# Combinatorics of Torus Diffeomorphisms

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**Abstract**

We construct dynamical partitions of the torus for a diffeomorphism that is isotopic to the identity. The existence and the combinatorics of the partitions is soleley deteremined by the rotation set of the diffeomorphism. When the rotation set consists of a single non-resonant vector, there is a whole hierarchy of partitions analogous to the partitions of the circle into the closest return intervals under an irrational circle rotation. In particular, all such torus maps are infinitely renormalizable in a natural sense.

## 1 Introduction

Taking $\mathbb{T}^d$ to be the $d$-dimensional torus obtained as the quotient $\mathbb{R}^d/\mathbb{Z}^d$ where the integer lattice $\mathbb{Z}^d \subset \mathbb{R}^d$ acts by translations, let $\mathbf{Diff}_0(\mathbb{T}^d)$ be the space of all $C^1$-diffeomorphisms $f : \mathbb{T}^d \to \mathbb{T}^d$ that are $C^1$-isotopic to the identity and let $\mathbf{Diff}_0(\mathbb{R}^d)$ be the space of all their lifts to $\mathbb{R}^d$. For $F \in \mathbf{Diff}_0(\mathbb{R}^d)$, *the rotation vector of* $F$ is defined as

$$\rho_F := \lim_{n\to\infty} \frac{F^n(p) - p}{n}$$

provided the limit exists and is independent on $p \in \mathbb{R}^d$. If $\rho_F$ fails to exist, one speaks of *the rotation set* $\rho(F)$ *of* $F$ defined as the set of all limit points of sequences of the form $\frac{F^{n}(p_k)-p_k}{n_k}$ where $n_k \to \infty$ and $p_k \in \mathbb{R}^d$ (see [17]).

In connection with the classical problem of stability and breakdown of quasi-periodic invariant tori, of particular interest are the *non-resonant torus maps*, that is $f \in \mathbf{Diff}_0(\mathbb{T}^d)$ with a lift $F \in \mathbf{Diff}_0(\mathbb{R}^d)$ such that $\rho_F = (r_1, \dots, r_d)$ is well defined and

$$\left(\sum_{i=1}^{d} \alpha_i r_i = \alpha_0, \quad \alpha_i \in \mathbb{Z}\right) \Rightarrow (\alpha_i = 0 \text{ for } i = 0, 1, \dots, d).$$

The fundamental question is that about dynamical classification of such maps; in particular, about existence of a conjugacy or a semiconjugacy to the simplest model provided by the quasi-periodic translation, $T_\rho : p \mapsto p + \rho$, $\rho = \rho_F$. In dimension $d = 1$, the resulting theory of circle diffeomorphisms with irrational rotation number is rather complete and, in a nutshell, amounts to the following [4]:

(i) (Poincaré) A semiconjgacy, $h : \mathbb{T} \to \mathbb{T}$, $T_\rho \circ h = h \circ f$, always exists.

(ii) (Denjoy) $h$ is a conjugacy if $f$ is sufficiently regular ($f'$ is BV).

(iii) (Arnold, Herman, Yoccoz, Katznelson and Ornstein ... ) The smoothness of $h$ is determined by the smoothness of $f$ and the number theoretical properties of $\rho_F$.

The case of dimension $d = 2$ and higher is much harder: the skew-product examples of Herman in [8, 9] (see also [7, 1]) show that the analogues of all three assertions above generally fail. (In particular, from 4.5 of [9] one can extract real-analytic $f \in \mathbf{Diff}_0(\mathbb{T}^2)$ with Diophantine $\rho_F$ that are not even semi-conjugated to the translation.) Beyond the local situation when $f$ is near translation and some sort of KAM technique applies, little is known in positive (c.f. [10]). The failure of (i) is the main culprit depriving one on the outset of the topological model on which the analytical arguments could be layered[1]. However, the success of (ii) and (iii) in the circle case depends not as much on (i) per se as on its corollary about the existence of the hierarchy of dynamical partitions of $\mathbb{T}$ associated to the process of Diophantine approximation of $\rho$ (see e.g. (1.6) page 26 in [4]). It is our main result that the analogous partitions exist in dimension $d = 2$ and that one can justifiably say that all non-resonant torus maps with a given rotation vector $\rho$ combinatorially look like the translation $T_\rho$. Thus we resolve the problem of finding a suitable generalization of (i) to $d = 2$ and offer a departure point for attempts to generalize (ii) and (iii).

To state the result, we first extend the notion of Farey neighboring fractions.

**Definition 1.1** *Suppose that we are given three rational vectors $\rho_1$, $\rho_2$, and $\rho_3$ written in the lowest terms $\rho_1 = (u_1/u_3, u_2/u_3)$, $\rho_2 = (v_1/v_3, v_2/v_3)$, and $\rho_3 =$*

[1] Assuming (i) one can make some progress see [19, 18].

$(w_1/w_3, w_2/w_3)$ with $u_3, v_3, w_3 > 0$. The vectors $(\rho_i)_{i=1,2,3}$, form a Farey triple iff

$$\det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} = 1.$$

*Given a subset $B \subset \mathbb{R}^2$, $(\rho_i)_{i=1,2,3}$ is a Farey triple for $B$ iff it is a Farey triple and $B$ is contained in the open simplex with vertices at $\rho_1$, $\rho_2$, and $\rho_3$. Moreover, if $B = \{\rho\}$ is a single point, we simply speak of $(\rho_i)_{i=1,2,3}$ as a Farey triple for $\rho$.*

**Theorem 1.2** *Suppose that $f \in \mathbf{Diff}_0(\mathbb{T}^2)$ and its lift $F \in \mathbf{Diff}_0(\mathbb{R}^2)$ are given together with $\rho_1 := (u_1/u_3, u_2/u_3)$, $\rho_2 = (v_1/v_3, v_2/v_3)$, and $\rho_3 := (w_1/w_3, w_2/w_3)$ such that $(\rho_i)_{i=1,2,3}$ is a Farey triple for $\rho(F)$, then there exist three smooth quadrilaterals (i.e. diffeomorphic images of closed Euclidean rectangles) $J_0$, $K_0$, and $L_0$ in $\mathbb{T}^2$ such that, denoting $J_i := f^i(J_0)$, $K_i := f^i(K_0)$, and $L_i := f^i(L_0)$, the family*

$$\{J_0, \dots, J_{w_3-1}, K_0, \dots, K_{u_3-1}, L_0, \dots, L_{v_3-1}\},$$

*is an essentially disjoint covering of $\mathbb{T}^2$ (i.e. it covers all of $\mathbb{T}^2$ and its elements have pairwise disjoint interiors). The combinatorics of this covering is independent of $f$ in the sense that, if $f$ and $\bar{f}$ are two maps that satisfy the above hypotheses, then there exists a homotopic to the identity homeomorphism $h : \mathbb{T}^2 \to \mathbb{T}^2$ that maps the covering for $f$ to the one for $\bar{f}$, and the following diagram commutes*

$$\begin{array}{ccc} \mathbb{T}^2 \setminus (J_{w_3-1} \cup K_{u_3-1} \cup L_{v_3-1}) & \xrightarrow{f} & \mathbb{T}^2 \setminus (J_0 \cup K_0 \cup L_0) \\ h\downarrow & & h\downarrow \\ \mathbb{T}^2 \setminus (\bar{J}_{w_3-1} \cup \bar{K}_{u_3-1} \cup \bar{L}_{v_3-1}) & \xrightarrow{\bar{f}} & \mathbb{T}^2 \setminus (\bar{J}_0 \cup \bar{K}_0 \cup \bar{L}_0) \end{array} . \tag{1.1}$$

*Moreover, if $f$ is a translation, $J_0$, $K_0$, and $L_0$ can be obtained as projections to $\mathbb{T}^2$ of three Euclidean parallelograms in $\mathbb{R}^2$ every two of which share a side (as in Figure 4.3).*

We shall refer to the covering in the theorem as *a dynamical tiling for $f$*. Figure 1.1 depicts example tilings for the translation $\bar{F} = T_\rho$ where $\rho = (\lambda, 1/\lambda - 1)$ with $\lambda^3 - \lambda^2 - \lambda - 1 = 0$ and for the area preserving version of *the standard map* $F = F_{a,b,\rho} = T_\rho \circ V_b \circ H_a$ where $H_a(x, y) = (x + \frac{a}{2\pi}\sin(2\pi y), y)$ with $a = 0.45$ and $V_b(x, y) = (x, y + \frac{b}{2\pi}\sin(2\pi x))$ with $b = 1.8$. Theorem 1.2 says that the two pictures are homeomorphic and that $f$ and $\bar{f}$ are conjugated save for three tiles. This of course yields no immediate conclusions about the asymptotic dynamics of $f$ (in a way consistent with existing counterexamples). On the other hand, for any non-resonant $f \in \mathbf{Diff}_0(\mathbb{T}^2)$, there is a sequence of Farey triples $(\rho_i^{(n)})_{i=1}^3$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \max_i |\rho - \rho_i^{(n)}| = 0$, $\rho := \rho_F$, and diagram (1.1) supplies a sequence $h_n$ of diffeomorphisms that conjugate $f$ to the translation onto a progressively bigger

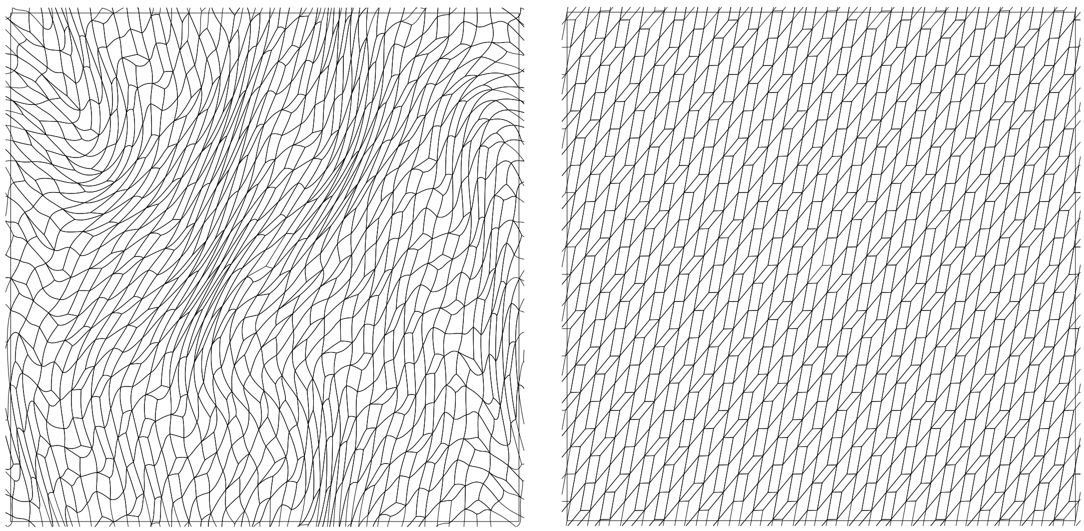

Figure 1.1: Dynamical tilings for a standard map and the translation with the same rotation vector associated to $u = (927, 778, 504)$, $v = (504, 423, 274)$, and $w = (274, 230, 149)$

subset of $\mathbb{T}^2$ so that $\lim_{n\to\infty} \max_{p\in\mathbb{T}^2} \operatorname{dist}(\bar{f} \circ h_n(p), h_n \circ f(p)) = 0$. Passage to the limit $\lim_{n\to\infty} h_n$ to obtain a conjugacy or a semiconjugacy is not possible in general without further control of the *geometry* of $J_i^{(n)}$'s. (It is only for $d = 1$ that the order structure on $\mathbb{T}$ readily secures Poincaré's semiconjugacy.)

Theorem 1.2 is more than an approximation result: it is a result on existence of renormalizations for $f$. Indeed, we shall show in Section 4 that each of the *tiles* $J_0$, $K_0$, and $L_0$ has a natural identification on the boundary making it into a torus, and that the return maps $J_0 \to J_0$, $K_0 \to K_0$, and $L_0 \to L_0$ yield again diffeomorphisms isotopic to the identity. In particular, the return map to $J_0 \cup K_0 \cup L_0 = J_{w_3} \cup K_{u_3} \cup L_{v_3}$ yields a map in $\mathbf{Diff}_0(\mathbb{T}^2)$ that can be rightfully called a $(u, v, w)$*-renormalization of* $f$ (after *rescaling* its domain back to $\mathbb{T}^2$). This is inspired and in perfect harmony with the circle case where the renormalization arises from the return map to a pair of suitable adjacent segments (see e.g. [4]). Following the established paradigm, convergence of the renormalizations (of which a whole sequence exists for non-resonant $f$) may lead to conjugacy results; although, implementation of this idea will be anything but easy and is left for future. We should not try to leave the impression that we are the first to attempt renormalization involving invariant two-tori — see e.g. [11, 14, 12]. At this point, we lack the expertise to compare these aproaches beyond noticing that ours is more geometrical and global (i.e. not confined to near translations) but is restricted (for now) to the context of a single torus.

The formulation of Theorem 1.2 by necessity omits many key aspects of the theory which emerge in all clarity only from the proof. For a quick overview, it may be worth skimming Section 2,3, and 4. Section 2 establishes the line of the argument and contains the key geometrical ideas. Section 3 provides detailed description of the combinatorics of the dynamical tiling by defining and analyzing their simplest models in the form of *the stepped translation* on *the stepped plane* and *the stepped torus*. Here we borrow and extend the terminology that we learned from R.F. Williams, who independently studied certain *stepped planes* in connection with DA tiling spaces (see [21]). Section 4 builds on Section 3 to offer the beginning of the renormalization theory while a more systematic development is relegated to [16]. Finally, Section 5 contains the most technical topological part of the proof of Theorem 1.2. Unlike Sections 2,3, and 4, which are easily translated to the context of $d > 2$, these arguments are intimately connected with the topology of two-dimensional submanifolds of $\mathbb{T}^3$ and their generalization to $d > 2$ is not obvious. We should also mention that this work continues and to some extent overlaps [15]. Reading the introduction to [15] may provide further perspective on our results. In particular, our Theorem 1.2 should be compared with Translation Loop Theorem (Theorem 1.3) in [15]. Finally, let us stress that our main objective in the present paper was to present the key ideas with a minimal burden of technicalities. For instance, our assumption about $C^1$-smoothness can almost surely be weakened but we adopted it to be able to invoke Fried's results on existence of cross sections.

*Note added in revision:* Robert MacKay pointed out that renormalizations on the $n$-dimensional torus were studied in [20]. The emphasis of [20] is on elucidating the combinatorial structure of the rigid translations by coding orbits via "even $n$-colourings of the integers". The coding domains are obtained via *a staircase construction* which appears to coincide with *the stepped plane* we use. In particular, our Section 3 and Theorem 2.3 find their counterpart in Theorem 1 of [20]. Coincidentally, similar ideas in various guises appear also in the study of *the generalized Sturmian sequences* and the associated tiling spaces by by G. Rauzy, P. Arnoux, S. Ito and M. Ohtsuki, V.Berthe and L. Vuillion and others — see Chapters 7 and 8 in [5] for more information and references. What sets our work apart is that we uncover combinatorial structure of maps that are not rigid translations — nor are assumed to be conjugated or *Denjoy-semi-conjugated* to such maps as in Theorem 2 of [20].

