Geometric theory of unimodular Pisot substitutions

Marcy Barge*

Jaroslaw Kwapisz†

Department of Mathematical Sciences

Montana State University

Bozeman MT 59717-2400

August 3, 2006

Abstract

We are concerned with the tiling flow T associated to a substitution ϕ over a finite alphabet. Our focus is on substitutions that are unimodular Pisot, i.e., their matrix is unimodular and has all eigenvalues strictly inside the unit circle with the exception of the Perron eigenvalue $\lambda > 1$. The motivation is provided by the (still open) conjecture asserting that T has pure discrete spectrum for any such ϕ. We develop a number of necessary and sufficient conditions for pure discrete spectrum, including: injectivity of the canonical torus map (the geometric realization), Geometric Coincidence Condition, (partial) commutation of T and the dual $\mathbb{R}^{d-1}$-action, measure and tiling properties of Rauzy fractals, and concrete algorithms. Some of these are original and some have already appeared in the literature — as *sufficient* conditions only — but they all emerge from a unified approach based on the new device: the strand space $\mathcal{F}_\phi$ of ϕ. The proof of the *necessity* hinges on determination of the discrete spectrum of T as that of the associated Kronecker toral flow.

*barge@math.montana.edu

†jarek@math.montana.edu

Key Notations

$\sim$ and $\sim_{t\omega}$ — coincidence and coincidence along $E^s + t\omega$ (page 22)

$\mathcal{A} = \{1, \ldots, d\}$ and $\mathcal{A}^*$ — the alphabet and the set of finite words (page 8)

A — the matrix of the substitution (page 9)

cr_ϕ — the coincidence rank of ϕ (page 22)

$\mathcal{C}^R$ — the R-cylinder about E^u (page 18)

c^R — the R-patch of a configuration $c \in \mathcal{F}^*$ (page 40)

$c \leftrightarrow \gamma$ — duality between configuration c and strand γ (page 43)

E^u, E^s — the unstable and stable spaces of A (page 9)

$\mathcal{F}$ — the space of all bi-infinite strands (page 10)

$\mathcal{F}_\phi$ — the strand space of ϕ (page 18)

$\mathcal{F}^*_\phi$ — the dual tiling space of ϕ (page 39)

$\overset{u}{\mathcal{F}}_\phi$ — the generic core of $\mathcal{F}_\phi$ (page 22)

$\overset{s}{\mathcal{F}^*}_\phi$ — the generic core of $\mathcal{F}^*_\phi$ (page 39)

$\overset{u}{\mathcal{F}^*}_\phi$, $\overset{s}{\mathcal{F}}_\phi$, $\overset{su}{\mathcal{F}^*}_\phi$, $\overset{su}{\mathcal{F}}_\phi$ — the other generic cores, superscripts signify E^u and E^s invariance, (pages 30 and 42)

$\phi_+, \phi_-, \phi_\pm$ — the right-, left-, central- "differentials" of ϕ (page 11)

Φ — substitution on strands induced by ϕ (page 10)

Φ_γ — pointwise (Φ-induced) map $\gamma \to \Phi(\gamma)$ (page 10)

Φ^* — dual substitution on collections of edges induced by ϕ (page 40)

G^u_ϕ — the image in $\mathbb{T}^d$ of $\overset{u}{\mathcal{F}}_\phi$, the generic part of $\mathcal{F}_\phi$ (E^u-invariant) (page 22)

G^s_ϕ — the image in $\mathbb{T}^d$ of $\overset{s}{\mathcal{F}^*}_\phi$, the generic part of $\mathcal{F}^*_\phi$ (E^s-invariant) (page 29)

$[\gamma]$ — the word of the strand γ (page 9)

$\hat{\gamma}$ — the state of strand γ (page 21)

$\gamma|^N_{-N}$ — the central (partial) substrand of length $2N$ (page 23)

h, h_ϕ — the canonical torus mappings $\mathcal{F} \to \mathbb{T}^d$ and $\mathcal{F}_\phi \to \mathbb{T}^d$ (page 18)

h^*_ϕ — the dual canonical torus mapping $\mathcal{F}^*_\phi \to \mathbb{T}^d$ (page 41)

I_i — the basic strand of the letter $i \in A$ (page 9)

λ — the Perron eigenvalue of A (page 9)

m_ϕ — the minimal degree of h_ϕ, $m_\phi = cr_\phi$, (page 21)

$N = N_\phi$ — the stabilizing iterate of ϕ (page 11)

pr^u — the projection onto E^u along E^s (page 9)

pr^s — the projection onto E^s along E^u (page 9)

$\mathrm{Per}^-(\Phi)$, $\mathrm{Per}^+(\Phi)$, $\mathrm{Per}^\pm(\Phi)$ — the forward-, backward-, bi-infinte Φ-periodic simple strands (page 12)

$\mathrm{Per}^\pm_{\mathrm{weld}}(\Phi)$, $\mathrm{Per}^\pm_{\mathrm{nweld}}(\Phi)$ — the welded and non-welded strands in $\mathrm{Per}^\pm(\Phi)$ (page 13)

$\mathbb{S}$, $\mathbb{S}_p$ — all states, all states over $p \in \mathbb{T}^d$, all such states in $\mathcal{C}^R$ (page 21)

$\mathcal{T}$ — the space of all tilings (page 10)

$\mathcal{T}_\phi$ — the tiling space of substitution ϕ (page 11)

$\mathcal{T}^{\min}_\phi$ — the unique translation minimal subset of $\mathcal{T}_\phi$ (page 15)

T^t — time t map of the tiling flow (page 12)

T^*_x — the translation by $x \in E^s$ of the dual $\mathbb{R}^{d-1}$-action (page 40)

$\mathcal{Z}^u_\gamma, \mathcal{Z}_\gamma$ — the homoclinic return times and the stabilizer of strand γ (page 31)

ω — the Perron eigenvector of A (page 9)

1 Introduction

We present a theory of the tiling flow $T^t : \mathcal{T}_\phi \to \mathcal{T}_\phi$, $t \in \mathbb{R}$, associated to a substitution $\phi : \mathcal{A} \to \mathcal{A}^*$ over a finite alphabet $\mathcal{A} = \{1, \ldots, d\}$ that is unimodular Pisot, i.e., the matrix of ϕ, $A = (a_{ij})_{i,j=1}^d$ where a_{ij} is the count of i in $\phi(j)$, is unimodular and has all its eigenvalues strictly inside the unit circle with the exception of the Perron eigenvalue $\lambda > 1$. Thus λ belongs to the class of Pisot-Vijayaraghavan numbers.

