof: Let U be an open ball of radius $\epsilon_0/2$ containing $\gamma(I)$ and $J_1, J_2, \ldots$ be the connected components of the open set $\alpha_x^{-1}(U \cap \mathcal{O}_x) \subset \mathbb{R}^m$.

Note that $\operatorname{diam}(J_k) \leq L_0$, so the closure $\overline{J}_k$ is compact making $(\alpha_x|_{\overline{J}_k})^{-1}$ continuous. Moreover, $\Gamma_k := \alpha_x(\overline{J}_k) \cap \gamma(I)$ is compact. Since $\alpha_x(\partial J_k) \subset \partial U$ and $\gamma(I) \subset U$, we also have $\Gamma_k = \alpha_x(J_k) \cap \gamma(I)$, so the Γ_k are disjoint (like the J_k). Thus $\bigcup_k \gamma^{-1}(\Gamma_k) = I$ is a disjoint union of compact sets, forcing $\gamma^{-1}(\Gamma_{k_0}) = I$ for some k_0 (by Sierpiński's theorem, [5]). In particular, $\alpha_x^{-1} \circ \gamma|_I = (\alpha_x|_{J_{k_0}})^{-1} \circ \gamma|_I$ is continuous. From $\alpha_x^{-1} \circ \gamma(I) \subset J_{k_0}$, $\operatorname{diam}(\alpha_x^{-1} \circ \gamma(I)) \leq \operatorname{diam}(J_{k_0}) \leq L_0$. $\square$

Corollary. *There is $C > 0$ such that, for any $x \in X$, $g_x := \alpha_x^{-1} \circ f \circ \alpha_x : \mathbb{R}^m \to \mathbb{R}^m$ is continuous and*

$$|g_x(t_1) - g_x(t_2)| \leq C|t_1 - t_2| + C, \quad t_1, t_2 \in \mathbb{R}^m. \tag{0.2}$$

Proof: Because $\alpha_x(t)$ is a uniformly continuous function of $x \in X$ and t in a compact neighborhood of 0, there is $\delta_0 \in (0,1)$ such that $\operatorname{diam}(f \circ \alpha_x(I)) < \epsilon_0/2$ provided I is a segment with length $|I| \leq \delta_0$. Cover the segment $[t_1, t_2]$ joining t_1 to t_2 by at most $|t_2 - t_1|/\delta_0 + 1$ subsegments I_j of length δ_0 and apply the lemma to each $\gamma_j := f \circ \alpha_x|_{I_j}$ to conclude that $\operatorname{diam}(\alpha_x^{-1} \circ f \circ \alpha_x([t_1, t_2])) \leq (|t_2 - t_1|/\delta_0 + 1)L_0$. $C := L_0/\delta_0$ is as desired. $\square$

Proof of the theorem: First, we claim that $G_r := \{x : \ |\phi(x)| \leq r\}$ is closed for any $r \geq 0$. Indeed, suppose that $x_n \in G_r$ converge to $x \in X$. There is a subsequence (n_k) and $t \in [-r, r]$ such that $\phi(x_{n_k}) \to t$ and $n_k \to \infty$. By passing to the limit in $f(x_{n_k}) = x_{n_k} + \phi(x_{n_k})$ we get $f(x) = x + t$. Since also $f(x) = x + \phi(x)$, the aperiodicity of T forces $\phi(x) = t$ thus placing x in G_r.

Furthermore, the restriction $\phi|_{G_r}$ is continuous, as otherwise we would have x and x_n as above and $\epsilon > 0$ with $|\phi(x_n) - \phi(x)| \geq \epsilon$, which yields $|t - \phi(x)| \geq \epsilon$ contary to $t = \phi(x)$.

It remains to show that $X = G_{r'}$ for some $r' > 0$ as then $\phi = \phi|_{G_{r'}}$, which we know is continuous. Since $X = \bigcup_{r \in \mathbb{N}} G_r$, the Baire theorem secures $r \in \mathbb{N}$ with $U := \operatorname{int}(G_r) \neq \emptyset$. By the minimality of T and the compactness of X, there is $R > 0$ such that any orbit piece $\{y + t : \ |t| < R\}$, $y \in X$, intersects U.

Consider an arbitrary $x \in X$. Take t with $|t| \leq R$ and $x + t \in U$ so that $|\phi(x + t)| \leq r$. Since $g_x(0) = \phi(x)$ and $g_x(t) = \phi(x + t) + t$, the corollary yields

$$|\phi(x)| \leq |\phi(x + t)| + |g_x(0) - g_x(t)| + |t| \leq r + CR + R + R. \tag{0.3}$$

[1] This hinges on absence of *small periods*, i.e. existence of $L_0 > 0$ such that $x + t = x$, $|t| < L_0 \implies t = 0$, which fails exactly when some x is stabilized by a connected subgroup of $\mathbb{R}^m$.

That is $x \in G_{r'}$ with $r' := r + CR + R + R$. $\square$

From another perspective (cf. [4]), the theorem is an assertion about regularity of the *coboundary* ϕ in the *cohomological equation*

$$\psi(x,t) = t + \phi(x+t) - \phi(x), \quad x \in X,\ t \in \mathbb{R}^m, \tag{0.4}$$

where the *cocycle* $\psi : X \times \mathbb{R}^m \to \mathbb{R}^m$ is related to f via $f(x+t) = f(x) + \psi(x,t)$ (which, if f^{-1} exists, means that ψ is the cocycle associated to the action $(t,x) \mapsto f^{-1} \circ T(t, f(x))$ considered as a *time change* of T). Here ψ is the *data* and it is always continuous. Indeed, by the corollary and $\psi(x,t) = g_x(t) - g_x(0)$, we have $\sup\{|\psi(x,t)| :\ x \in X,\ |t| \leq r\} \leq Cr + C < +\infty$. Thus, if $x_n \to x$, $t_n \to t$, and $\psi(x_n, t_n) \to s$, then $f(x_n + t_n) = f(x_n) + \psi(x_n, t_n)$ converges to $f(x+t) = f(x) + s$, so $\psi(x,t) = s$. More subtly, the theorem guarantees that the solution to (0.4), our displacement function ϕ, is continuous if T is minimal.

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