Let us also briefly comment on the interplay between *renormalization* and *flow-equivalence*. It is known that the two effect the same scaling of *the frequencies of motions* and therefore coincide up to conjugacy. However, the connection is typically formalized (see e.g. Prop. 1.1 in [13]) in terms of $\mathbb{Z}^{d+1}$ actions on $\mathbb{R}^d$ (or commuting $d+1$-tuples of maps), which obscures the existence of the dynamical partitions we construct.

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## 2 Outline of Proof

Let us outline the strategy of the proof of Theorem 1.2 while relegating detailed arguments to the subsequent sections.

Suppose that the hypothesis of Theorem 1.2 are satisfied. As a first step, we use the fact that $f$ is in fact smoothly isotopic to the identity to suspend $f$ into a $C^1$-flow $\phi : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{T}^3$. Here, we think of the universal cover $\mathbb{R}^2$ of $\mathbb{T}^2$ as the $(x, y)$-plane $\tilde{\mathcal{S}} := \{z = 0\}$ embedded in the $(x, y, z)$-space $\mathbb{R}^3$ and of $\mathbb{T}^2$ as the corresponding two torus $\mathcal{S} := \pi_{\mathbb{T}^3}(\tilde{\mathcal{S}})$, where $\pi_{\mathbb{T}^3} : \mathbb{R}^3 \to \mathbb{T}^3$ is the canonical projection. The suspended flow induces $f$ as the return map to $\mathcal{S}$. The (unique) lifted flow $\tilde{\phi} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ has flow lines crossing $\tilde{\mathcal{S}}$ transversally with $z$-coordinate changing the sign from $-$ to $+$, and $F : \tilde{\mathcal{S}} \to \tilde{\mathcal{S}}$ given by

$$F = T^{-1}_{(0,0,1)} \circ \tilde{\phi}^1|_{\tilde{\mathcal{S}}} \tag{2.1}$$

is a lift of $f$. We shall assume that this is the original $F$ satisfying the hypothesis of the theorem (as this can be always achieved by adjusting the initial isotopy used to construct the suspension).

Let $A \in \mathbb{S}L_3(\mathbb{Z})$ be the matrix

$$A := \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}. \tag{2.2}$$

The hypothesis on the rotation set of $f$ guarantees that *the set of homological directions of $\phi$* — as defined in [6] — is contained in the cone $A\mathbb{R}^3_+$, where $\mathbb{R}_+ = (0, \infty)$. Fried's theorem (Theorem D in [6]) supplies therefore global cross-sections $J$, $K$, and $L$ that are 2-tori homologous to the affine tori $\pi_{\mathbb{T}^3}(\mathrm{lin}(u, v))$, $\pi_{\mathbb{T}^3}(\mathrm{lin}(v, w))$, $\pi_{\mathbb{T}^3}(\mathrm{lin}(w, u))$ — respectively. (How Fried's theory specializes to the context of torus maps is explained in some more detail in Section 2 of [15].)

The cross-sections may, a priori, intersect inside $\mathbb{T}^3$ in a very complicated manner. The heart of our argument lies in assuring that $J$, $K$, and $L$ form *a clean triple of cross-sections* in the sense of the following definition (c.f. Definition 1.1 in [15]).

**Definition 2.1** *A triple of $C^1$-embedded tori $J$, $K$, and $L$ in $\mathbb{T}^3$ is called* clean *if $J \cap K$, $K \cap L$, and $L \cap J$ are simple closed loops spanning the homology $H_1(\mathbb{T}^3)$, $J \cap K \cap L$ is a single point, and all the above intersections are transversal. If additionally $J$, $K$, and $L$ are cross-sections to a flow, then $J$, $K$, and $L$ are called* a clean triple of cross-sections.

This roughly says that tori forming a clean triple have the simplest possible topology of their mutual intersections permitted by their cohomology classes (as exemplified by the affine tori). The key theorem below is shown in Section 5 and it extends Theorem 3.6 in [15] dealing with the much simpler case of two cross sections.

**Theorem 2.2 (Cleaning)** *Suppose that $\phi : \mathbb{T}^3 \times \mathbb{R} \to \mathbb{T}^3$ is a $C^1$-smooth flow on $\mathbb{T}^3$. If there are three (global) cross sections $J$, $K$, and $L$ whose cohomology classes form a basis of the first cohomology of $\mathbb{T}^3$ over $\mathbb{Z}$, such cross sections exist that form a clean triple of cross sections. Moreover, the clean triple can be obtained by modifying each of $J$, $K$, and $L$ via an isotopy in $\mathbb{T}^3$.*

It is left to derive all the assertions of Theorem 1.2 from the existence of a clean triple of cross sections. To achieve a clear geometrical picture, let us conjugate $\phi$ (and its lift $\tilde{\phi}$) by a diffeomorphism isotopic to the identity so that the three clean cross sections $J$, $K$, and $L$ are *straightened out* to the affine tori obtained as the quotients (under $\mathbb{Z}^3$) of the planes

$$\tilde{J} := \text{lin}(u, v), \qquad \tilde{K} := \text{lin}(v, w), \qquad \tilde{L} := \text{lin}(w, u).$$

This step, formalized in Lemma 6.2, is quite intuitive since one expects that cutting $\mathbb{T}^3$ along $J$, $K$, and $L$ yields a 3-ball that can be mapped to $[0, 1]^3$ so that the natural boundary identifications transform into those induced by $\mathbb{Z}^3$.

Let us retain the old notation after the conjugation. Note that, typically, $\mathcal{S}$ is no longer an affine torus nor is $\tilde{\mathcal{S}}$ a plane anymore. The flow $\tilde{\phi}$ (after the conjugacy) is transversal to the planes $\tilde{J}$, $\tilde{K}$, and $\tilde{L}$. Those planes bound the cone $A\mathbb{R}^3_+$, which is carried by the flow into itself (see Figure 2.1). The rest of the argument is based on the very simple idea that the flow lines on $\mathbb{T}^3$, having their lifts trapped in $A\mathbb{R}^3_+$, are approximately determined by the way $A\mathbb{R}^3_+$ projects to $\mathbb{T}^3$; and the approximation is better if the cone $A\mathbb{R}^3_+$ is narrow — which happens whenever $u_i$, $v_i$, and $w_i$ are large. More precisely, for $p \in \mathbb{R}^3$, the flow line $\tilde{\phi}^{\mathbb{R}}(p)$ intersects each of $\tilde{\mathcal{S}}$, $\tilde{J}$, $\tilde{K}$, and $\tilde{L}$ exactly once so that the three faces of the fundamental parallelepiped

$$P := \{\alpha u + \beta v + \gamma w : \ 0 \le \alpha, \beta, \gamma \le 1\}$$

that meet at the vertex $(0, 0, 0)$ — see Figure 2.1 — map diffeomorphically, by the holonomy along the flow lines, to three *quadrilaterals* in $\tilde{\mathcal{S}}$:

$$\tilde{J}_0 := \{p \in \tilde{\mathcal{S}} : \ \tilde{\phi}^{\mathbb{R}}(p) \cap \tilde{J} \cap P \neq \emptyset\},$$
$$\tilde{K}_0 := \{p \in \tilde{\mathcal{S}} : \ \tilde{\phi}^{\mathbb{R}}(p) \cap \tilde{K} \cap P \neq \emptyset\},$$
$$\tilde{L}_0 := \{p \in \tilde{\mathcal{S}} : \ \tilde{\phi}^{\mathbb{R}}(p) \cap \tilde{L} \cap P \neq \emptyset\}.$$

For $i \in \mathbb{N}$, let us set

$$\tilde{J}_i := F^i(\tilde{J}_0), \qquad \tilde{K}_i := F^i(\tilde{K}_0), \qquad \tilde{L}_i := F^i(\tilde{L}_0),$$

and write $J_i$, $K_i$, and $L_i$ for the corresponding projections to $\mathcal{S}$. In order to complete the proof of Theorem 1.2, we shall prove in the next section the following more precise result.

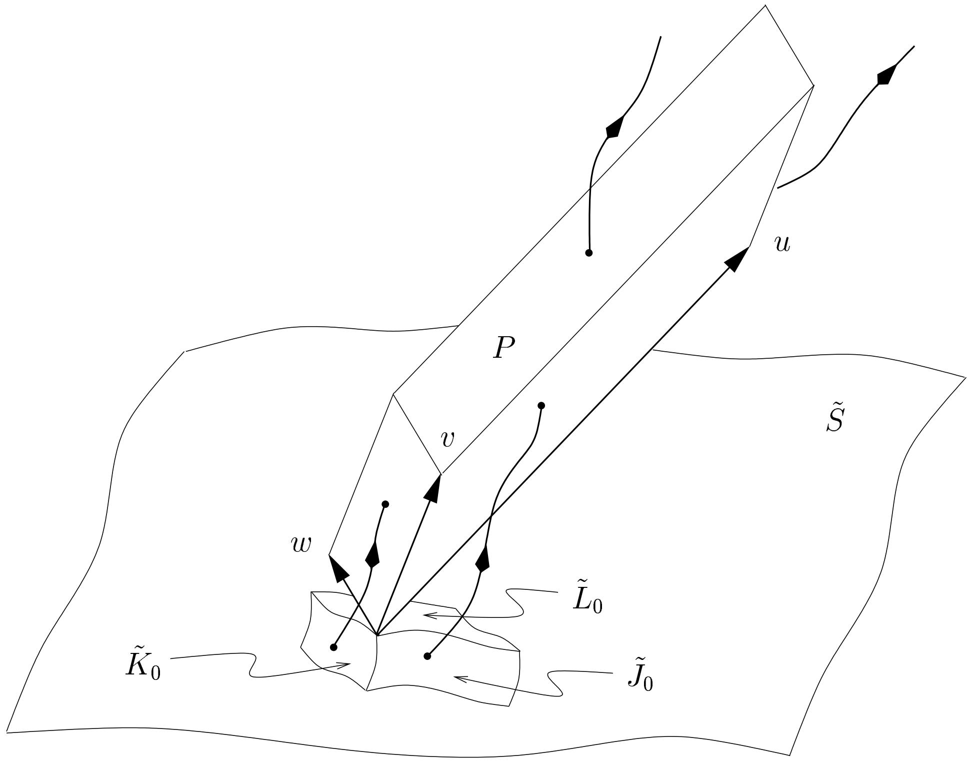


Figure 2.1: The Cone $A\mathbb{R}^3_+$ and the flow (after straightening the cross-sections).

**Theorem 2.3 (Combinatorial Part of Theorem 1.2)** *The family of quadrilaterals*

$$\mathcal{F}_{f,A} := \{J_0, \dots, J_{w_3-1}, K_0, \dots, K_{u_3-1}, L_0, \dots, L_{v_3-1}\},$$

*obtained as above from a clean triple of cross sections to a suspension flow of* $f \in \mathbf{Diff}_0(\mathbb{T}^2)$*, is an essentially disjoint covering of* $\mathbb{T}^2$ *that satisfies the assertions of Theorem 1.2.*

We note that, unlike Theorem 1.2, Theorem 2.3 and its proof generalize to dimensions $d > 2$ at a mere cost of complicating the notations.

# 3 The Stepped Translation

This section explores the link between clean triples of cross-sections and dynamical tilings and culminates in a proof of Theorem 2.3. *The stepped translation* will be a special map serving as a combinatorial model for all $f$'s satisfying the assumptions of Theorem 2.3.

For convenience, besides *the straightening conjugacy* performed in Section 2, we further conjugate the flow via the linear map induced by the matrix $A^{-1}$, which carries $(u, v, w)$ onto the standard basis $(e_1, e_2, e_3)$. Thus, in the place of $\tilde{\phi}^t$, we consider

$$A^{-1} \circ \tilde{\phi}^t \circ A.$$

As before, we shall retain the pre-conjugacy notations. Observe that the the cross-section $\mathcal{S}$ (over which the suspension was built) is now cohomologous to the affine torus

$$\hat{\Pi} := \{dz \circ A = u_3 dx + v_3 dy + w_3 dz = 0\},$$

and the lift $\tilde{\mathcal{S}}$ is an embedded topological plane in $\mathbb{R}^3$ that is $\mathbb{Z}^3$-equivariantly homotopic to

$$\tilde{\Pi} := \{u_3 x + v_3 y + w_3 z = 0\}.$$

The *time translation*, formerly $T_{(0,0,1)}$, is represented now by $T_{A^{-1}e_3}$, and (2.1) becomes

$$F = T^{-1}_{A^{-1}e_3} \circ \tilde{\phi}^1|_{\tilde{\mathcal{S}}}. \tag{3.1}$$

Our strategy is to exploit that the return maps to cohomologous cross sections are conjugated and replace $\mathcal{S}$ by a section that comes with a natural dynamical tiling. For a technical reason — the new section being degenerate in $\mathbb{T}^3$ — we choose to work exclusively with lifts. In particular, we shall need the cyclic (suspension) covering of $\phi$,

$$\hat{\phi} : \mathbb{R}^3/\Lambda \times \mathbb{R} \to \mathbb{R}^3/\Lambda$$

where the sublattice $\Lambda \subset \mathbb{Z}^3$ is that of *the spatial translations:*

$$\Lambda := \mathbb{Z}A^{-1}e_1 + \mathbb{Z}A^{-1}e_2. \tag{3.2}$$

Having fixed any connected lift $\hat{\mathcal{S}}$ of $\mathcal{S}$ to $\mathbb{R}^3/\Lambda$, we see that

$$\hat{f} = \hat{T}^{-1}_{A^{-1}e_3} \circ \hat{\phi}^1|_{\hat{\mathcal{S}}}, \tag{3.3}$$

were $\hat{T}_v$ denotes the quotient by $\Lambda$ of $T_v : \mathbb{R}^3 \to \mathbb{R}^3$, is conjugated to $f : \mathcal{S} \to \mathcal{S}$ via the canonical projection from $\mathbb{R}^3/\Lambda$ to $\mathbb{T}^3$.

The hyperplane $\tilde{\Pi}$ cuts $\mathbb{R}^3$ into two open half spaces: $\tilde{\Pi}^-$ and $\tilde{\Pi}^+$, where $\tilde{\Pi}^-$ is the half space containing $-A^{-1}e_3$, i.e. the one exited by the flow through $\tilde{\Pi}$. Referring to any $\mathbb{Z}^3$-translate of $[0,1]^3$ as *a lattice cube*, we define $H^-$ to be the union of all closed lattice cubes with their interiors entirely contained in $\tilde{\Pi}^-$,

$$H^- := \bigcup \{Q \text{ a lattice cube} : \ \operatorname{int}(Q) \subset \tilde{\Pi}^-\}.$$

Also, let $\tilde{\Theta}$ be the boundary of $H^-$,

$$\tilde{\Theta} := \partial H^-.$$

Note that $\tilde{\Pi}$ and $\tilde{\Theta}$ are invariant under $\Lambda$ so that we have a well defined quotient

$$\hat{\Theta} := \tilde{\Theta}/\Lambda.$$

**Definition 3.1** *With any $A = (u, v, w) \in \mathbb{S}L_3(\mathbb{Z})$ we associate the following objects:*

- $\tilde{\Theta}$ *called* the stepped plane,
- $\hat{\Theta}$ *called* the stepped torus,
- $\mathcal{T} := T_{-A^{-1}e_3}|_{\tilde{\Theta}} : \tilde{\Theta} \to \mathbb{R}^3$ *called* the stepped plane translation,
- *the quotient of $\mathcal{T}$, $\hat{\mathcal{T}} : \hat{\Theta} \to \mathbb{R}^3/\Lambda$, called* the stepped torus translation.