This particular choice of the hypotheses is dictated by the key conjecture in the theory of substitutions of non-constant length (cf. [26]).

Pure Discrete Spectrum Conjecture (PDSC) *The tiling flow T of a unimodular Pisot substitution ϕ has pure discrete spectrum.*

While we are unable to prove or disprove PDSC, we attempt to provide a launching pad for attacks on the problem. Our approach is somewhat unorthodox from the outset and giving precise statements of the results in this introduction would be impractical. Instead, let us start by presenting two highlights that give some measure of the advance we have to offer.

To be more specific about T, we fix Perron eigenvectors ω and ω^* of A and its transpose: $A\omega = \lambda\omega$ and $A^T\omega^* = \lambda\omega^*$ normalized so that $|\omega| = 1$ and their scalar product $\langle\omega|\omega^*\rangle = 1$. The length of tiles corresponding to the letter i is ω_i^*, $i = 1, \ldots, d$; and T^t translates the tilings making up $\mathcal{T}_\phi$ by distance t (to the left). For those familiar with the $\mathbb{Z}$-action generated by the left shift $\sigma : X_\phi \to X_\phi$ on the *substitutive system* X_ϕ associated to ϕ, the flow T is *the special flow of* σ under the function taking value ω_i^* on the ith cylinder set.

Theorem 1.1 (Theorem 9.3) *The set of eigenvalues of T (discrete spectrum) coincides with the subgroup of $\mathbb{R}$ generated by the components of ω, that is*

$$\sigma_d(T) = \left\{ \sum_{i=1}^{d} k_i\omega_i : \; k_i \in \mathbb{Z} \right\}. \tag{1.1}$$

The "$\supset$" inclusion above has been known for quite some time; "$\subset$" is new.

Theorem 1.2 (Corollary 9.4) *T has pure discrete spectrum iff ϕ satisfies a certain combinatorial condition called the* Geometric Coincidence Condition (GCC).

We have yet to explain what the *Geometric Coincidence Condition* is (in Section 7). For now, it suffices to say that it is algorithmically decidable for any particular ϕ. We mention that other characterisations of pure discrete spectrum involve *homoclinic return times* (Corollary 12.6), commutation of T with the *dual* $\mathbb{R}^{d-1}$ *action* (Propositions 16.1 and 16.3), *Rauzy fractals* obtained from *iterated function systems* (Remarks 18.3 and 18.5), and concrete algorithms (Propositions 17.1 and 17.4). Many of these conditions, in one form or another, already appear in the literature as sufficient for pure discrete spectrum — see the overview below. Beside showing their necessity, we place them within a unified geometric framework based on the concept of the *strand space of a substitution*, $\mathcal{F}_\phi$. On another level, a pivotal role in making PDSC more tangible is played by two results. First, Theorem 7.3 links GCC with a.e. injectivity of the *canonical torus* $h_\phi : \mathcal{T}_\phi \to \mathbb{T}^d$ (also called the *geometric realization*). Second, Theorem 1.1 (Theorem 9.3) above asserts that the discrete spectrum of T coincides with that of the Kronecker (linear) toral flow with frequency vector ω (which is to say that the substitutive system of ϕ has no non-trivial coboundaries). This implies that for T to have pure discrete spectrum h_ϕ must be an isomorphism.

The literature on substitution dynamics is rather sizable and we shy away from giving an overview as we could not possibly rival [11] (see also [30, 34]). However, let us go over the main elements of our theory, comment on their role and attempt to trace their origin. We note that the bulk of the publications is devoted to the shift map $\sigma : X_\phi \to X_\phi$ on the substitutive system associated to ϕ, which is (essentially) the canonical Poincaré section to the flow T. As a rule, the results can be easily transported between the two contexts; in particular, pure discrete spectrum for T is equivalent to pure discrete spectrum for σ (via [9], see also Corollary 5.7). Whether one works with the map or the flow is to some extent a matter of taste; although, the theory for the latter seems to be more "rounded" and amenable to generalizations (say, to higher dimensions).

The *tiling space* $\mathcal{T}_\phi$ and the *strand space* $\mathcal{F}_\phi$ (see Sections 3 and 5) are both defined as global attractors of the "hyperbolic" inflation-substitution action Φ induced by ϕ. They are homeomorphic. While $\mathcal{F}_\phi$ has no precedence in the literature, the notion of a tiling space does (see [36, 27, 14, 15, 34]), and our $\mathcal{T}_\phi$ deviates from the convention by possibly including a finite number of extra orbits that are wandering under T. (That may be annoying at first but many of our arguments naturally play out in the ambient spaces $\mathcal{F}$ and $\mathcal{T}$ in which the attractors live.)

The *geometric realization/canonical torus* (Sections 5 and 6) whereby $\mathcal{T}_\phi$ is mapped onto a geometric model in the form of the d-dimensional torus $\mathbb{T}^d$, is our main tool rooted in Rauzy's treatment of the Tribonacci substitution in [25]. Other examples (or narrow classes) were analysed by many authors but, for substitutive systems, Arnoux-Ito [2] and Canterini-Siegel [7, 8] developed approaches applicable to all Pisot substitutions. The basic idea is to invert the obvious construction associating to a Markov partition for a hyperbolic system with 1-dimensional unstable foliation a substitution reflecting the way the Markov boxes map over each other. That is how one can practically construct Markov partitions (see 7.6.2 in [11], [16, 21] and the references therein). We prefer the term *canonical torus* because the rightful geometric realization might be a manifold other than $\mathbb{T}^d$. See [4] for an example of a *reducible Pisot* substitution whose canonical torus is a.e. 2-to-1 and factors into an a.e. 1-to-1 map onto a genus two surface (which might properly be called geometric realization) and a branched covering of the torus.

Coincidence conditions (Section 7) also have a long history. Dekking [10] introduced one for constant length substitutions and successfully used it to characterise pure discrete spectrum in this case. Dekking's condition was generalized to the non-constant length case by Arnoux and Ito [2] and independently by Hollander and Solomyak [12]. Our condition GCC is stronger and emerged from analyzing the fiber of the canonical torus map. The combinatorial nature of GCC readily yields *algorithms* (Section 17) for verifying pure discrete spectrum and we include a short section to that effect for completeness. Other algorithms in the literature seem to be always forged from some sort of *coincidence* and the reader should consult [20, 34, 32, 30, 31] for more details (and broader scope going beyond the one-dimensional unimodular Pisot case). One original insight we have to offer here is that, in the balanced pair algorithm, it suffices to test a single elementary initial balanced pair of the form (ij, ji) (Proposition 17.3).