We shall see that $\tilde{\Theta}$ and $\hat{\Theta}$ are indeed topologically a plane and a torus but neither $\mathcal{T}(\tilde{\Theta}) \subset \tilde{\Theta}$ nor $\hat{\mathcal{T}}(\hat{\Theta}) \subset \hat{\Theta}$. As we already mentioned in the introduction, *stepped planes* can be found in [20] and in the literature on *tiling spaces* (see e.g. [21, 5]).

One way to understand $\tilde{\Theta}$ (and progress towards the proof of Theorem 2.3) is by analyzing the family $\hat{\mathcal{Q}}$ of all the lattice cubes in $H^-/\Lambda$ that meet $\hat{\Theta}$ and the family of all their lifts $\tilde{\mathcal{Q}}$, which are the lattice cubes in $H^-$ that meet $\tilde{\Theta}$. Figure 3.1 is a 2-d projection of an example $\tilde{\mathcal{Q}}$. Note that the six faces of any lattice cube $Q$ split into two families: the three *entrance faces* sharing the vertex $\min Q := (\min_Q x, \min_Q y, \min_Q z)$ through which the flow enters $Q$, and the three *exit faces* sharing the vertex $\max Q$ through which the flow exits $Q$. (In Figure 3.1, only the exit faces are visible — c.f. also Figure 4.2.) It is easy to see that $\tilde{\Theta}$ does not meet the entrance faces of $Q \in \tilde{\mathcal{Q}}$: the negative cone, $\max Q - \mathbb{R}^3_+$, is disjoint from $\tilde{\Pi}$, so a cube $Q'$ neighboring with $Q$ across an entrance face $D$ cannot hit $\tilde{\Pi}$, which means that $D$ is not contained in $\tilde{\Theta}$.

**Fact 3.2** *$\tilde{\Theta}$ constitutes a cross-section to $\tilde{\phi}$ in the sense that every flow line meets $\tilde{\Theta}$ exactly once (as it exits $H^-$), and $\tilde{\Theta}$ is a topological plane homeomorphic to $\tilde{\mathcal{S}}$ via the natural holonomy*

$$\tilde{h} : \tilde{\Theta} \to \tilde{\mathcal{S}}$$

*that sends $p \in \tilde{\Theta}$ to the unique point $\tilde{h}(p) \in \tilde{\mathcal{S}}$ on the flow line $\tilde{\phi}^{\mathbb{R}}(p)$.*

*Proof:* Because $\tilde{\phi}$ is a lift of a suspension flow, given $p \in \mathbb{R}^3$, there are $t_- < t_+$ such that $\tilde{\phi}^{(-\infty,t_-)}(p) \subset \tilde{\Pi}^-$ and $\tilde{\phi}^{(t_+,\infty)}(p) \subset \tilde{\Pi}^+$, and $\lim_{t\to\pm\infty} \operatorname{dist}(\tilde{\phi}^t(p), \tilde{\Pi}) = \infty$. Thus $\tilde{\phi}^{\mathbb{R}}(p)$ hits both $H^-$ and its complement, and we conclude that $\tilde{\Theta} = \partial H^-$ meets all the flow lines. That each flow line intersects $\tilde{\Theta}$ exactly once follows from the fact that $\tilde{\Theta}$ (exclusively made of the exit faces of $\tilde{\mathcal{Q}}$) is topologically transverse to the flow with all flow lines exiting $H^-$ through $\tilde{\Theta}$. Hence, we have shown that $\tilde{\Theta}$ is a cross-section. The assertion about the holonomy follows readily. □

We turn now our attention to the natural tiling of $\tilde{\Theta}$ into the three families of cubic faces, each collecting the faces $\mathbb{Z}^3$-congruent to the unit square in one of the planes $\{z = 0\}$, $\{y = 0\}$, or $\{x = 0\}$. Descending to $\hat{\Theta}$, we denote by $\hat{\mathcal{J}}$, $\hat{\mathcal{K}}$, and $\hat{\mathcal{L}}$ the three corresponding families that tile $\hat{\Theta}$ (see Figure 3.1).

**Fact 3.3** *Writing $\#A$ for the cardinality of a set $A$, we have*

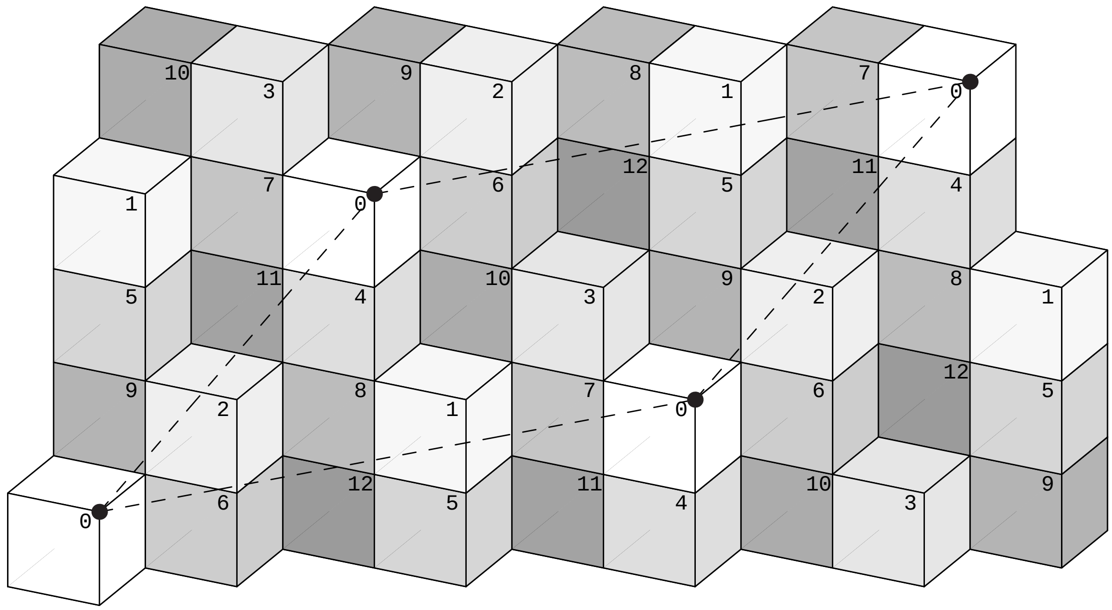


Figure 3.1: *The Stepped Plane* $\tilde{\Theta}$ for $A = [u, v, w]$ with $u = (24, 20, 13)$, $v = (13, 11, 7)$, and $w = (7, 6, 4)$ as viewed from the point $(2000, 1000, 1000)$. (For the right perspective think about the marked vertices as *sticking out*.) The dashed lines outline a fundamental parallelogram in the plane $\tilde{\Pi}$ of the sublattice $\Lambda$. The only (mod $\Lambda$) cube $Q_0$ that meets $\tilde{\Pi}$ is white; all other cubes are *just behind* $\tilde{\Pi}$ and the shade darkens with increasing distance from $\tilde{\Pi}$. Modulo $\Lambda$, there are $13 + 7 + 4$ cubes touching $\tilde{\Theta}$, of which only $13 = \max\{13, 7, 4\}$ are showing a face. The number $k$ (by the max vertex) identifies a cube as a lift of $\hat{\mathcal{T}}^k Q_0$, where $\hat{\mathcal{T}} = T_{-A^{-1}e_3} = T_{(1,4,-4)} \equiv T_{(1,-2,0)} \pmod{\Lambda}$ is *the stepped torus translation*. The flow (not pictured) pierces $\tilde{\Theta}$ transversally *from behind* and moves roughly toward the viewer. If the perspective is ignored, the two dimensional picture shows the dynamical tiling for the translation by $(68/37, 57/37)$, the suspension of which is the constant flow in the direction $(2, 1, 1)$.

*(i)* $\#\hat{\mathcal{J}} = w_3$*,* $\#\hat{\mathcal{K}} = u_3$*, and* $\#\hat{\mathcal{L}} = v_3$*.*

*(ii)* $\#\hat{\mathcal{Q}} = u_3 + v_3 + w_3$*.*

*(iii)* $\hat{\mathcal{T}}(\tilde{\mathcal{Q}}) \subset H^-$ *and* $\#(\hat{\mathcal{Q}} \setminus \hat{\mathcal{T}}(\hat{\mathcal{Q}})) = 1$*.*

*Proof:* Fix the orientation on $\tilde{\Pi}$ determined by the (outward) normal $(u_3, v_3, w_3)$.

(i) Consider the projection to $\mathbb{R}^3/\Lambda$ of the constant vector field everywhere equal to $e_3$. The flux through $\hat{\Theta}$ is clearly equal to $\#\hat{\mathcal{J}}$. On the other hand, the fundamental parallelogram for the torus $\hat{\Pi}$ is spanned by $\{A^{-1}e_1, A^{-1}e_2\}$ and the cross-product $A^{-1}e_1 \times A^{-1}e_2 = (u_3, v_3, w_3)$, so the flux through $\hat{\Pi}$ is equal to the Euclidean scalar product $\langle (u_3, v_3, w_3), e_3 \rangle = w_3$. The divergence being zero, the two fluxes are equal by the Stokes Theorem. Thus $\#\hat{\mathcal{J}} = w_3$. The analogous arguments show the two other equalities.

(ii) Consider on $\mathbb{R}^3/\Lambda$ the quotient of the constant vector field $(-1, -1, -1)$. The flux through $\hat{\Theta}$ (and any of its translates) is the same as the flux through $\hat{\Pi}$ and thus equals $\langle (u_3, v_3, w_3), (-1, -1, -1) \rangle = -u_3 - v_3 - w_3$.

At the same time, since all the faces in $\tilde{\Theta}$ are exit faces, $T_{(-1,-1,-1)}$ maps $H^-$ strictly into itself and therefore $\hat{\mathcal{Q}}$ consists exactly of the lattice cubes between $\hat{\Theta}$ and $\hat{T}_{(-1,-1,-1)}(\hat{\Theta})$. Thus $\#\hat{\mathcal{Q}}$ can be computed as the volume trapped between $\hat{\Theta}$ and $T_{(-1,-1,-1)}(\hat{\Theta})$:

$$\#\hat{\mathcal{Q}} = \left| \int_0^1 [\text{The flux through } T_{t(-1,-1,-1)}(\hat{\Theta})] \, dt \right| = (u_3 + v_3 + w_3) \cdot 1$$

where we used the classical interpretation of "flux" known as the Reynolds' Transport Theorem (see (6.1) in Theorem 6.1 in the appendix).

(iii) That $\mathcal{T}(\tilde{\mathcal{Q}}) \subset H^-$ follows immediately from the definitions of $H^-$ and $\tilde{\mathcal{Q}}$ and the fact that $\mathcal{T}(\tilde{\Pi}) \subset \tilde{\Pi}^-$. Thus $\hat{\mathcal{T}}(\hat{\mathcal{Q}}) \subset H^-/\Lambda$, and the cardinality $\#(\hat{\mathcal{Q}} \setminus \hat{\mathcal{T}}(\hat{\mathcal{Q}}))$ coincides with the volume $V$ trapped between $\hat{\mathcal{T}}(\hat{\Theta})$ and $\hat{\Theta}$. Taking the constant vector field equal to $-A^{-1}e_3$, we compute the flux through $\hat{\Pi}$ to be 1 (since $\det A = 1$). Therefore,

$$V = \int_0^1 [\text{The flux through } T_{-tA^{-1}e_3}(\hat{\Theta})] \, dt = 1 \cdot 1.$$

□

Finally, we investigate $\mathcal{T}$ to bring out the key fact that, although $\mathcal{T}(\tilde{\Theta}) \not\subset \tilde{\Theta}$, $\mathcal{T}$ has an interesting *action* on $\tilde{\Theta}$ that leaves $\tilde{\Theta}$ *nearly invariant*. Let $Q_0 \in \hat{\mathcal{Q}}$ be the cube with $\max Q_0 = (0, 0, 0) \pmod{\Lambda}$; we call $Q_0$ *the fundamental cube of* $\hat{\Theta}$. Also, let $\hat{J}_0$, $\hat{K}_0$, and $\hat{L}_0$ be the exit faces of $Q_0$ — all three are contained in $\hat{\Theta}$ (c.f. Figure 3.1).

**Fact 3.4**

*(i)* $Q_0$ *is the unique cube in* $\hat{\mathcal{Q}}$ *that intersects* $\hat{\Pi}$*;*

*(ii)* $\hat{\mathcal{Q}} \setminus \hat{\mathcal{T}}(\hat{\mathcal{Q}}) = Q_0$*;*

*(iii)* $\hat{\mathcal{J}} \setminus \hat{\mathcal{T}}(\hat{\mathcal{J}}) = \hat{J}_0$*,* $\hat{\mathcal{K}} \setminus \hat{\mathcal{T}}(\hat{\mathcal{K}}) = \hat{K}_0$*,* $\hat{\mathcal{L}} \setminus \hat{\mathcal{T}}(\hat{\mathcal{L}}) = \hat{L}_0$*;*

*(iv)* $\hat{\mathcal{Q}} = \{Q_0, \hat{\mathcal{T}}(Q_0), \ldots, \hat{\mathcal{T}}^{u_3+v_3+w_3-1}(Q_0)\}$*;*

*(v)* $\hat{\mathcal{J}} = \{\hat{\mathcal{T}}^j(\hat{J}_0)\}_{j=0}^{w_3-1}$*,* $\hat{\mathcal{K}} = \{\hat{\mathcal{T}}^j(\hat{K}_0)\}_{j=0}^{u_3-1}$*, and* $\hat{\mathcal{L}} = \{\hat{\mathcal{T}}^j(\hat{L}_0)\}_{j=0}^{v_3-1}$.

*Proof:* (i) For $Q \in \tilde{\mathcal{Q}}$, $\mathrm{int}(Q) \cap \tilde{\Pi} = \emptyset$ by definition of $H^-$, so that if $Q \cap \tilde{\Pi} \neq \emptyset$, then $Q \cap \tilde{\Pi}$ contains a vertex (in fact, $Q \cap \tilde{\Pi} = \max Q$). It follows that the cubes in $\tilde{\mathcal{Q}}$ hitting $\tilde{\Pi}$ are in 1-1 correspondence with the lattice points on $\tilde{\Pi}$. We are done by recalling that $A^{-1}e_1$ and $A^{-1}e_2$ span $\Lambda = \mathbb{Z}^3 \cap \tilde{\Pi}$ so that the origin 0 (mod $\Lambda$) is the sole element of $\hat{\Pi} \cap \mathbb{Z}^3/\Lambda$, the only lattice point in $\hat{\Pi}$.

(ii) By (iii) of Fact 3.3, we know $\hat{\mathcal{Q}} \setminus \hat{\mathcal{T}}(\hat{\mathcal{Q}})$ is a single lattice cube. Because $A^{-1}e_3 \in \tilde{\Pi}^+$, $\mathcal{T}^{-1}([-1,0]^3) \not\subset \tilde{\Pi}^-$. Since $[-1,0]^3$ is a lift of $Q_0$, this implies that $Q_0 \notin \hat{\mathcal{T}}(\hat{\mathcal{Q}})$ by the definition of $\hat{\mathcal{Q}}$ (and $\tilde{\Theta}$).

(iii) $\hat{\mathcal{J}} \cup \hat{\mathcal{K}} \cup \hat{\mathcal{L}}$ tiles $\hat{\Theta}$ and $\hat{\mathcal{T}}(\hat{\mathcal{J}} \cup \hat{\mathcal{K}} \cup \hat{\mathcal{L}})$ tiles $\hat{\mathcal{T}}(\hat{\Theta})$ into cubic faces. By (ii), there is exactly one cube, $Q_0$, between $\hat{\mathcal{T}}(\hat{\Theta})$ and $\hat{\Theta}$. Therefore, $\hat{\mathcal{T}}(\hat{\Theta})$ and $\hat{\Theta}$ coincide save for the faces of $Q_0$. The three exit faces of $Q_0$, $\hat{J}_0$, $\hat{K}_0$, and $\hat{L}_0$, belong to $\hat{\Theta}$, which leaves the three remaining entrance faces in $\hat{\mathcal{T}}(\hat{\Theta})$ — (iii) follows.

(iv) Consider $Q \in \hat{\mathcal{Q}}$. From $\lim_{n\to\infty} \mathrm{dist}(\tilde{\Pi}, \mathcal{T}^{-n}(\tilde{\Pi})) = \infty$, $\hat{\mathcal{T}}^{-n}Q \notin \hat{\mathcal{Q}}$ for some $n > 0$; furthermore, we claim that $\hat{\mathcal{T}}^{-n+1}(Q) = Q_0$ for the minimal such $n$ denoted $n(Q)$. This is because (ii) asserts that $Q_0$ is the only cube leaving $\hat{\mathcal{Q}}$ under $\hat{\mathcal{T}}^{-1}$. Clearly, (iv) shall follow if $n(Q_*) = u_3 + v_3 + w_3$ for some $Q_* \in \hat{\mathcal{Q}}$. Since $\hat{\mathcal{T}}^{-k}(Q) \neq \hat{\mathcal{T}}^{-j}(Q)$ for $k \neq j$, $n(Q) \in \{1, \ldots, \#\hat{\mathcal{Q}}\}$ and the map $n : \hat{\mathcal{Q}} \to \{1, \ldots, \#\hat{\mathcal{Q}}\}$ is injective. Consequently, $n$ is surjective, and we can set $Q_* := n^{-1}(\#\hat{\mathcal{Q}})$. We are done by (ii) of Fact 3.3.

(v) By using (iii), the arguments for (iv) adapt word by word. □

We are ready to give a proof of Theorem 2.3.

*Proof of Theorem 2.3:* Recall the homeomorphism $\tilde{h} : \tilde{\Theta} \to \tilde{\mathcal{S}}$ supplied by Fact 3.2 and let $\hat{h} : \hat{\Theta} \to \hat{\mathcal{S}}$ be the quotient by $\Lambda$. The curvilinear parallelograms $J_0$, $K_0$, and $L_0$ (defined in the previous section) are given by

$$J_0 = A \circ \hat{h}(\hat{J}_0), \qquad K_0 = A \circ \hat{h}(\hat{K}_0), \qquad L_0 = A \circ \hat{h}(\hat{L}_0)$$

where $A$ is the initial linear conjugacy performed at the beginning of this section — which we agreed to suppress, so we shall work with $\hat{h}(\hat{J}_0)$, $\hat{h}(\hat{K}_0)$, and $\hat{h}(\hat{L}_0)$. Set $\hat{J}_i := \hat{\mathcal{T}}^i(\hat{J}_0)$, $\hat{K}_i := \hat{\mathcal{T}}^i(\hat{K}_0)$, and $\hat{L}_i := \hat{\mathcal{T}}^i(\hat{L}_0)$. From part (v) of Fact 3.4,

$$\{\hat{h}(\hat{J}_0), \ldots, \hat{h}(\hat{J}_{w_3-1}), \hat{h}(\hat{K}_0), \ldots, \hat{h}(\hat{K}_{u_3-1}), \hat{h}(\hat{L}_0), \ldots, \hat{h}(\hat{L}_{v_3-1})\}$$

constitutes an essentially disjoint covering of the torus $\hat{\Theta}$. Writing $\mu(p)$ for *the flight time* of $p \in \hat{\Theta}$ to $\hat{\mathcal{T}}^{-1}(\hat{\Theta})$ so that $\hat{\phi}^{\mu(\cdot)} = \hat{\phi}_{\hat{\Theta}}^{\hat{\mathcal{T}}^{-1}(\hat{\Theta})}$ is the holonomy from $\hat{\Theta}$ to

$\hat{\mathcal{T}}^{-1}(\hat{\Theta})$, we see that the following diagram commutes

$$\begin{array}{ccc} \hat{\Theta} & \xrightarrow{\hat{\mathcal{T}}\circ\hat{\phi}^{\mu(\cdot)}} & \hat{\Theta} \\ \hat{h}\big\downarrow & & \hat{h}\big\downarrow \\ \hat{\mathcal{S}} & \xrightarrow{\hat{\mathcal{T}}\circ\hat{\phi}^{1}} & \hat{\mathcal{S}} \end{array} \tag{3.4}$$

However, $\hat{f} = \hat{\mathcal{T}} \circ \hat{\phi}^1|_{\hat{\mathcal{S}}}$ by (3.3), and $\mu(\cdot) = 0$ everywhere except on $\hat{J}_{w_3-1} \cup \hat{K}_{u_3-1} \cup \hat{L}_{v_3-1}$ as it follows from (v) of Fact 3.4. Hence,

$$\begin{array}{ccc} \hat{\Theta} \setminus (\hat{J}_{w_3-1} \cup \hat{K}_{u_3-1} \cup \hat{L}_{v_3-1}) & \xrightarrow{\hat{\mathcal{T}}} & \hat{\Theta} \setminus (\hat{J}_0 \cup \hat{K}_0 \cup \hat{L}_0) \\ \hat{h}\big\downarrow & & \hat{h}\big\downarrow \\ \hat{\mathcal{S}} \setminus (\hat{h}(\hat{J}_{w_3-1}) \cup \hat{h}(\hat{K}_{u_3-1}) \cup \hat{h}(\hat{L}_{v_3-1})) & \xrightarrow{\hat{f}} & \hat{\mathcal{S}} \setminus (\hat{h}(\hat{J}_0) \cup \hat{h}(\hat{K}_0) \cup \hat{h}(\hat{L}_0)) \end{array}$$

Because $\hat{f}$ is naturally conjugated to $f$, this last diagram translated to the original (pre-conjugacy) setting yields

$$\begin{array}{ccc} \hat{\Theta} \setminus (\hat{J}_{w_3-1} \cup \hat{K}_{u_3-1} \cup \hat{L}_{v_3-1}) & \xrightarrow{\hat{\mathcal{T}}} & \hat{\Theta} \setminus (\hat{J}_0 \cup \hat{K}_0 \cup \hat{L}_0) \\ \big\downarrow & & \big\downarrow \\ \mathcal{S} \setminus (J_{w_3-1} \cup K_{u_3-1} \cup L_{v_3-1}) & \xrightarrow{f} & \mathcal{S} \setminus (J_0 \cup K_0 \cup L_0) \end{array} \tag{3.5}$$

The commuting diagram in Theorem 2.3 is obtained by putting together two copies of the above diagram, one for $f$ and another for $\bar{f}$. □

# 4 Renormalization

In this section, we make precise our claim from the introduction that Theorem 1.2 can be viewed as a result on existence of renormalization for maps in $\mathbf{Diff}_0(\mathbb{T}^2)$. The main point is that renormalization is *global* (i.e. not restricted to perturbations of the translations). Our considerations parallel those in [16] dealing with a simpler renormalization suitable for annulus maps and maps in $\mathbf{Diff}_0(\mathbb{T}^2)$ whose rotation set is localized only in one direction. More detailed discussion is relegated to [16].

**Definition 4.1** *Given $A \in \mathbb{S}L_3(\mathbb{Z})$,*

$$A = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix},$$

*with $u_3, v_3, w_3 > 0$, we say that $f \in \mathbf{Diff}_0(\mathbb{T}^2)$ is A-renormalizable iff the stepped torus $\hat{\Theta}$ associated to $A$ can be mapped homeomorphically via some $h : \hat{\Theta} \to \mathcal{S} = \mathbb{T}^2$ so that, taking $J_i := h(\hat{J}_i)$, $K_i := h(\hat{K}_i)$, and $L_i := h(\hat{L}_i)$, the diagram (3.5) commutes.*

The idea is that $\mathbb{T}^2$ has a tiling that maps under $f$ in the same way as the canonical tiling of the stepped torus under $\hat{\mathcal{T}}$. (As it will become clear $h$ may well be required to be piecewise smooth.)

Since $\hat{J}_0$, $\hat{K}_0$, and $\hat{L}_0$ are lattice cube faces in $\mathbb{R}^3$, their opposite sides are naturally identified by $\mathbb{Z}^3$; and the homeomorphism $h$ transports those identifications to the topological quadrilaterals $J_0$, $K_0$, and $L_0$.

**Fact 4.2** *The identifications of the opposite sides of $J_0$, $K_0$, and $L_0$ are effected by iterates of $f$ and are as depicted in Figure 4.1.*

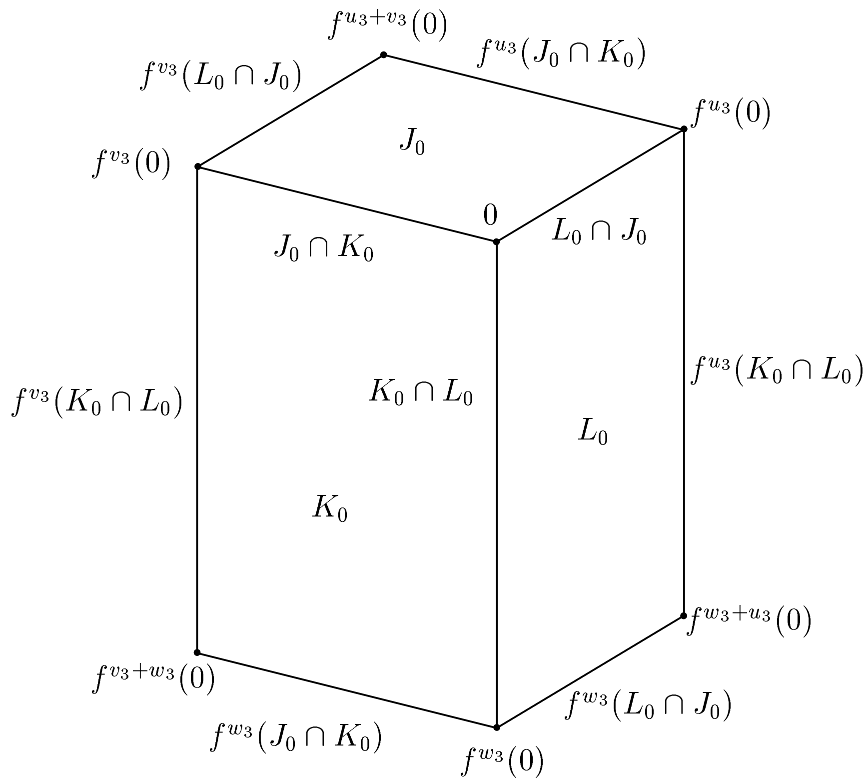


Figure 4.1: Identifications: the parallel sides map to one another under an iterate of $f$.

*Proof:* $\hat{J}_0$, $\hat{K}_0$, and $\hat{L}_0$ were defined as the exit faces of the fundamental lattice cube $Q_0$ and they are the three faces sharing the vertex 0 (cf. Fact 3.4 and Figure 3.1). From (ii) and (v) of Fact 3.4, $\hat{J}_{w_3} = \hat{\mathcal{T}}^{w_3}(\hat{J}_0)$, $\hat{K}_{u_3} = \hat{\mathcal{T}}^{u_3}(\hat{K}_0)$, and $\hat{L}_{v_3} = \hat{\mathcal{T}}^{v_3}(\hat{L}_0)$ are the three entrance faces of the cube $Q_0$ (see Figure 4.2). Since $\hat{\mathcal{T}}$ is a quotient of a translation, the arrangement of the faces in $Q_0$ easily reveals the iterates of $\hat{\mathcal{T}}$ identifying pairs of parallel sides of $\hat{J}_0$, $\hat{K}_0$, and $\hat{L}_0$. Diagram 3.5 assures that the same iterates of $f$ identify the opposite sides of $J_0$, $K_0$, and $L_0$. □

Let $J_0/\sim$, $K_0/\sim$, and $L_0/\sim$ be the tori obtained by identifying the opposite sides of $J_0$, $K_0$, and $L_0$.

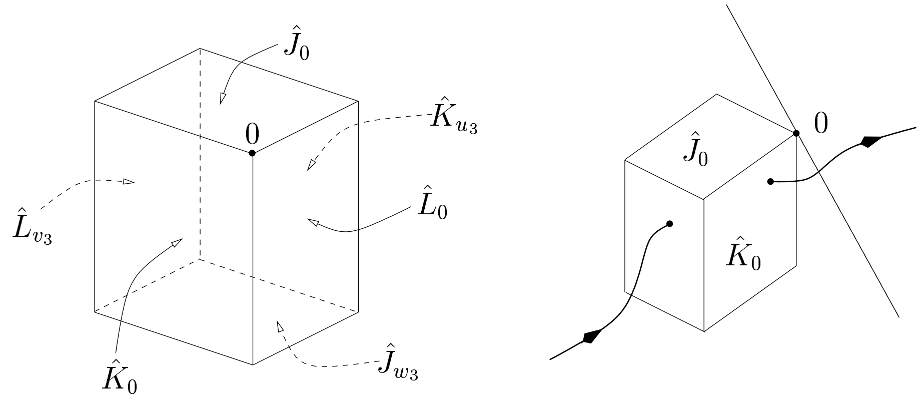


Figure 4.2: The cube $Q_0$ and the holonomy $\psi$ from the entrance faces to the exit faces.

**Proposition 4.3** *Suppose that $f \in \mathbf{Diff}_0(\mathbb{T}^2)$ is $A$-renormalizable. Then the suspension flow of $f$ has a well defined cross-section $J$ in the cohomology class represented by* $\text{lin}(u, v)$ *(mod $\mathbb{Z}^3$); and the map $\mathcal{R}_{J_0}(f) : J_0/\sim \to J_0/\sim$ induced on the torus $J_0/\sim$ by the first return to $J_0$ under $f$ is conjugated to the Poincaré return map $\phi_J : J \to J$ for the flow. The conjugacy is via a map isotopic to the identity. The analogous statements are also true for $K_0$ and $L_0$.*

Observe that the existence of the cross sections in the cohomology classes of $\text{lin}(u, v)$, $\text{lin}(v, w)$, and $\text{lin}(w, u)$ (mod $\mathbb{Z}^3$) guarantees (see Section 2 in [15]) that the set of homological directions of the flow is contained in the cone spanned by $\{u, v, w\}$, which implies that $\rho_1 = (u_1/u_3, u_2/u_3)$, $\rho_2 = (v_1/v_3, v_2/v_3)$, and $\rho_3 = (w_1/w_3, w_2/w_3)$ form a Farey triple for the rotation set of $f$. This complements Theorem 2.3 and yields the following characterization of $A$-renormalizability.