For computation of the discrete spectrum of T, we employ *homoclinic return times* (Section 11) where, by definition, $t \in \mathbb{R}$ is such a time for a tiling γ iff $T^t(\gamma)$ and γ are in the same stable set of Φ. This concept is closely related to the return time as used in [13, 34] — [34] served as our inspiration here — but is of utility in the general context of T and Φ satisfying

$$\Phi \circ T^t = T^{\lambda t} \circ \Phi. \tag{1.2}$$

The basic premise is that of the *Fourier duality* between frequencies and periods: the module of eigenvalues α is the dual (in $\mathbb{Q}(\lambda)$) of the module generated by all homoclinic return times t; and the latter we can compute (Section 12). The proof is intimately connected with a refinement of the classical Pisot theory regarding convergence $\lambda^n x \to 0 \mod \mathbb{Z}$, see [17]. (Incidentally, GCC is equivalent with the set of homoclinic return times being a subgroup of $\mathbb{R}$ for some γ, see Corollary 12.6.)

Rauzy fractals (Section 18), historically, arose as a device for constructing a geometric realization onto a torus, for which they formed a fundamental domain (under suitable hypotheses). Our development does not rely on this idea and simplicity is well served by letting the fractals surface only at the very end as the (stable faces of) canonical Markov boxes in a certain geometric model of $\mathcal{T}_\phi$, called the *cylinder model* (Section 18). From that standpoint, *the generalized domain exchange transformation* associated to ϕ and regularity properties of Rauzy fractals are rather immediate (see Proposition 18.1). In fact, the hyperbolic map on the cylinder model induced by ϕ is *the natural extension* of the *(Graph) Iterated Function System* that is often used to establish the properties of Rauzy fractals. A recent reference with some of the same results from the IFS point of view is [33] — see also comments in 7.5.2 of [11] and [38].

As a byproduct of our approach, we also obtain tilings of $\mathbb{R}^{d-1}$ that are dual to the original tiling of $\mathbb{R}$. Those tilings exist a priori (i.e., without the coincidence hypotheses on ϕ) and give raise to the *dual tiling space* $\mathcal{F}^*_\phi$ of the substitution ϕ (Section 14). Conceptually, $\mathcal{F}^*_\phi$ shares with $\mathcal{F}_\phi$ an equiprivileged central place in the theory, although we limited its analysis to the utilitarian minimum. Here again efficiency is well served by dispensing with Rauzy fractals and defining $\mathcal{F}^*_\phi$ as appropriate collections of segments intersecting the stable space of A, $E^s \simeq \mathbb{R}^{d-1}$. (Roughly, these are the equivalence classes of the coincidence relation along E^s.) As it turns out, $\mathcal{F}_\phi^*$ comes with a natural *inflation-substitution* dynamics closely related to the generalized dual substitution Θ devised by Arnoux and Ito [2] (see also [3] and 8.2 in [11]). One could say that $\mathcal{F}_\phi^*$ is the tiling space of Θ.

Finally, $\mathcal{F}_\phi^*$ has bearing on our understanding of $\mathcal{F}_\phi$, thanks to the *duality isomorphism between* $\mathcal{F}_\phi$ and $\mathcal{F}_\phi^*$ (Section 15), which is a measurable (and a.e. continuous) map conjugating the ϕ induced dynamics on both spaces. Its chief role is to transport the $\mathbb{R}$ action on $\mathcal{F}_\phi$ and the $\mathbb{R}^{d-1}$ action on $\mathcal{F}_\phi^*$ to the same space, say $\mathcal{F}_\phi$. GCC is equivalent to commutation of the two actions. In fact, just "partial" commutation is already enough, which is a useful reduction of GCC from the point of view of algorithm development (see Theorem 16.3 and Propositions 17.3 and 17.4).

***

Let us turn to the organization of the paper. Sections 2–4 introduce the fundamentals of the tiling spaces for primitive aperiodic substitutions. Sections 5–8 develop the strand space, the canonical torus map and the coincidence. Sections 9–13 are devoted to computation of the eigenvalues and equivalence of GCC and PDSC. Sections 14–16 are concerned with the dual tiling space and the dual $\mathbb{R}^{d-1}$ action. Sections 17 and 18 are meant to complete the picture by linking our theory with algorithms for checking GCC and Rauzy fractals, respectively. Section 19 integrates into the theory the result — shown earlier in [12, 32] via [5] — that GCC holds for unimodular Pisot substitutions over a two letter alphabet.

To keep the exposition self-contained, we have added an appendix (Section 20) with our account of Mossé's recognizability, i.e., the injectivity of the inflation-substitution map Φ on the tiling space $\mathcal{T}_\phi$. (In fact, Theorem 20.1 shows the injectivity on some neighborhood of $\mathcal{T}_\phi$.) The other appendix (Section 21) is meant to serve as a bridge to the literature studying tiling spaces as inverse limits (in the spirit of Williams and Anderson and Putnam).

List of Sections

Acknowledgments: We are grateful to Anne Siegel and Valerie Berthé for their support and help with literature references and to the referee for a thorough review. The second author would like to acknowledge partial support of NSF grant DMS-0201600.

2 Preliminaries: words, substitutions, strands, and tilings

Let $\mathcal{A}$ be a finite set, say $\mathcal{A} = \{1, \dots, d\}$ for some $d \in \mathbb{N}$. To avoid triviality we assume that $d \geq 2$. We shall think of $\mathcal{A}$ as an alphabet and of sequences in $\mathcal{A}$ as *pointed words*. Here by a sequence we understand any mapping $x : \{k \in \mathbb{Z} : M < k < N\} \to \mathcal{A}$ where $M, N \in \mathbb{Z} \bigcup \{-\infty, \infty\}$. The adjective "pointed" is supposed to be a reminder that sequences with the same ordered sets of elements but different domains are different; as for instance in the case of x and y given by $x(-1) = 1$, $x(0) = 2$, $x(1) = 1$ and $y(0) = 1$, $y(1) = 2$, $y(2) = 1$. In practice, it is convenient to represent pointed words as juxtapositions of their letters with a period preceding the first letter, as illustrated by $x = 1.21$ and $y = .121$. (The oddities like . 121 hardly ever cause a problem.) Disregarding the period leads to the concept of a *word*; technically, a *word* is an equivalence class of pointed words where x and y are equivalent iff $x = y \circ s$ for some order preserving bijection s. Thus 1.21 and .121 determine the same word 121. What is meant by *finite, forward infinite, backward infinite, and bi-infinite* word or pointed word should be clear. Also, we shall use $|a|$ for the *length of a word* a and $\overline{a}$ for the *reverse of* a; e.g. $|122| = 3$ and $\overline{122} = 221$.

Denote by $\mathcal{A}^*$ the set of all nonempty finite words. A substitution on $\mathcal{A}$ is a map $\phi : \mathcal{A} \to \mathcal{A}^*$. Such ϕ extends to all of $\mathcal{A}^*$ by the formula $\phi(w_1 \dots w_b) := \phi(w_1) \dots \phi(w_b)$. Much of what follows can be ultimately viewed as an attempt to understand the behavior of the iterates $\phi^n := \phi \circ \dots \circ \phi$ as n tends to ∞; although, a well rounded theory requires more definitions.

The *matrix of* ϕ is $A = (a_{ij})_{i,j=1}^d$ where a_{ij} is the number of occurrences of i in $\phi(j)$. ϕ is called *primitive* iff A is a primitive matrix (i.e. A^k has all positive entries for some $k > 0$). Since nearly all interesting substitutions can be understood via primitive ones, **let us assume from now on that ϕ is primitive**.

The matrix A has a simple Perron eigenvalue $\lambda > 1$ with a strictly positive (right column) eigenvector $\omega \in \mathbb{R}^d$; $A\omega = \lambda\omega$. Set $E^u = \mathrm{lin}(\omega)$ and let E^s be the the $d-1$ dimensional invariant space complementary to E^u so that $\mathbb{R}^d = E^u \oplus E^s$. We have the projection $\mathrm{pr}^u : \mathbb{R}^d \to E^u$ with $\ker(\mathrm{pr}^u) = E^s$; concretely, $\mathrm{pr}^u(p) := \langle p, \omega^* \rangle \omega$ where ω^* is the Perron eigenvector of the adjoint, $A^T \omega^* = \lambda \omega^*$, normalized so that $\langle \omega, \omega^* \rangle = 1$. (The one parameter freedom in choosing $\omega, \omega^* > 0$ satisfying this normalization corresponds to scaling the time of the tiling flow and is not important. Taking ω of unit Euclidean length conveniently assures that the ω_i^* are the tile lengths on E^u.) Let also $\mathrm{pr}^s : \mathbb{R}^d \to E^s$ be the complementary projection along E^u.

Let $(e_i)_{i=1,\dots,d}$ be the standard basis of $\mathbb{R}^d$. A *finite pointed strand* is a piecewise isometric $\tilde{\gamma} : [a,b] \to \mathbb{R}^d$ where $a, b \in \mathbb{Z}$ and for any integer $k \in [a,b)$ there is $i \in \mathcal{A}$ such that $\tilde{\gamma}(k+1) - \tilde{\gamma}(k) = e_i$. A *finite strand* is a subset of $\mathbb{R}^d$ of the from $\tilde{\gamma}([a,b])$ for some pointed strand $\tilde{\gamma}$. The points of $\tilde{\gamma}(\mathbb{Z} \cap [a,b])$ are the *vertices of* γ, and $\tilde{\gamma}([k,k+1])$, $k \in \mathbb{Z} \cap [a,b)$ are its *edges*. Upon replacing $[a,b]$ above by $[a,\infty)$, $(-\infty,b]$, or (∞,∞) we get the analogous concepts of *forward-infinite, backward-infinite, or bi-infinite strands*. Any (finite or infinite) pointed strand $\tilde{\gamma}$ determines a pointed word $[\tilde{\gamma}] = (w_k)$ given by $\tilde{\gamma}(k) - \tilde{\gamma}(k-1) = e_{w_k}$. Likewise, a strand γ determines a word denoted by $[\gamma]$.

Finite or forward-infinite strands beginning at 0, backward-infinite strands ending at 0 or bi-infinite strands containing 0 as a vertex shall be referred to as *simple strands*. Note that for $x \in \mathcal{A}^*$ there is a unique simple strand γ_x beginning at 0 with $x = [\gamma_x]$; and we have a unique preferred parametrization $\tilde{\gamma}$ of γ such that $\tilde{\gamma}(0) = 0$. Thus one may think of $\mathcal{A}^*$ as embedded in the space of strands or pointed strands; for instance, $i \in \mathcal{A}$ corresponds to the single edge $I_i := \{se_i : \; s \in [0,1]\}$. This allows for an easy but critical step of extending the substitution ϕ to a map

Φ on strands and a map $\tilde{\Phi}$ on pointed strands so that, for $x \in \mathbb{R}^d$, $\tilde{\gamma}$ a pointed strand, and γ a strand, we have

$$\tilde{\Phi}(\tilde{\gamma} + x) = \tilde{\Phi}(\tilde{\gamma}) + Ax, \qquad \Phi(\gamma + x) = \Phi(\gamma) + Ax. \tag{2.1}$$

Moreover, if $\gamma = \Phi(\eta)$ there is a natural map $\Phi_\eta : \eta \to \gamma$ characterized by

$$\mathrm{pr}^u \circ \Phi_\eta = \lambda \cdot \mathrm{pr}^u. \tag{2.2}$$

We forgo explicit formulas for Φ, $\tilde{\Phi}$, and Φ_η in favor of Figure 2.

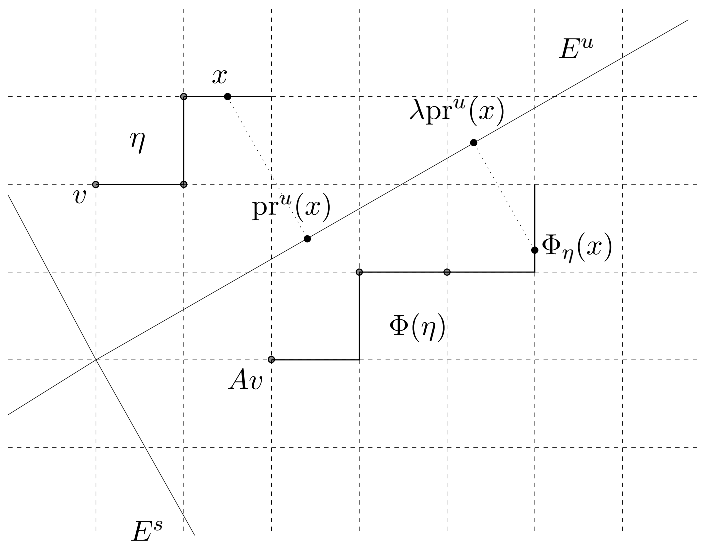

Figure 2.1: The maps Φ and Φ_η for the Fibonacci substitution $\phi : 1 \mapsto 12,\ 2 \mapsto 1$.