**Corollary 4.4** *A mapping $f \in \mathbf{Diff}_0(\mathbb{T}^2)$ is $A$-renormalizable iff $(\rho_1, \rho_2, \rho_3)$, defined above, is a Farey triple for $\rho(F)$ for some lift $F \in \mathbf{Diff}_0(\mathbb{R}^2)$ of $f$.*

*Proof of Proposition 4.3.* The plan is to first construct the conjugacy under the additional hypothesis that $J_0$, $K_0$, and $L_0$ are obtained from a clean triple of cross sections $(J, K, L)$ via the construction described in Section 3. In the second part of the proof we shall show that the hypothesis can be satisfied when $f$ is an $A$-renormalizable map.

Thus we first assume that $\mathbb{T}^2$ is embedded as $\hat{\mathcal{S}}$ in the cyclic covering of the suspension flow of $f$ (i.e. $\mathbb{R}^3/\Lambda \simeq \mathbb{T}^2 \times \mathbb{R}$, see (3.2)) and that $\hat{h} : \hat{\Theta} \to \mathbb{T}^2 = \hat{\mathcal{S}}$ is given by the holonomy between $\hat{\mathcal{S}}$ and the stepped torus $\hat{\Theta}$. The three clean cross sections are just the quotients to $\mathbb{T}^3$ of $\hat{J}_0$, $\hat{K}_0$, and $\hat{L}_0$. Denoting by $f_A$ and $\phi_A$ the return maps to the set $A$ under $f$ and $\phi$, we see that, tautologically, $f_{J_0}$ coincides with the return map to $J_0$ under $f_{J_0 \cup K_0 \cup L_0}$, and likewise $\phi_J$ coincides with the return map to $J$ under $\phi_{J \cup K \cup L}$. (Every point of $\hat{J}_0$ has to return to $\hat{J}_0$ because $\hat{J}_0$ is a lift of a cross section; and the same holds for $\hat{K}_0$, and $\hat{L}_0$.) It suffices then to show that $f_{J_0\cup K_0\cup L_0}$ induces on the union of three tori $(J_0/\sim)\cup(K_0/\sim)\cup(L_0/\sim)$ a map that is conjugated to $\phi_{J\cup K\cup L}$ via the homeomorphism induced by $\hat{h}$.

To see that, recall the diagram (3.4) from Section 3 to the effect that $\hat{h}$ conjugates $f$ to $\bar{f} := \hat{\mathcal{T}} \circ \hat{\phi}_{\hat{\Theta}}^{\hat{\mathcal{T}}^{-1}(\hat{\Theta})} = \hat{\phi}^{\hat{\Theta}}_{\hat{\mathcal{T}}(\hat{\Theta})} \circ \hat{\mathcal{T}}$. Hence, $f_{J_0\cup K_0\cup L_0}$ is conjugated by $\hat{h}$ to $\bar{f}_{\hat{J}_0\cup\hat{K}_0\cup\hat{L}_0}$. However, $\hat{\phi}^{\hat{\Theta}}_{\hat{\mathcal{T}}(\hat{\Theta})} = \mathrm{Id}$ except on $\hat{J}_{w_3}\cup\hat{K}_{u_3}\cup\hat{L}_{v_3}$ where it equals $\psi := \hat{\phi}^{\hat{J}_0\cup\hat{K}_0\cup\hat{L}_0}_{\hat{J}_{w_3}\cup\hat{K}_{u_3}\cup\hat{L}_{v_3}}$ (cf. Figure 4.2). Therefore, on $\hat{\Theta}\setminus(\hat{J}_{w_3-1}\cup\hat{K}_{u_3-1}\cup\hat{L}_{v_3-1})$, $\bar{f}$ coincides with $\hat{\mathcal{T}}$, and one easily computes by using (v) of Fact 3.4 that

$$\bar{f}_{\hat{J}_0\cup\hat{K}_0\cup\hat{L}_0}(p) = \begin{cases} \psi\circ\hat{\mathcal{T}}^{w3}(p) & \text{if } p\in\hat{J}_0; \\ \psi\circ\hat{\mathcal{T}}^{u3}(p) & \text{if } p\in\hat{K}_0; \\ \psi\circ\hat{\mathcal{T}}^{v3}(p) & \text{if } p\in\hat{L}_0. \end{cases} \tag{4.1}$$

Upon descending to $\mathbb{T}^3$, $\hat{J}_0$, $\hat{K}_0$, and $\hat{L}_0$ yield the cross sections $J$, $K$, and $L$; and $\psi$ yields the return map $\phi_{J\cup K\cup L}$. The above formula shows then that $\bar{f}_{\hat{J}_0\cup\hat{K}_0\cup\hat{L}_0}$ descends to the return map $\phi_{J\cup K\cup L}$. Thus $f_{\hat{J}_0\cup\hat{K}_0\cup\hat{L}_0}$ is conjugated to $\phi_{J\cup K\cup L}$, as needed.

To finish the proof, we have to argue that if $f$ is $A$-renormalizable then the suspension flow of $f$ has a triple of clean cross sections $J$, $K$, and $L$ (in the appropriate cohomology classes) so that the map $h$ (in the definition of renormalizability) can be realized as the holonomy between $\hat{\Theta}$ and $\hat{\mathcal{S}}$ (as constructed in Section 3). Let $\bar{f} := h\circ f\circ h^{-1} : \hat{\Theta}\to\hat{\Theta}$. By definition of renormalizability, $\bar{f} = \hat{\mathcal{T}}$ on $\hat{\Theta}\setminus(\hat{J}_{w_3-1}\cup\hat{K}_{u_3-1}\cup\hat{L}_{v_3-1})$. Take $g : \hat{J}_{w_3}\cup\hat{K}_{u_3}\cup\hat{L}_{v_3}\to\hat{J}_0\cup\hat{K}_0\cup\hat{L}_0$ so that $\bar{f}|_{\hat{J}_{w_3-1}\cup\hat{K}_{u_3-1}\cup\hat{L}_{v_3-1}} = g\circ\hat{\mathcal{T}}$. By a routine construction there is a flow on $Q_0$ such that it enters $Q_0$ through $\hat{J}_{w_3}\cup\hat{K}_{u_3}\cup\hat{L}_{v_3}$, exits through $\hat{J}_0\cup\hat{K}_0\cup\hat{L}_0$, and realizes $g$ as the associated holonomy. After extending the flow equivariantly from the fundamental domain $Q_0$ to all of $\mathbb{R}^3/\Lambda$, we obtain nothing else than the cyclic covering of the suspension flow of $\bar{f}$. The quotient flow $\phi$ on $\mathbb{T}^3$ is then conjugated to the suspension flow of $f$. By construction, the quotients $J := \hat{J}_0/\mathbb{Z}^3$, $K := \hat{K}_0/\mathbb{Z}^3$, and $L := \hat{L}_0/\mathbb{Z}^3$ are the sought after clean cross sections. □

Any map of the form $\eta\circ\mathcal{R}_{J_0}(f)\circ\eta^{-1} : \mathbb{T}^2\to\mathbb{T}^2$ (where $\eta : J_0/\sim\to\mathbb{T}^2$ is a homeomorphism) is called a $(u,v)$-*renormalization of* $f$. Often it is more natural is to consider the torus $(J_0\cup K_0\cup L_0)/\sim$ obtained by identifying the sides of the *hexagon* $J_0\cup K_0\cup L_0$ (c.f. Figure 4.1) and the map $\mathcal{R}_{J_0\cup K_0\cup L_0} : (J_0\cup K_0\cup L_0)/\sim\to(J_0\cup K_0\cup L_0)/\sim$ induced by the return map $f_{J_0\cup K_0\cup L_0}$ to $J_0\cup K_0\cup L_0$ (c.f. (4.1)). Any rescaling of $\mathcal{R}_{J_0\cup K_0\cup L_0}$ to the original $\mathbb{T}^2$ is called an $A$-*renormalization of* $f$. Figure 4.3 compares the renormalization on $\mathbb{T}^2$ with the renormalization on $\mathbb{T}$. (Here, $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$ are two consecutive continued fraction convergents of the rotation number $\rho$, i.e. *a Farey pair* for $\rho$.) The picture for $d>2$ is readily obtained from a generic projection of the $d+1$-dimensional cube into $\mathbb{R}^d$.

The proof of Proposition 4.3, specifically (4.1), relates $A$-renormalizations of $f$ to the holonomy between the entrance and exit faces of the fundamental domain

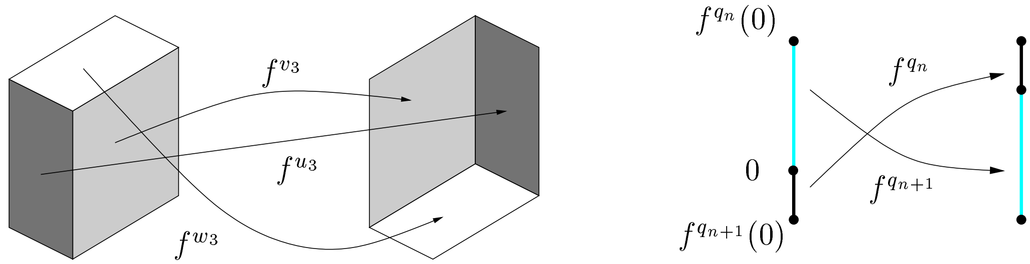


Figure 4.3: Renormalization in dimension 2 and 1.

$Q_0$ and yields the following corollary.

**Proposition 4.5** *In the context of Proposition 4.3, any A-renormalization is conjugated to the Poincaré return map to a cross section cohomologous to the affine torus that is the quotient of the plane through u, v, and w.*

We leave the proof as an exercise and remark only that a suitable cross section can be readily built by cutting $J$, $K$, and $L$ along their mutual intersections and by gluying (and slightly deforming) the resulting pieces togeather.

To finish, let us comment that any concrete *renormalization scheme* would entail selecting representatives from the whole conjugacy class of $A$-renormalizations. In that connection, we note that when $f$ is conformal (a translation), then there is a preferred conformal conjugacy class within all $A$-renormalizations:

**Remark 4.6** *If the torus $\mathbb{T}^2$ acted upon by an A-renormalizable map $f \in \mathbf{Diff}_0(\mathbb{T}^2)$ is equipped with a conformal structure with respect to which $f$ is conformal, then there are natural induced conformal structures on the renormalized tori $J_0/\sim$, $K_0/\sim$, $L_0/\sim$, and $(J_0 \cup K_0 \cup L_0)/\sim$. These structures are independent of the map $h$ in Definition 4.1.*

*Sketch of Proof of Remark 4.6.* By the measurable Riemann mapping theorem, it suffices to define the conformal structure almost everywhere. The interiors of $J_0$, $K_0$, and $L_0$ carry the conformal structure induced from $\mathbb{T}^2$ by the inclusion embedding. This induces a.e. conformal structure on $J_0/\sim$, $K_0/\sim$, $L_0/\sim$, and $(J_0 \cup K_0 \cup L_0)/\sim$. It is left to see that these conformal structures are independent of the choice of $h$ in Definition 4.1. Consider $J_0$ and $\bar{J}_0$ arising from two different such choices. By Proposition 4.3, $\mathcal{R}_{J_0}$ is conjugated to the return map $\phi_J$ to the cross-section $J$, and the conjugacy is induced by the holonomy from $J_0$ to $\hat{J}_0$. In the suspension covering $\mathbb{R}^3/\Lambda$, there is a unique flow-line through each interior point of $J_0$. Therefore, by considering all the flow-lines in $\mathbb{R}^3/\Lambda$ that avoid the boundaries of $J_0$ and $\bar{J}_0$, we get a natural a.e. defined mapping between $J_0$ and $\bar{J}_0$. This mapping is conformal. Similar arguments work for $K_0$ and $L_0$. □

## 5 Clean Triples of Cross Sections

Following the outline of the proof of Theorem 1.2 in Section 2, we turn to the existence of clean triples of cross sections for flows on $\mathbb{T}^3$ and prove Theorem 2.2.

Below we continue to use the standard identification of $H_1(\mathbb{T}^3)$ and $H^1(\mathbb{T}^3)$ with $\mathbb{R}^3$ of Cartesian coordinates $x, y, z$. Let us assume that the cohomology classes of sections $J$, $K$, and $L$ in the formulation of Theorem 2.2 are represented by the 1-forms $dz$, $dx$, and $dy$, respectively. (This can always be assured by conjugating the flow by a linear automorphism of $\mathbb{T}^3$.) We may also fix on $J$, $K$, and $L$ the orientation induced by those cohomology classes and assume that the flow pierces $J$, $K$, and $L$ in the positive direction. This forces the rotation set to be in *the positive octant*:

$$\rho(\phi) \subset \{(x, y, z) :\ x, y, z > 0\}, \tag{H}$$

where $\rho(\phi)$ is defined as the rotation set of the time-one map of the lifted flow $\tilde{\phi} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$; namely, $\rho(\phi) := \rho(\tilde{\phi}^1)$ (c.f. Section 2 in [15]).

In preparation for the proof of Theorem 2.2, note that, by virtue of Thom's transversality theorem, $J$, $K$, and $L$ can always be perturbed so that they are mutually transversal. In such case, $J \cap K$, $K \cap L$, and $L \cap J$ are 1-dimensional compact manifolds, i.e. finite unions of disjoint simple loops, which we call *intersection loops*. Likewise, the set $J \cap K \cap L$ consists of finitely many points, which we call *triple points*. Consider for a moment just one pair of cross sections, say $L$ and $J$, and let $\gamma_k$'s be the connected components of $L \cap J$ so that $L \cap J = \sum_{k=1}^{r} \gamma_k$. Observe that each $\gamma_k$ is a simple loop that is either null-homotopic or of homology type $\pm(1, 0, 0)$, common to both $J$ and $L$. In fact, if $\bullet : H_2(\mathbb{T}^3) \times H_2(\mathbb{T}^3) \to H_1(\mathbb{T}^3)$ is the intersection product, we have

$$\sum_{k=1}^{r} [\gamma_k]_{H_1(\mathbb{T}^3)} = [L \cap J]_{H_1(\mathbb{T}^3)} = [L]_{H_2(\mathbb{T}^3)} \bullet [J]_{H_2(\mathbb{T}^3)} = (1, 0, 0). \tag{5.1}$$

Hence, among the essential (i.e. homotopically non-trivial) intersection loops $\gamma_k$, all but one occur in homologically opposing pairs. Roughly speaking, those pairs result from $L$ and $J$ getting *folded through each other* as depicted in Figure 5.2. The null-homotopic $\gamma_k$'s result, in turn, from $L$ and $J$ *bumping into each other* as depicted in Figure 5.1. Elimination of such *folds* and *bumps* was carried out in [15] to show existence of clean pairs of cross sections. Here we extend the argument to handle the more complex intersection possibilities arising for three cross sections. The strategy will be to inductively diminish *the complexity of a triple* $(J, K, L)$ defined as $c(J, K, L) = (c_0, c_1)$ with

$$c_0 := b_0(J \cap K \cap L) - 1 \tag{5.2}$$

$$c_1 := b_0(J \cap K) + b_0(K \cap L) + b_0(L \cap J) - 3 \tag{5.3}$$

where $b_0$ stands for the number of connected components (the $0^{\text{th}}$-Betti number). Clearly, $(J, K, L)$ is clean if and only if $c(J, K, L) = (0, 0)$.