Any arc $K \subset \eta$ can be measured either by its arc-length $|K|$ or the arc length $|K|_u$ of $\mathrm{pr}^u(K)$, called the *u-length of* K (where "u" stands for the "unstable", of course). The latter has the advantage of $|\Phi_\eta(K)|_u = \lambda |K|_u$. There is of course $C > 1$ (depending only on A) such that

$$C^{-1}|K| \leq |K|_u \leq C|K|. \tag{2.3}$$

Nearing the end of this long string of definitions, we consider $\tilde{\mathcal{F}}$, the *space of bi-infinite pointed strands* $\tilde{\gamma} : \mathbb{R} \to \mathbb{R}^d$. We take $\tilde{\mathcal{F}}$ equipped with the compact-open topology on maps $\mathbb{R} \to \mathbb{R}^d$. It is easy to see that $\tilde{\mathcal{F}}$ is locally compact. The *space* $\mathcal{F} := \{\tilde{\gamma}(\mathbb{R}) : \ \tilde{\gamma} \in \tilde{\mathcal{F}}\}$ *of bi-infinite strands* can be topologized by viewing it as the quotient of $\tilde{\mathcal{F}}$ by the discrete action of $\mathbb{Z}$ on $\tilde{\mathcal{F}}$ via $\mathbb{Z} \ni k : \tilde{\gamma}(\cdot) \to \tilde{\gamma}(\cdot + k)$. By taking only the bi-infinite strands containing 0 or, equivalently and preferably, bi-infinite strands up to translation along E^s, we obtain

$$\mathcal{T} := \{\gamma \in \mathcal{F} : \ 0 \in \gamma\} \equiv \mathcal{F}/E^s \qquad \tilde{\mathcal{T}} := \{\tilde{\gamma} \in \tilde{\mathcal{F}} : \ 0 \in \tilde{\gamma}(\mathbb{R})\} \equiv \tilde{\mathcal{F}}/E^s,$$

called the *space of all tilings* and the *space of all pointed tilings*, respectively. We leave it for the reader to see that $\mathcal{T}$ is compact and that Φ naturally induces a self-map of $\mathcal{T}$, which at minimal risk of harm can be denoted by $\Phi : \mathcal{T} \to \mathcal{T}$.

***

Before we go, let us justify the "tiling" terminology. Consider $\gamma \in \mathcal{T}$ and let $\mathcal{E}$ be the set of pairs (S, i) where S is the projection to E^u of an edge of γ and i is the letter of that edge. Clearly, the segments S are mutually disjoint save for the endpoints, and $E^u = \bigcup_{(S,i)\in\mathcal{E}} S$. It is easy to see that $\mathcal{E}$ fully determines γ. If one chooses to encode letters with colors, then $\mathcal{E}$ is perceived as a tiling of E^u into colored segments (each congruent to one of $\{\mathrm{pr}^u(I_i) : \ i \in \mathcal{A}\}$). Note that the segments of $\mathcal{E}$ are ordered on E^u but there is no natural way to distinguish one tile as "the first" because $\mathcal{E}$ is just a set. For $\tilde{\gamma} \in \tilde{\mathcal{T}}$, the analogous construction leads to a sequence $\tilde{\mathcal{E}} = ((S_k := \tilde{\gamma}([k, k+1]), i_k))_{k\in\mathbb{Z}}$, which may be perceived as a colored tiling with a distinguished point (say $\min S_0$).

3 Tiling space and flow; and the ϕ-periodic words

Given a primitive substitution ϕ as in the previous section, the *tiling space of* ϕ is the global attractor of $\Phi : \mathcal{T} \to \mathcal{T}$:

$$\mathcal{T}_\phi := \bigcap_{n\in\mathbb{N}} \Phi^n(\mathcal{T}). \tag{3.1}$$

Our terminology is somewhat unorthodox: typically the name tiling space of ϕ is attached to a smaller set that we call *the minimal tiling space of* ϕ (see (3.5) below).

Observe that $\mathcal{T}_\phi$ is non-empty and compact. As we shall see, $\mathcal{T}_\phi$ is typically fairly complex (contains an indecomposable continuum) but much of its structure is encoded in its simplest elements that are the simple strands corresponding to the bi-infinite pointed words that are periodic under ϕ. We describe those first. For $i \in \mathcal{A}$, let $\phi_+(i)$ be the first letter of $\phi(i)$ and $\phi_-(i)$ be the last letter of $\phi(i)$. Also, set $\phi_\pm(ij) := \phi_-(i)\phi_+(j)$. Being maps of finite sets

$$\phi_-, \phi_+ : \mathcal{A} \to \mathcal{A} \qquad \phi_\pm : \mathcal{A}^2 \to \mathcal{A}^2 \tag{3.2}$$

they have non-empty sets of periodic points, $\mathrm{Per}(\phi_-)$, $\mathrm{Per}(\phi_+)$, and $\mathrm{Per}(\phi_\pm)$, respectively. Moreover, there is $N \in \mathbb{N}$ such that

$$\phi_\pm^N \circ \phi_\pm^N = \phi_\pm^N. \tag{3.3}$$

We call the minimal such N for which also A^N has all positive entries the *stabilizing iterate of* ϕ, denoted N_ϕ.

Observe a simple fact underpinning many of the arguments to follow.

Fact 3.1 *If $\phi_+^m(i) = i$, then $\Phi^{km}(I_i) \subset \Phi^{(k+1)m}(I_i)$ and $\gamma := \bigcup_{k\in\mathbb{N}} \Phi^{km}(I_i)$ is a forward-infinite simple strand with $\Phi^m(\gamma) = \gamma$.*

The fact has variants for ϕ_- and $\phi_\pm$ and is not hard to elaborate into the following. Denote by $\mathrm{Per}^+(\Phi)$, $\mathrm{Per}^-(\Phi)$, and $\mathrm{Per}^\pm(\Phi)$ the sets of Φ-periodic simple forward infinite strands, simple backward infinite strands, and simple bi-infinite strands, respectively.

Proposition 3.2 *(i) Given $i \in \mathrm{Per}(\phi_+)$ of period m, there is a unique $\gamma_+ \in \mathrm{Per}^+(\Phi)$ beginning with I_i; its period is m.*

(ii) Given $j \in \mathrm{Per}(\phi_-)$ of period n, there is a unique $\gamma_- \in \mathrm{Per}^-(\Phi)$ ending with $-I_j$; its period is n.

(iii) The strand $\gamma := \gamma_-\gamma_+$ is a unique strand in $\mathrm{Per}^\pm(\Phi)$ containing $I_i \cup (-I_j)$; its period is $nm/(n,m)$ — the smallest common multiple of m and n.

(iv) Any Φ-periodic simple strand γ arises as above.

(v) Given any simple strand η, $\lim_{n\to\infty} \Phi^{nN_\phi}(\eta) = \gamma$ for some Φ-periodic simple strand γ (either forward, backward, or bi-infinite)

Proof: We shall only prove (iv) and leave the rest as an exercise. Let γ be a Φ-periodic simple strand with minimal period m, $\Phi^m(\gamma) = \gamma$. For $n \in \mathbb{N}$, set $\gamma_n := \Phi^n(\gamma)$ and let J_n^- and J_n^+ be the edges of γ_n abutting at 0, J_n^- ending at 0 and J_n^+ starting at 0. Denoting by i_n the type of J_n^- and by j_n the type of J_n^+, we see that $\phi_-^m(i_0) = i_0$ and $\phi_+^m(j_0) = j_0$. In particular, $\gamma = \bigcup_{k\in\mathbb{N}} \Phi^{km}(J_0^- \cup J_0^+)$ as in Fact 3.1. It is left to see that m equals the least common multiple of the periods of i_0^+ and i_0^-, for which we write M. That M divides m is clear. By construction, $\phi_\pm^M((i_0^-, i_0^+)) = (i_0^-, i_0^+)$, and so $J_0^- \cup J_0^+ \subset \Phi^M(J_0^- \cup J_0^+)$. By applying Φ^{km}, we get $\Phi^{km}(J_0^- \cup J_0^+) \subset \Phi^{M+km}(J_0^- \cup J_0^+)$, which yields $\gamma = \Phi^M(\gamma)$ by taking unions over $k \in \mathbb{N}$. Hence, m divides M making the two equal. □

The translation action of $\mathbb{R}^d$ on strands induces a natural translation action of $\mathbb{R} \equiv E^u \equiv \mathbb{R}^d/E^s$ on $\mathcal{T} \equiv \mathcal{F}/E^s$. Concretely, for $t \in \mathbb{R}$, we define $T^t : \mathcal{F} \to \mathcal{F}$ by $T^t(\gamma) := \gamma + t\omega$. We shall abuse notation and use $T^t : \mathcal{T} \to \mathcal{T}$ for the corresponding action on strands modulo E^s. Observe that

$$\Phi \circ T^t = T^{\lambda t} \circ \Phi. \tag{3.4}$$

As a consequence, $\mathcal{T}_\phi$ is T^t-invariant: $T^t(\mathcal{T}_\phi) = \mathcal{T}_\phi$ for $t \in \mathbb{R}$. In particular, beside the basic Φ-periodic strands supplied by Proposition 3.2, $\mathcal{T}_\phi$ contains the closures of their translation orbits. That this is all of $\mathcal{T}_\phi$ is shown by (iii) of the following proposition. Below, a word a being a subword of word b is denoted $a \subset b$.

Proposition 3.3 *Given $\eta \in \mathcal{T}$, the following are equivalent*

(i) $\eta \in \mathcal{T}_\phi$

(ii) for any finite substrand $\xi \subset \eta$, there is $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then there are $i, j \in \mathcal{A}$ with $[\xi] \subset \phi^n(ij)$.

(iii) η is in the closure $\mathrm{cl}\{T^t\gamma : \ \gamma \in \mathrm{Per}^{\pm}(\Phi), \ t \in \mathbb{R}\}$.

Proof: Set $N = N_\phi$.
(i) $\Rightarrow$ (ii): For every $n \in \mathbb{N}$, let $\eta_{-n} \in \mathcal{F}$ be such that $\Phi^n(\eta_{-n}) = \eta$. Set $\delta := \min\{|I_i|_u = |\mathrm{pr}^u(I_i)| : \ i \in \mathcal{A}\}$. By taking an edge of η_{-n} intersecting E^s together with one of the two adjacent edges, one obtains a length two substrand μ_{-n} of η_{-n} such that $(-\delta/2, \delta/2)\omega \subset \mathrm{pr}^u(\mu_{-n})$. Since $\lambda^n(-\delta/2, \delta/2)\omega \subset \mathrm{pr}^u(\Phi^n(\mu_{-n}))$, we have that $\xi \subset \Phi^n(\mu_{-n})$ for sufficiently large n (cf. Fact 3.1). Thus $[\xi] \subset \phi^n(ij)$ where $ij = [\mu_{-n}]$.

(ii) $\Rightarrow$ (iii): First observe that, for $i, j \in \mathcal{A}$ and $n > N$, $\phi^n(ij) = \phi^{n-N}(\phi^N(ij))$ contains $\phi^{n-N}(i'j')$ where $i' := \phi_-^N(i) \in \mathrm{Per}(\phi_-)$ and $j' := \phi_+^N(j) \in \mathrm{Per}(\phi_+)$. Moreover, $\phi^{n-N}(i'j')$ constitutes a *definite* (central) part of $\phi^n(ij)$ in the sense that $|\phi^{n-N}(i'j')|/|\phi^n(ij)| \geq c$ where $c > 0$ is independent of n, i, j. As a consequence, if (ii) holds, it does so with an additional requirement that $i \in \mathrm{Per}(\phi_-)$ and $j \in \mathrm{Per}(\phi_+)$.