*Proof of Theorem 2.2:* We assume that $J$, $K$, and $L$ are transversal as in the preceding discussion. We shall describe four modifications, called *reductions*, each yielding another transversal triple of cross sections with either $c_1$ or $c_0$ diminished. Each reduction will be applicable under a different hypothesis, but they all will isotope one of the cross sections inside a judiciously chosen topological 3-disk or a solid torus inside $\mathbb{T}^3$. The proof will culminate in combining the reductions to lower the complexity to $(0, 0)$.

The first three reductions go back to [15] so we give here a somewhat less detailed exposition. A null-homotopic intersection loop is called *minimal* if it bounds a disk in one of $J$, $K$, or $L$ that contains no other null-homotopic intersection loops.

**Simple Bump Reduction.** *Hypothesis: There is a minimal null-homotopic intersection loop with no triple points.*
To fix attention, we assume that the null-homotopic loop, call it $\gamma$, appears among the components of $L \cap J$ and the disk $D_J$ bounded by $\gamma$ in $J$ contains no other null-homotopic intersection loops. Other cases are treated completely analogously.

Let $D_L$ be the disk bounded by $\gamma$ in $L$. $D_L \cup D_J$ forms a piecewise-smoothly embedded 2-sphere in $\mathbb{T}^3$ and thus bounds an embedded 3-disk $B$ in $\mathbb{T}^3$. This last assertion follows from a version of the classical Schönflies Theorem found in the appendix in [15].

Since the flow is transversal to $L$ and $J$, it either enters or exits $B$ across the entire two-dimensional interiors $D_L^o$ and $D_J^o$; and it cannot enter (or exit) through both sets because the hypothesis (H) on the rotation set precludes (forward or backward) invariance of $B$. By reversing the flow if necessary, we may assume then that the flow enters $B$ via $D_L^o$ and exits $B$ via $D_J^o$.

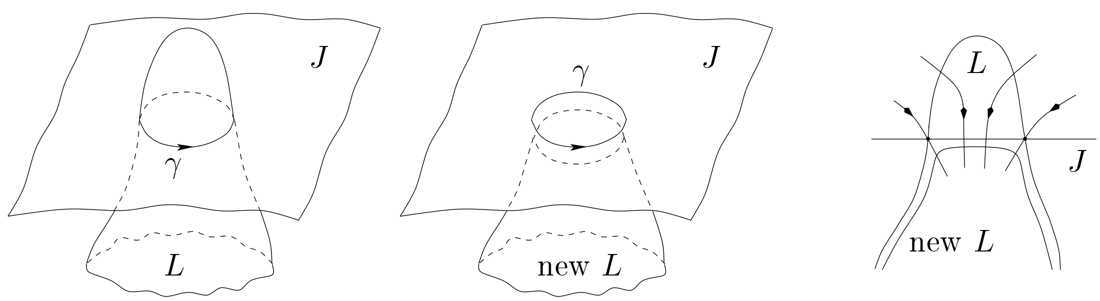


Figure 5.1: A Bump Removal.

We are now ready to modify $L$ in three steps as follows (see Figure 5.1).

- *Step 1. [Surgery] Replace $D_L$ with $D_J$.* Note that this modification can be effected by an isotopic deformation along the flow lines supported in $B$: a point $p$ of $D_L$ is connected inside $B$ with a unique point of $D_J$ by the flow line through $p$. (In particular, the isotopy class of $L$ in $\mathbb{T}^3$ is unaffected.)

- *Step 2. [Detachment] Replace $L$ with its image under the time-$\epsilon$-map $\phi^\epsilon$ for some small $\epsilon > 0$ to be specified later (in the proof of Claim 5.1).* This is meant to detach $L$ from $D_J$ (and thus restore transversality of $L$ and $J$).

- *Step 3. [Smoothing] Perturb $L$ to a smooth torus while preserving transversality to the flow and the two other cross sections.* What needs to be done here is *a rounding* of the (edge) singularity along $\gamma^\epsilon := \phi^\epsilon(\gamma)$ generated by Step 1. The perturbation must be $C^0$-small and $C^1$-small away from $\gamma^\epsilon$; how small is determined in the proof of Claim 5.1. We shall skip the details of this tedious but routine step (best performed after diffeomorphically straightening a small tubular neighborhood of $\gamma$ onto a neighborhood of the $\gamma$ depicted in Figure 5.1.)

The goal of simple bump reduction is summarized in the following claim.

**Claim 5.1** *The modification following steps 1 through 3 leaves complexity $c_0$ unchanged and diminishes complexity $c_1$; specifically, the number of the minimal null-homotopic intersection loops is diminished by at least one.*

*Proof of Claim 5.1:* First note that $D_J^o \cap K = \emptyset$ because (by the minimality of $\gamma$) all the loops in $J \cap K$ intersecting $D_J^o$ would have to be homotopically non-trivial and thus would have to hit $\gamma = \partial D_J$, contradicting the hypothesis on the absence of triple points on $\gamma$. By transversality, some neighborhood $V$ of the disk $D_J$ is free of $K$. Consequently, the modification of $L$ effected by Steps 1 and 2 does not affect $K \cap L$ or $J \cap K \cap L$ as long as $\epsilon > 0$ is small enough so that $\phi^t(D_J)$ is contained in $V$ for $0 < t < \epsilon$. And this is still true after Step 3 if the perturbation is sufficiently small. In particular, $c_0$ is indeed unaffected by this reduction.

At the same time, $\phi^\epsilon(J) \cap J = \emptyset$ for all small $\epsilon > 0$. In particular, $\phi^\epsilon(D_J) \cap J = \emptyset$ so that, after Step 2, $L$ no longer meets $J$ in some neighborhood of $B$; somewhat imprecisely, we say that $\gamma$ is removed form $L \cap J$ (perhaps together with some other loops in $D_L \cup J$). Again, this persists through Step 3 if the perturbation is sufficiently small. As a result, $c_1$ is diminished because, in the complement of a small neighborhood of $B$, the modification of $L$ amounts to a small $C^1$-perturbation so that the number of connected components of $L \cap J$ (in that complement) is preserved by virtue of transversality. □

**Complex Bump Reduction.** *Hypothesis: There is a minimal null-homotopic intersection loop that contains triple points.*
This reduction proceeds exactly as the just described *Simple Bump Reduction* and we assume that $\gamma$, $D_J$, $D_L$ and $B$ are as before with the only difference that now $\gamma$ contains a triple point. Thus $D_J$ intersects $K$, and $J \cap K \cap L$ will be affected.

**Claim 5.2** *Complexity $c_0$ is diminished.*

Note that we make no claim about $c_1$ (and $c_1$ may in fact increase).

*Proof of Claim 5.2:* The reasoning is very similar to that showing Claim 5.1, so we are brief. First, transversality assures that $L$ will not pick up new points in $J \cap K \cap L$ outside some small neighborhood of $B$ provided $\epsilon > 0$ and the perturbation in Step 3 are sufficiently small. Second, because $\phi^\epsilon(J) \cap J = \emptyset$, the modified $L$ no longer meets $J$ in some neighborhood of $B$; in particular, the points of $\gamma \cap K$ are removed from $J \cap K \cap L$. Thus $c_0$ is diminished. □

**Fold Reduction.** *Hypothesis:* $c_1 > 0$ *and there are no null-homotopic intersection loops.* All intersection loops are essential and one of the intersections $L \cap J$, $J \cap K$, and $K \cap L$ contains more than one loop. For specificity (and with no loss of generality), we assume that this is $L \cap J$ and we denote the components of $L \cap J$ by $\gamma_1, \ldots, \gamma_r$. Viewed on the two-torus $J$ (with $H_1(J)$ identified with $\mathbb{R}^2$ by taking the $x$ and $y$-directions as the basis), $\gamma_k$'s are disjoint $(1,0)$-loops, which cut $J$ into a number of annuli. The situation is analogous for $L$.

**Lemma 5.3** *There are components $\gamma_i$ and $\gamma_j$ of $L \cap J$ and closed annuli $A_J$ in $J$ and $A_L$ in $L$ with $\partial A_J = \partial A_L = \gamma_i \cup \gamma_j$ such that $A_J \cup A_L$ bounds a solid torus $B$ in $\mathbb{T}^3$. Moreover, one can require that $A_J^o \cap L = \emptyset$*

*Proof:* The idea is to make sure that $A_J \cup A_L$ is a 2-torus that deforms to a loop in $\mathbb{T}^3$. The plan is to select $\gamma_i$ and $\gamma_j$ so that $J$ and $L$ intersect along $\gamma_i$ and $\gamma_j$ with *opposite signs* (in the sense of intersection homology). This is easily done at the level of lifts as follows. Consider some (connected) lifts $\tilde{J}$ and $\tilde{L}$ of $J$ and $L$. Note that $\tilde{J} = \tilde{J} + \mathbb{Z} \times \mathbb{Z} \times 0$ and $\tilde{L} = \tilde{L} + \mathbb{Z} \times 0 \times \mathbb{Z}$, so if $p \in \tilde{L} + (k_1, k_2, k_3) \cap \tilde{J} + (l_1, l_2, l_3) = \tilde{L} + (0, k_2, 0) \cap \tilde{J} + (0, 0, l_3)$ for some $k, l \in \mathbb{Z}^3$, then $p - (0, k_2, l_3) \in \tilde{L} - (0, 0, l_3) \cap \tilde{J} - (0, k_2, 0) = \tilde{L} \cap \tilde{J}$. Thus the total preimage of $L \cap J$ under the natural projection to $\mathbb{T}^3$ is

$$(\tilde{L} + \mathbb{Z}^3) \cap (\tilde{J} + \mathbb{Z}^3) = (\tilde{L} \cap \tilde{J}) + 0 \times \mathbb{Z} \times \mathbb{Z}.$$

It follows that $\tilde{L} \cap \tilde{J}$ must have more than one connected component because $L \cap J$ is not a single loop.

Consider then two distinct components $\tilde{\gamma}_i$ and $\tilde{\gamma}_j$ of $\tilde{L} \cap \tilde{J}$ and the *strips* (i.e. topological $[0,1] \times \mathbb{R}$) $\tilde{A}_J$ and $\tilde{A}_L$ they bound in $\tilde{J}$ and $\tilde{L}$. We may take $\tilde{\gamma}_i$ and $\tilde{\gamma}_j$ *adjacent* in $\tilde{J}$ in the sense that no other component of $\tilde{L} \cap \tilde{J}$ sits in $\tilde{A}_J^o$. Upon descending to $\mathbb{T}^3$, we get two components $\gamma_i$ and $\gamma_j$ of $L \cap J$, and annuli $A_J \subset J$ and $A_L \subset L$ with $\partial A_J = \partial A_L = \gamma_i \cup \gamma_j$. Also, $\gamma_i$ and $\gamma_j$ are *adjacent* in $J$ so that $A_J$ is a connected component of $J \setminus L$, which guarantees $A_J^o \cap L = \emptyset$. Now, $\tilde{A_J} \cup \tilde{A_L}$ is a piecewise-smooth cylinder in $\mathbb{R}^3$ that is equivariant under $\mathbb{Z}(0,1,0)$ (by the construction). Therefore, $A_J \cup A_L$ is a a piecewise-smooth 2-torus in $\mathbb{T}^3$ embedded so that the induced map $H_1(\mathbb{T}^2) \to H_1(\mathbb{T}^3)$ has rank one. By a version of Alexander Torus Theorem (Theorem 8.10 [15]), $A_J \cup A_L$ bounds a solid torus $B$ in $\mathbb{T}^3$. □

The flow is transversal to $A_J$ and $A_L$, and (by reversing time if necessary) we may assume that it enters $B$ via $A_L^o$ and exits via $A_J^o$. This hinges on the fact that $B$ cannot contain a forward or backward semi-orbit because $\pm(0,1,0)$ is precluded from the rotation set $\rho(\phi)$ by the hypothesis (H).

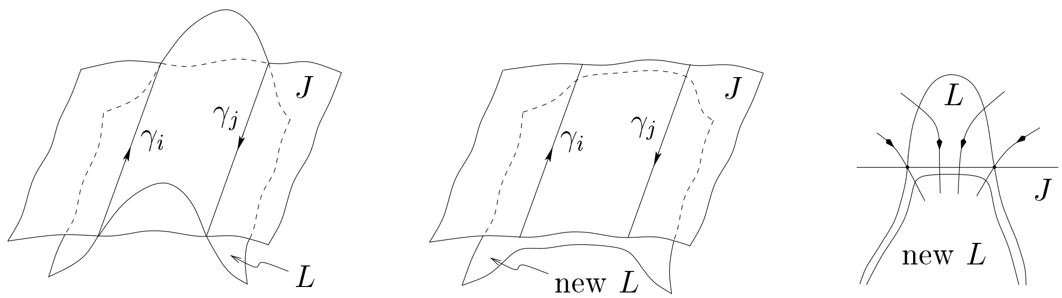


Figure 5.2: A Fold Removal.

We modify $L$ in three steps analogous to those in the previous two reductions (see Figure 5.2).

- *Step 1. Replace $A_L$ with $A_J$ (via an isotopy in $B$ along the flow lines).*
- *Step 2. Replace $L$ with its image under the time-$\epsilon$-map for small $\epsilon > 0$.*
- *Step 3. Smoothen $L$ by a small perturbation.*

**Claim 5.4** *The complexity $c_0$ is diminished.*

*Proof of Claim 5.4.* By transversality, for small $\epsilon > 0$, $L$ will pick up no new points in $J \cap K \cap L$ outside some small neighborhood of $B$ where the modification amounts to a small $C^1$-perturbation. At the same time, $\gamma_i$ and $\gamma_j$ are removed from $L \cap J$; therefore, the points of $(\gamma_i \cup \gamma_j) \cap K$ are removed from $J \cap K \cap L$ and $c_0$ is diminished. Here we used that $(\gamma_i \cup \gamma_j) \cap K$ is nonempty (in fact, contains at least two points) because $\gamma_i$ and $J \cap K$ have a non-zero intersection number in $J$, and so do $\gamma_j$ and $J \cap K$. □

Finally we describe the most involved reduction, which has no analogue in [15] so we proceed more carefully.

**Triple Reduction.** *Hypothesis: $c_1 = 0$ and $c_0 > 0$.*
Since $c_1 = 0$, $J \cap K$, $K \cap L$, and $L \cap J$ are simple closed loops. Note that where any two of these loops intersect all three must intersect, and the intersection points form the set of triple points, $T := J \cap K \cap L$. As in the previous reductions, the plan is to find a suitable 3-disk in $\mathbb{T}^3$ on a neighborhood of which one cross section will be modified to diminish $\#T = c_0 + 1$.