To prove (iii) we may well adjust η by translation so that it has a vertex at 0. Fix an arbitrary $m \in \mathbb{N}$ and take the ***central substrand*** $\xi_m \subset \eta$ with $|\xi_m| = 2m$, i.e., ξ_m is a substrand of η with m edges on both sides of $0 \in \eta$. By our initial observation, there are $i_m \in \mathrm{Per}(\phi_-)$, $j_m \in \mathrm{Per}(\phi_+)$ and $k_m \in \mathbb{N}$ such that $[\xi_m] \subset \phi^{k_m N}(i_m j_m)$, and we can pick a substrand $\xi'_m \subset \Phi^{k_m N}(-I_{i_m} \cup I_{j_m})$ with $[\xi'_m] = [\xi_m]$. We have $\gamma_m := \bigcup_{s \in \mathbb{N}} \Phi^{sN + k_m N}(-I_{i_m} \cup I_{j_m})$ that is fixed by Φ^N and thus in $\mathrm{Per}^{\pm}(\phi)$. Taking $t_m \in \mathbb{R}$ so that $\xi'_m + t_m\omega = \xi_m \mod E^s$, we see that $T^{t_m}(\gamma_m)$ and η coincide on the central substrand of length $2m$. By passing to a subsequence we may assure that γ_m is constant and equal to some γ so that $\lim_{m\to\infty} T^{t_m}(\gamma) = \eta$.

(iii) $\Rightarrow$ (i): Clearly, $\mathrm{Per}^{\pm}(\phi) \subset \mathcal{T}_\phi$. Thus (i) follows by translation invariance of $\mathcal{T}_\phi$. □

Our next task is to inspect the recurrence properties of the translation flow T on $\mathcal{T}_\phi$. We shall see that primitivity of ϕ — which was not invoked in Proposition 3.3 — implies that T is minimal after discarding from $\mathcal{T}_\phi$ orbits of certain pathological $\gamma \in \mathrm{Per}^{\pm}(\Phi)$. To this end, let us brand $\gamma \in \mathrm{Per}^{\pm}(\Phi)$ as a ***welded simple*** Φ***-periodic strand*** if the word ij of the central length two substrand of γ is not a subword of any of the words $\phi^{2N}(l)$ where $l \in \mathcal{A}$ and $N = N_\phi \in \mathbb{N}$ is the stabilizing iterate. (It will only be clear from the proof below why $2N$ and not N.) The set of all welded γ's will be denoted $\mathrm{Per}^{\pm}_{\mathrm{weld}}(\Phi)$, and we set $\mathrm{Per}^{\pm}_{\mathrm{nweld}}(\Phi) = \mathrm{Per}^{\pm}(\Phi) \setminus \mathrm{Per}^{\pm}_{\mathrm{weld}}(\Phi)$. We remark that if $\mathrm{Per}^{\pm}_{\mathrm{weld}}(\Phi) = \emptyset$, then (ii) of Proposition 3.3 is true with ij replaced by a single letter i.

Example (of a welded strand): Under $\phi : 1 \mapsto 121, 2 \mapsto 2212$, 11 is easily seen not to be a subword of any word obtained by inflating a single letter. Thus the simple strand γ corresponding to the fixed bi-infinite word of ϕ of the form $\ldots 1.1 \ldots$ is welded. (The substitution $\phi : 1 \mapsto 12221, 2 \mapsto 21212212$ has the same property and is unimodular Pisot.)

Below we denote the set of forward and backward limit points of a set B under the flow T by $\omega^+(B)$ and $\omega^-(B)$, respectively. Also, $\omega(B) := \omega^+(B) \cup \omega^-(B)$.

Proposition 3.4 *Let $\gamma \in \mathrm{Per}^{\pm}(\Phi)$. The following are equivalent:*

(i) $\eta \in \mathcal{T}_\phi \setminus \{T^t\gamma : \ \gamma \in \mathrm{Per}^{\pm}_{\mathrm{weld}}(\Phi), \ t \in \mathbb{R}\}$;

(ii) for any finite substrand $\xi \subset \eta$ there is $n_0 \in \mathbb{N}$ so that if $n \geq n_0$ then $[\xi] \subset \phi^n(i)$ for some $i \in \mathcal{A}$.

(iii) $\eta \in \omega^-(\gamma)$;

(iv) $\eta \in \omega^+(\gamma)$.

Proof: Again take $N := N_\phi$.
(i) $\Rightarrow$ (ii): In the context of the argument for (i) $\Rightarrow$ (ii) of the previous proposition, let p_{-n} be the middle vertex of μ_{-nN}. Note that $q_n := A^{nN}(p_{-n})$ is a vertex of η. We have two possibilities:

(B) $|\mathrm{pr}^u(q_n)|$ is bounded as $n \to \infty$;

(U) $|\mathrm{pr}^u(q_n)|$ is unbounded as $n \to \infty$.

Suppose (B) holds. Along some subsequence $n_k \to \infty$, we have $q_{n_k} = q$ and $[\mu_{-n_kN}] = ij$ for some $q \in \gamma$, $i, j \in \mathcal{A}$. Since $\mu_{-n_kN} - p_{-n_k}$ is a length two strand with the middle vertex at 0, (iii) of Proposition 3.2 yields $\gamma := \bigcup_{k\to\infty} \Phi^{n_kN}(\mu_{-n_kN} - p_{-n_k}) \in \mathrm{Per}^{\pm}(\Phi)$. At the same time, $\bigcup_{k\to\infty} \Phi^{n_kN}(\mu_{-n_kN} - p_{-n_k}) = \Phi^{n_kN}(\mu_{-n_kN}) - q = \eta - q$ so that, if $t \in \mathbb{R}$ is such that $t\omega = \mathrm{pr}^u(-q)$, then $T^t(\gamma) = \eta$, which contradicts (i) and excludes (B).
Suppose now (U) holds and $|\mathrm{pr}^u(q_n)| \to \infty$ along some subsequence $n_k \to \infty$. Depending on whether $\langle \mathrm{pr}^u(q_{n_k})|\omega\rangle$ converges to $+\infty$ or $-\infty$ let I_k be the first or the second edge of μ_{-n_kN}, respectively. Then for any substrand $\xi \subset \eta$ we have $\xi \subset \Phi^{n_kN}(I)$ for large enough k.