Viewed in the torus $J$, $J \cap K$ and $L \cap J$ are two simple loops of homology classes $(0,1)$ and $(1,0)$ that intersect transversally along $T$. Because $\#T > 1$, we claim that there must exist two subarcs $\alpha \subset J \cap K$ and $\gamma \subset L \cap J$ such that $\alpha \cup \gamma$ bounds a disk $D_J$ in $J$ (c.f. Figure 5.3). Moreover, by taking such disk to be minimal with respect to inclusion, we may require that $L \cup J$ meets $\alpha$ and $J \cap K$ meets $\gamma$ only at the common endpoints of $\alpha$ and $\gamma$, which endpoints we further denote by $t$ and $t'$. The above two facts are easily shown via the Jordan Theorem applied in the plane covering $J$. (In particular, $t$ and $t'$ can be found as the *adjacent* intersection points between some connected lifts of $J \cap K$ and $L \cap J$ — c.f. the proof of Lemma 5.3.)

By similar considerations of the way $\alpha$ intersects $K \cap L$ in $K$, one can see that of the two subarcs of the simple loop $K \cap L$ that terminate at $t$ and $t'$, one — call it $\beta$ — is such that $\alpha \cup \beta$ bounds a disk $D_K$ in $K$ (c.f. Figure 5.3). Finally, being the boundary of the disk $D_J \cup D_K$ in $\mathbb{T}^3$, $\beta \cup \gamma$ is a contractible loop and therefore bounds some disk $D_L$ in $L$. The union $D_J \cup D_K \cup D_L$ is a piecewise smooth sphere in $\mathbb{T}^3$, which we denote by $\Delta$.

There is a 3-disk $B$ bounded by $\Delta$ in $\mathbb{T}^3$. Observe the following restrictions on how the interior $B^o$ can meet $J \cup K \cup L$ (which may happen since nothing prevents $\beta$ from containing many triple points).

**Fact 5.5** *(i)* $D_J^o \cap (L \cup K) = \emptyset$;

*(ii)* $D_L^o \cap K = \emptyset$;

*(iii)* $D_K^o \cap L = \emptyset$;

*(iv)* $\Delta \cap T = \beta \cap T$.

*Proof:* (i) Suppose that $D_J^o \cap L \neq \emptyset$. This means that the curve $L \cap J$ crosses the boundary of $D_J$: $L \cap J \cap (\alpha \cup \gamma) \neq \emptyset$. However, being simple, $L \cap J$ cannot cross its own subarc $\gamma$, and $L \cap J \cap \alpha^o = \emptyset$ by the construction of $\alpha$ (where $\alpha^o$ stands for the one-dimensional interior of $\alpha$). Likewise, one can see that $D_J^o \cap K \neq \emptyset$ would contradict $\alpha \subset J \cap K$ and $J \cap K \cap \gamma^o = \emptyset$.
(ii) As above, $\partial D_L = \beta \cup \gamma$, and $K \cap L$ can enter $D_L$ neither through $\beta$ because $\beta \subset K \cap L$ nor through $\gamma^o$ because that would put a triple point in $\gamma^o$ (contradicting $J \cap K \cap \gamma^o = \emptyset$).
(iii) Otherwise, the curve $K \cap L$ would have to cross $\partial D_K = \alpha \cup \beta$, which is impossible for a similar reason as in (ii).
(iv) Parts (i), (ii), (iii), imply that $\Delta \cap T \subset (\alpha \cup \beta \cup \gamma) \cap T$. However, $(\alpha \cup \gamma) \cap T = \{t, t'\} \subset \beta$ because $\alpha^o$ and $\gamma^o$ contain no triple points by construction. The equality (iv) follows. □

We set out to analyze $\Delta \cap J$, which may be quite complicated if $\beta^o \cap T \neq \emptyset$, as illustrated in Figure 5.3. The goal is to find another 2-sphere $\Delta_1 \subset B$ that bounds a 3-disk $B_1$ in $\mathbb{T}^3$ with a property that ${B_1}^o \cap (J \cup K \cup L) = \emptyset$. Let $\mathcal{D}$ be the family of all the connected components of $B \cap J$ with the exception of $D_J$ — which is the only component of $B \cap J$ entirely contained in $\Delta$. The transversality between $J$, $K$, and $L$, assures that $D \in \mathcal{D}$ is a compact surface with a piecewise smooth boundary contained in $\Delta$.

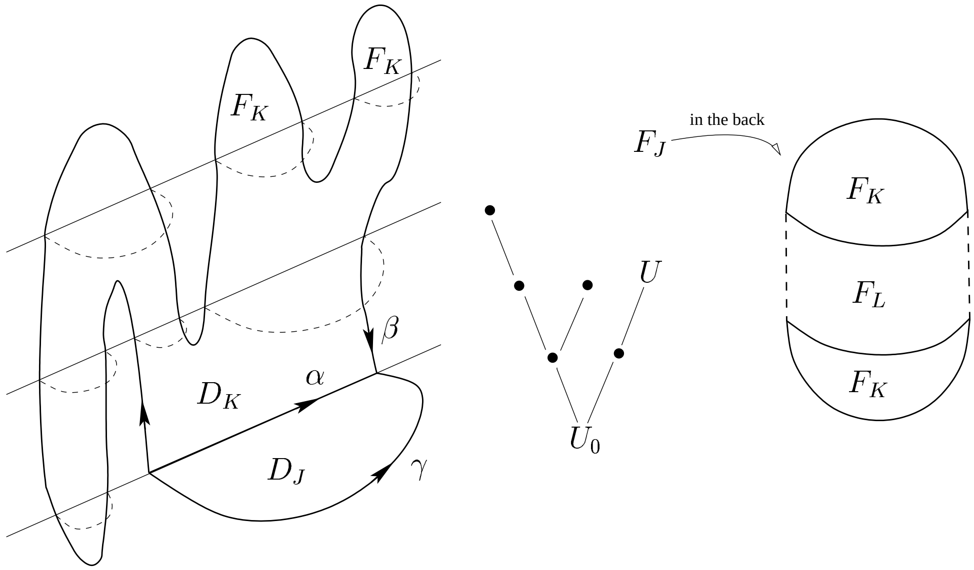


Figure 5.3: The 3-disk $B$ sliced by lifts of $J$, the graph $\mathcal{G}$, and the sphere $\Delta_1$. ($J$ and $K$ are depicted as affine tori. The dashed intersection lines reveal the locus of $L$.)

**Fact 5.6** *Each $D \in \mathcal{D}$ is a closed topological 2-disk.*

*Proof:* Let $\tilde{B} \subset \mathbb{R}^3$ be a connected lift of $B$. In $\tilde{B}$, which is a topological 3-disk, there are also contained unique lifts $\tilde{\Delta}$, $\tilde{D}_J$, $\tilde{D}_K$, and $\tilde{D}_L$ of the sphere $\Delta$ and $D_J$, $D_K$, and $D_L$, respectively.

The plane $\tilde{K}$ that is the connected lift of $K$ containing $\tilde{D}_K$ separates $\mathbb{R}^3$. By (i) and (ii) of Fact 5.5, $\tilde{D}_J^o$ and $\tilde{D}_L^o$ do not intersect $\tilde{K} + \mathbb{Z}(1,0,0)$ and thus the sphere $\tilde{\Delta} = \partial\tilde{B} = \tilde{D}_J \cup \tilde{D}_K \cup \tilde{D}_L$ intersects $\tilde{K} + \mathbb{Z}(1,0,0)$ solely along the disk $\tilde{D}_K \subset \tilde{K}$. It follows that $\tilde{B}$ is entirely contained on one side of $\tilde{K}$ and that $\tilde{B}$ avoids $\tilde{K} + (m,0,0)$ for $m \neq 0$ (c.f. Figure 5.3).

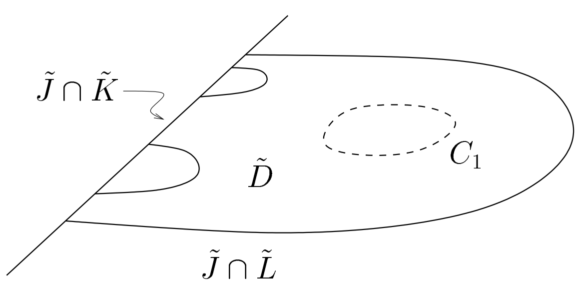


Figure 5.4: A typical element of $\mathcal{D}$ in $\tilde{J}$ (depicted as the horizontal plane).

Now, fix $D \in \mathcal{D}$. We have to show that $D$ is a topological disk. Let $\tilde{D} \subset \tilde{B}$ be a connected lift of $D$ and $\tilde{J}$ be the connected lift of $J$ containing $\tilde{D}$. By definition of $\mathcal{D}$, $\partial D \subset (\Delta \setminus D_J) \cap J \subset (J \cap K) \cup (L \cap J)$ so that $\partial \tilde{D} \subset (\tilde{J} \cap (\tilde{K} + \mathbb{Z}(1,0,0))) \cup ((\tilde{L} + \mathbb{Z}(0,1,0)) \cap \tilde{J})$ where $\tilde{L}$ is a connected lift of $L$. By our previous argument, $\partial \tilde{D} \subset (\tilde{J} \cap \tilde{K}) \cup ((\tilde{L} + \mathbb{Z}(0,1,0)) \cap \tilde{J})$, and $\tilde{D}$ is entirely contained on one side of $\tilde{J} \cap \tilde{K}$ in $\tilde{J}$ — see Figure 5.4. Therefore, on the force of the Jordan Theorem, there is no $\tilde{J} \cap \tilde{K}$ inside the Jordan curve $C$ constituting the boundary of the unbounded component of $\tilde{J} \setminus \tilde{D}$. Hence, should there be another component $C_1$ of $\partial \tilde{D}$, we would have $C_1 \subset (\tilde{L} + \mathbb{Z}(0,1,0)) \cap \tilde{J}$ (since $C_1$ sits inside $C$). This contradicts $L \cap J$ being an essential loop. It follows that $C$ is the sole component of $\partial \tilde{D}$, which makes $\tilde{D}$ a disk. $\Box$

Thus each $D \in \mathcal{D}$ is a 2-disk in $B$ that intersects $\Delta = \partial B$ along its boundary circle and thus separates $B$ into two sets the closures of which are again 3-disks. Any one of those two *new* 3-disks may be further separated in an analogous manner by another member of $\mathcal{D}$ and so on. (The above assertions of course depend on the Schönflies Theorem for piecewise smooth embeddings; however, we leave off their routine proofs.) We need some rudimentary understanding of the combinatorics of the resulting partition of $B$ into 3-disks.

To be more precise, let $\mathcal{U}$ be the family of the closures of the connected components of $B \setminus J$. We shall say that $U_1 \in \mathcal{U}$ is ***adjacent*** to $U_2 \in \mathcal{U}$ along $D \in \mathcal{D}$ iff $D \subset \partial_B U_1 \cap \partial_B U_2$, where the subscript $B$ indicates that the boundary is taken relative to $B$. I view of the preceding remarks, $\mathcal{D}$ collects exactly the connected components of the relative boundaries $\partial_B U$ for all $U \in \mathcal{U}$; and $U_1 \in \mathcal{U}$ is adjacent to $U_2 \in \mathcal{U}$ along some $D \in \mathcal{D}$ iff $U_1 \cap U_2 \neq \emptyset$. Let $\mathcal{G}$ be the (undirected) abstract graph with vertices $\mathcal{U}$ and edges $\mathcal{D}$ where an edge $D \in \mathcal{D}$ is joining $U_1$ and $U_2$ iff $U_1$ is adjacent to $U_2$ along $D$ (see Figure 5.3).

**Fact 5.7** *$\mathcal{G}$ is a tree.*

*Proof:* We have to show that $\mathcal{G}$ has no non-trivial cycles. Consider a non-backtracking path in $\mathcal{G}$, i.e. a sequence of vertices $U_0, U_1, \ldots, U_{m+1}$ and a sequence of edges $D_1, D_2, \ldots, D_m$ such that $D_i$ joins $U_i$ to $U_{i+1}$ and $D_i \neq D_{i+1}$ for $i = 1, \ldots, m$. Recall that $D_i$ separates $B$ into two topological 3-disks whose closures we denote $W_i^-$ and $W_i^+$ so that $W_i^-$ contains $U_i$ and $W_i^+$ contains $U_{i+1}$. Any two of the four 3-disks $\{W_i^\pm, W_{i+1}^\pm\}$ either are disjoint or are contained in one another. Since $D_{i+1} \subset W_{i+1}^+$ and $D_{i+1} \subset U_{i+1} \subset W_i^+$, we have $W_i^+ \cap W_{i+1}^+ \neq \emptyset$; and therefore $W_{i+1}^+ \subsetneq W_i^+$ or $W_{i+1}^+ \supsetneq W_i^+$. The latter inclusion is however precluded by $U_{i+1} \subset W_i^+$ and $U_{i+1} \not\subset W_{i+1}^+$. Thus $W_{i+1}^+ \subsetneq W_i^+$ and we see that $W_1^+ \supsetneq W_2^+ \supsetneq \ldots \supsetneq W_m^+$. It follows that $D_1 \neq D_m$, which shows that the path cannot be a cycle. $\Box$

The graph $\mathcal{G}$ always has a vertex $U_0 \in \mathcal{U}$ that contains $D_J$; we refer to $U_0$ as *the root of* $\mathcal{G}$. Unless $\mathcal{G}$ consists of a single vertex, it also has *ends*, i.e. vertices other than $U_0$ that belong to only one edge.

**Fact 5.8** *Suppose that $U \in \mathcal{U}$ is an end of $\mathcal{G}$ and that $F_J = U \cap J$, $F_K = U \cap K$, and $F_L = U \cap L$.*

*(i) $U$ is a topological 3-disk, $U^o \cap (J \cup K \cup L) = \emptyset$, $\partial U = F_J \cup F_K \cup F_L$, and $F_J$ is a 2-disk;*

*(ii) for each of the sets $F^o_J$, $F^o_K$, $F^o_L$, the flow either exists $U$ across the whole set or enters $U$ across the whole set.*

Note that $F_K$ and $F_L$ may be disconnected (as the $F_K$ in Figure 5.3).

*Proof:* (i) $U$ being an end of $\mathcal{G}$ means that there is a single $D \in \mathcal{D}$ such that $D = \partial_B U$ and therefore $F_J = U \cap J = \partial_B U = D$ by the definition of $\mathcal{U}$. Thus $U$ is the 3-disk bounded by the two 2-sphere formed in the union $F_J \cup (\Delta \setminus D_J)$.

Let us argue that $U^o \cap (J \cup K \cup L) = \emptyset$. That $U^o \cap J = \emptyset$ follows from the definition of $\mathcal{U}$. If $U^o \cap L \neq \emptyset$, then $B^o \cap L \neq \emptyset$, and we could use $\partial B = \Delta = D_J \cup D_K \cup D_J$ and transversality of $J$, $K$, and $L$ to infer $L \cap (D^o_J \cup D^o_K) \neq \emptyset$, which contradicts (i) and (iii) of Fact 5.5. Similarly, $U^o \cap K = \emptyset$ because $K \cap (D^o_L \cup D^o_J) \neq \emptyset$ would contradict (i) and (ii) of Fact 5.5.

Finally, definition of $\mathcal{U}$ implies that $\partial U \subset J \cup K \cup L$, and so $\partial U = F_J \cup F_K \cup F_L$ follows immediately from $U^o \cap (J \cup K \cup L) = \emptyset$.
(ii) The flow either enters or exits $B$ across the whole 2-disk $D^o_K$ by transversality and connectedness of $D_K$. Because $U \subset B$ with $F^o_K \subset D^o_K$, at the points of $F^o_K$, the flow enters (exits) $U$ iff it enters (exits) $B$, which implies the assertion (ii) regarding $F_K$. The arguments for $F_L$ is analogous. The case of $F_J$ follows by transversality and connectedness[2] of $F_J = D$ (supplied by (i)). □

How the flow intersects $U$ is further constrained by Ważewski's theorem (see e.g. [3].) Let $U^+$ be *the exit set of* $U$, i.e., $U^+ := \{p \in U : \forall_{\epsilon>0} \exists_{0<t<\epsilon}\ \phi^t(p) \notin U\}$. Also, let $U^-$ be *the entrance set of* $U$, i.e., the exit set under the reversed flow.

**Fact 5.9** *(i) $U^+$ and $U^-$ are closed subsets of $U$;*

*(ii) $U$ is a Ważewski set under the flow and its inverse*[3]*;*

*(iii) $U^-$ and $U^+$ are topological 2-disks, each coinciding with one set or a union of two sets from the collection $\{F_J, F_K, F_L\}$.*

*Proof:* (i) We shall argue only for $U^+$ as the same arguments can be applied to $U_-$ after reversing the flow. It suffices to show that $W^+ := \{p \in \partial U : \exists_{\epsilon>0} \forall_{0<t<\epsilon}\ \phi^t(p) \in U\}$ is open in $\partial U$. That $W^+ \cap (F^o_J \cup F^o_K \cup F^o_L)$ is open follows from transversality (of the flow to $J$, $K$, and $L$), which leaves the openness at the points of the singular locus: $(F_J \cap F_K) \cup (F_K \cap F_L) \cup (F_L \cap F_J)$.

We consider first a (triple) point $p \in F_J \cap F_K \cap F_L$. By routine analytics, there is a small neighborhood $V$ of $p$ on which $J \cap V$, $K \cap V$, and $L \cap V$ can be *straightened* via a diffeomorphism of $V$ into subsets of the three standard coordinate planes in

[2]This is the only place where it is critical that $U$ is an endpoint, not some other element of $\mathcal{U}$ — all of which are topological 3-disks.

[3]That is $(U, U^+)$ constitutes an index pair in the sense of Conley

$\mathbb{R}^3$. In particular, the complement of $J \cup K \cup L$ in $V$ has eight components, each occupying (mapping to) a different *octant* in $\mathbb{R}^3$. Observe that $U \cap V$ occupies only one of the *octants* because $U^o$ is free of $J \cup K \cup L$ by (i) of Fact 5.8. By transversality of the flow to $J$, $K$, and $L$, if $p \in W^+$, then the vector generating the flow at $p$ must point strictly inside the octant occupied by $U$, and it follows that $W^+$ is open at $p$.

Now consider $p \in (F_J \cap F_K) \cup (F_K \cap F_L) \cup (F_L \cap F_J) \setminus (F_J \cap F_K \cap F_L)$. Again, two of the sets $J$, $K$, and $L$ meet at $p$, and there is a neighborhood $V$ of $p$ on which these two sets *straighten out* into subsets of two of the standard coordinate planes in $\mathbb{R}^3$. Arguing as before — this time in terms of the *quadrants* associated with the two planes — we see that $U \cap V$ must occupy only one of the *quadrants* and that, if $p \in W^+$, then $W^+$ is open at $p$.
(ii) This amounts to realizing that, since $U$ and $U^{\pm}$ are closed, the axioms a) and b) defining Ważewski sets in definition 2.2 on page 24 in [3] are automatically satisfied.
(iii) As a consequence of the hypothesis (H) on the rotation set, the maximal invariant subset of $U$ is empty, be that under the flow or its inverse. Ważewski's theorem, applicable on the force of (ii), asserts that both $U^+$ and $U^-$ are deformation retracts of $U$. Hence, $U^+$ and $U^-$ are both contractible, which makes them 2-disks. The assertion (iii) follows by taking into account that each of the sets $F_J^o$, $F_K^o$, and $F_L^o$ is contained in either $U^+$ or $U^-$ by (iii) of Fact 5.8. □

We are now ready to carry out the reduction. To define a suitable 3-disk $B_1$ (for localizing the isotopy), we consider the tree $\mathcal{G}$. Either $\mathcal{G}$ has only one vertex $B$, in which case we set $B_1 := B$, or there is an end $U$ of $\mathcal{G}$ as in Fact 5.9, in which case we set $B_1 := U$. The boundary of $B_1$ is made of three sets $F_J = B_1 \cap J$, $F_K = B_1 \cap K$, and $F_L = B_1 \cap L$. (If $B_1 = B$ all three sets are 2-disks, whereas if $B_1 = U$ only $F_J$ must be a 2-disk and $F_K$ and $F_L$ may be disconnected.) In any case, taking into account Fact 5.9, the flow enters and then exits $B_1$ across two 2-disks, each coinciding with one set or a union of two sets from the collection $\{F_J, F_K, F_L\}$. For specificity, after perhaps reversing the flow, we shall assume that $\phi$ enters $B_1$ via $F_L^o$ and exists via $(F_J \cup F_K)^o$. The other cases are handled completely analogously by simply permuting the notation pertaing to $J$, $K$, and $L$. Note that $\partial F_L$ cannot be entirely contained in any one of the essential loops $L \cap J$ or $K \cap L$ and therefore $\partial F_L$ contains some (at least two) triple points. We shall modify $L$ so that those triple points are removed via the familiar three steps:

- *Step 1.* Replace $F_L$ with $F_J \cup F_K$ via an isotopy in $B_1$ (along the flow lines).
- *Step 2.* Replace $L$ with its image under the time-$\epsilon$-map for small $\epsilon > 0$.
- *Step 3.* Smoothen $L$.

**Claim 5.10** *The modification following the steps 1 through 3 reduces $c_0$.*

The proof of the claim is virtually the same as that of Claim 5.2, we skip it.

*Conclusion of the proof of Theorem 2.2:* We finish the proof by combining the four reductions described above to yield a triple of cross sections with $c_0 = c_1 = 0$ (i.e. a clean triple). By construction, the reductions preserve the isotopy classes of the cross sections (as well as their mutual transversality).

First consider the special case when already $c_0 = 0$. Then each of $J \cap K$, $K \cap L$, or $L \cap J$ must contain exactly one essential loop by considerations of intersection homology (c.f. (5.1)). If also $c_1 = 0$ we are done. Otherwise, $c_1 > 0$ and there are null-homotopic intersection loops, all free of triple points because $c_0 = 0$. After picking a minimal such loop, *simple bump reduction* lowers $c_1$ while preserving $c_0 = 0$ (see Claim 5.1). Iteration of this process leads to $c_0 = c_1 = 0$.

We now turn attention to the case when $c_0 > 0$. In view of the preceding discussion, we will be done by induction if we succeed in lowering $c_0$ (while perhaps increasing $c_1$). If $c_1 = 0$, then $c_0$ is lowered by *triple reduction* as asserted by Claim 5.10. If $c_1 > 0$, then *simple bump reduction* can be used to iteratively remove the null-homotopic loops that are free of triple points without affecting $c_0$ (see Claim 5.1). This may yield $c_1 = 0$, a case we already resolved, or we may still have $c_1 > 0$. If $c_1 > 0$, then either there is a minimal null-homotopic intersection loop with triple points and we can apply *complex bump reduction* to reduce $c_0$ (see Claim 5.2), or there are no null-homotopic intersection loops and we are in position to lower $c_0$ via *fold reduction* (see Claim 5.4). □

# 6 Appendix: Auxiliary Results

In Section 3, we use the following classical connection between flux and volume (which, for lack of a convenient literature reference[4], we supply with a proof).

**Theorem 6.1 (Reynolds' Transport Theorem)** *Let $M$ be a Riemannian manifold with the volume form $\sigma$ and let $\eta : N \to M$ be a smooth submanifold of codimension 1. Suppose that $\phi : \mathbb{R} \times M \to M$ is a flow of a vector field $\mathcal{X}$ on $M$ and $N^t := \phi^t \circ \eta(N)$. If*

$$\Gamma := \bigcup_{t=0}^{t=t_0} N^t \times \{t\} \subset M \times \mathbb{R}$$

*and $\pi : M \times \mathbb{R} \to M$ is the natural projection, then*

$$\int_\Gamma \pi_*(\sigma) = \int_0^{t_0} \left[\text{Flux of } \mathcal{X} \text{ through } N^t\right] dt. \tag{6.1}$$

[4]c.f. page 11 in *Fundamental Mechanics of Fluids* by I.G. Currie

If $\pi|_\Gamma : \Gamma \to M$ is an embedding, then the left hand side of the formula (6.1) can be interpreted as the signed volume (trapped between $N^0$ and $N^{t_0}$), which is how we use the result in Section 3.

*Proof.* Let $\eta^t := \phi^t \circ \eta$ and $g : N \times \mathbb{R} \to M \times \mathbb{R}$ be given by $(y,t) \mapsto (\eta^t(y), t)$. We have

$$\int_\Gamma \pi^*(\sigma) = \int_{N\times[0,t_0]} g^* \circ \pi^*(\sigma).$$

The flux through $N^t$ equals[5]

$$\int_{N^t} i_{\mathcal{X}}(\sigma) = \int_N (\eta^t)^*(i_{\mathcal{X}}(\sigma))$$

where $i_{\mathcal{X}}$ is the inner multiplication by $\mathcal{X}$. It suffices then to show the following equality of $n$-forms on $N \times \mathbb{R}$:

$$g^* \circ \pi^*(\sigma) = (\eta^t)^*(i_{\mathcal{X}}(\sigma)) \wedge dt. \tag{6.2}$$

At a point $(y,t) \in N \times \mathbb{R}$, a tangent vector to $N \times \mathbb{R}$ has the form $v + a\frac{\partial}{\partial t}$ where $v$ is tangent to $N$. The image of $v + a\frac{\partial}{\partial t}$ under the differential of $\pi \circ g$ is

$$D(\pi \circ g)(y,t)\left(v + a\frac{\partial}{\partial t}\right) = D\eta^t v + a\mathcal{X}$$

where the spatial derivative $D\eta^t$ is evaluated at $(y,t)$ and $\mathcal{X}$ is taken at $\eta^t(y)$. Thus we verify (6.2) by evaluating the forms on $n$-tuples of vectors:

$$g^* \circ \pi^*(\sigma)\left(\left(v_i + a_i\frac{\partial}{\partial t}\right)_{i=1}^n\right) = \sigma\left(\left(D\eta^t v_i + a_i\mathcal{X}\right)_{i=1}^n\right)$$

$$= \sum_{j=1}^n \sigma\left(D\eta^t v_1, \ldots, \mathcal{X}, \ldots, D\eta^t v_n\right) a_j = ((\eta^t)^*(i_{\mathcal{X}}(\sigma)) \wedge dt)\left(\left(v_i + a_i\frac{\partial}{\partial t}\right)_{i=1}^n\right).$$

□

In Section 2, we used the following lemma from differential topology.

**Lemma 6.2** *If $(J, K, L)$ is a clean triple of $C^1$-smoothly embedded tori in $\mathbb{T}^3$, then there exist a homotopic to the identity $C^1$-diffeomorphism $h : \mathbb{T}^3 \to \mathbb{T}^3$ that maps each of $J$, $K$, and $L$ into an affine two torus in $\mathbb{T}^3$.*

This result can be derived from the classical Schönflies Theorem as suggested below.

*Sketch of Proof:* From the definition of a clean triple there is a basis $u, v, w \in \mathbb{Z}^3$ of $\mathbb{Z}^3$ over $\mathbb{Z}$ such that $J$, $K$, and $L$ are cohomologous to the affine tori obtained

[5]See page 411 in *Manifolds, Tensor Analysis, and Applications* by Abraham, Marsden, Ratiu.

as the quotients of the planes $\text{lin}(u, v)$, $\text{lin}(v, w)$, and $\text{lin}(w, u)$, respectively. With no loss of generality we assume that $u = (1, 0, 0)$, $v = (0, 1, 0)$, and $w = (0, 0, 1)$.

Let $j, k, l : \mathbb{T}^2 \to \mathbb{T}^3$ be embeddings such that $j(\mathbb{T}^2) = J$, $k(\mathbb{T}^2) = K$, and $l(\mathbb{T}^2) = L$. The preimages $j^{-1}(J \cap K)$ and $j^{-1}(L \cap J)$ that are two transversal essential loops in $\mathbb{T}^2$ that intersect transversally at a single point. By perhaps precomposing $j$ with an appropriate diffeomorphism of $\mathbb{T}^2$, we may secure that $j^{-1}(L \cap J)$ and $j^{-1}(J \cap K)$ are the affine loops covered by $\text{lin}((1, 0))$ and $\text{lin}((0, 1))$, respectively. (This amounts to an analogue of the lemma under consideration in one lower dimension, with a similar but simpler proof than what follows.) Likewise, we shall require that $k^{-1}(J \cap K)$ and $k^{-1}(K \cap L)$ are covered by $\text{lin}((1, 0))$ and $\text{lin}((0, 1))$ and $l^{-1}(K \cap L)$ and $l^{-1}(L \cap J)$ are covered by $\text{lin}((1, 0))$ and $\text{lin}((0, 1))$, respectively.

Let $Q = [0, 1]^3$ and $\tilde{j}, \tilde{k}, \tilde{l} : \mathbb{R}^2 \to \mathbb{R}^3$ be the lifts of $j, k, l$ normalized so that $j(0) = k(0) = l(0) = 0$. Consider the map $g : \partial Q \to \mathbb{R}^3$ defined by the following equalities where $x, y \in [0, 1]$

$$g(x, y, 1) - (0, 0, 1) = g(x, y, 0) = \tilde{j}(x, y) \tag{6.3}$$

$$g(1, x, y) - (1, 0, 0) = g(0, x, y) = \tilde{k}(x, y) \tag{6.4}$$

$$g(x, 1, y) - (0, 1, 0) = g(x, 0, y) = \tilde{l}(y, x). \tag{6.5}$$

Roughly speaking, $g$ is obtained by pasting together two copies of each $j$, $k$, and $l$. The fact that $(J, K, L)$ is a clean triple assures that $g$ is a piecewise smooth embedding of the piecewise smooth sphere $\partial Q$ into $\mathbb{R}^3$. Therefore, $g$ is not *wild*, say it extends to a bicollared neighborhood of $\partial Q$, and the Schönflies Theorem (see [2]) assures that $g$ extends to an embedding $G : Q \to \mathbb{R}^3$. Furthermore, $G$ extends to a $\mathbb{Z}^3$-equivariant map $F : \mathbb{R}^3 \to \mathbb{R}^3$ by setting $F(p) := G(p + v)$ for all $p \in \mathbb{R}^3$ where $v \in \mathbb{Z}^3$ is taken such that $p + v \in Q$. Because $G(Q)$ is a fundamental domain for the action of $\mathbb{Z}^3$, $F$ is a homeomorphism. In fact, little extra care in extending $g$ guarantees that $F$ is a diffeomorphisms. The $\mathbb{Z}^3$-quotient of $F$, $f : \mathbb{T}^3 \to \mathbb{T}^3$ is a homotopic to the identity torus diffeomorphism. The the sought after diffeomorphisms is $h := f^{-1}$. $\square$

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