(ii) $\Rightarrow$ (iii): First observe that, by primitivity of ϕ, "for some $i \in \mathcal{A}$" can be replaced by "for any $i \in \mathcal{A}$" in the formulation of (ii). The rest of the argument is similar to that for (ii) $\Rightarrow$ (iii) of Proposition 3.3. First adjust η by translation so that it has a vertex at 0. Fix an arbitrary $m \in \mathbb{N}$ and take the central substrand $\xi_m \subset \eta$ with $|\xi_m| = 2m$. Let $i_m j_m$ be the word of the central length two substrand of γ. Pick $k_m \in \mathbb{N}$ so that $[\xi_m]$ is a subword of $\phi^{k_mN}(j_m) \subset [\gamma]$ and pick a substrand $\xi'_m \subset \Phi^{k_mN}(I_{j_m})$ with $[\xi'_m] = [\xi_m]$. There is $t_m > 0$ — actually $t_m \geq C^{-1}2m/2$ where C is as in (2.3) — such that $\xi'_m - t_m\omega = \xi_m \mod E^s$. Since $T^{-t_m}(\gamma)$ and η coincide mod E^s on the central substrand of length $2m$, we have $\lim_{m\to\infty} T^{-t_m}(\gamma) = \eta$, i.e. $\eta \in \omega^-(\gamma)$.

(ii) $\Rightarrow$ (iv): Use an analogous argument to the above where $[\xi_m]$ is found as a subword of $\phi^{k_mN}(i_m) \subset [\gamma]$.

(iii) or (iv) $\Rightarrow$ (i): It suffices to show that $\eta \in \omega^+(\gamma) \cup \omega^-(\gamma)$ implies that $\eta \notin \mathrm{Per}^{\pm}_{\mathrm{weld}}(\Phi)$. We deal with the ω^+ case. Suppose then that $\eta \in \mathrm{Per}^{\pm}(\Phi) \cap \omega^+(\gamma)$. Let ij be the word of the central length two substrand of η. We have to show that η is not welded. For some $m \geq 0$, we shall construct length two substrands $\xi_0, \ldots, \xi_{-m}$ of γ such that the corresponding words $i_0 j_0, \ldots, i_{-m} j_{-m}$ satisfy $ij = i_0 j_0$, $\phi^N_{\pm}(i_{-r-1} j_{-r-1}) = i_{-r} j_{-r}$ for $r = 0, \ldots, m-1$, and $i_{-m} j_{-m} \subset \phi^N(l)$ for some letter l in $\mathcal{A}$. Since, by definition of N_ϕ, $ij = \phi^{mN}_{\pm}(i_{-m} j_{-m}) = \phi^N_{\pm}(i_{-m} j_{-m})$, this would guarantee that $ij \subset \phi^{2N}(l)$, which would show that η is not welded, as desired.

From $\eta \in \mathrm{cl}\{T^t\gamma : \ t > 0\}$, ij is a word of some length two substrand ξ_0 of γ. Because we take closure over $t > 0$, ξ_0 may be chosen so that it does not intersect E^s; in particular, its middle vertex $q_0 \neq 0$.

If ξ_0 is a substrand of some $\Phi^N(I_l)$ where l is a letter in γ, we stop after setting $m = 0$. Otherwise, there is a length two substrand ξ_{-1} of γ such that $\Phi^N(\xi_{-1})$ contains ξ_0 as a substrand and $A^N q_{-1} = q_0$ where q_{-1} is the middle vertex of ξ_{-1}. This means that $ij = \phi^N_{\pm}(i_{-1} j_{-1})$ where $i_{-1} j_{-1}$ is the word of ξ_{-1}. If ξ_{-1} is a substrand of some $\Phi^N(I_{l_{-1}})$ where l_{-1} is a letter in γ, we stop after setting $m = 1$ and $l = l_{-1}$. Otherwise, we repeat the step leading from ξ_0 to ξ_{-1} to obtain ξ_{-2} from ξ_{-1}; etc. To see that the process must stop, assume that this is not the case. The middle vertex q_{-n} of ξ_{-n} is given by $A^{-nN}(q_0)$ and thus converges to E^s. Since q_{-n} is a vertex of γ, this is only possible if $q_{-n} = 0$ for large enough n. But then $q_0 = A^{nN} q_{-n} = A^{nN} 0 = 0$ contradicting the construction of ξ_0. $\square$

Proposition 3.4 guarantees that the following is an unambiguous definition: The *minimal tiling space of* ϕ is

$$\mathcal{T}^{\min}_\phi := \omega^+(\gamma) = \omega^-(\gamma) = \mathcal{T}_\phi \setminus \{T^t\gamma : \ \gamma \in \mathrm{Per}^{\pm}_{\mathrm{weld}}(\Phi), \ t \in \mathbb{R}\} \tag{3.5}$$

where $\gamma \in \mathrm{Per}^{\pm}(\Phi)$.

Proposition 3.5 *$\mathcal{T}^{\min}_\phi$ is the unique minimal subset of $\mathcal{T}_\phi$.*

Proof: First let $\gamma \in \mathrm{Per}^{\pm}_{\mathrm{nweld}}(\Phi)$.
Claim: If ξ is any finite substrand of γ then there is $L > 0$ such that any substrand $\mu \subset \gamma$ of length $|\mu|_u \geq L$ contains a substrand $\xi' \subset \mu$ with $[\xi'] = [\xi]$.

Indeed, by (iii) of Proposition 3.3, there is $l \in \mathcal{A}$ and $k \in \mathbb{N}$ such that $\phi^k(l)$ contains $[\xi]$. By primitivity, l appears in every $\phi^m(j)$, $j \in \mathcal{A}$, for $m \geq N$. In particular, we may assume that k is a multiple of N. More importantly, in view of $\Phi^N(\gamma) = \gamma$, the word of every substrand of γ of u-length exceeding $2\max\{|\phi^N(j)|_u : j \in \mathcal{A}\}$ contains l. This and $\gamma = \Phi^k(\gamma)$ further imply that every $\mu \subset \gamma$ of u-length exceeding $L := 2\max\{|\phi^{N+k}(j)|_u : \ j \in \mathcal{A}\} = 2\lambda^{N+k}\max\{|j|_u : \ j \in \mathcal{A}\}$ contains $\phi^k(l)$ and thus also $[\xi]$ — which shows the claim.

To demonstrate minimality of $\omega^-(\gamma) = \omega^+(\gamma)$, it suffices to show that $\gamma \in \omega^+(\eta)$ for any $\eta \in \omega^+(\gamma)$. For convenience, let us adjust η by translation so that it has a vertex at 0. Fix an arbitrary central substrand $\xi \subset \gamma$. There is $t > 0$ such that

$T^t\gamma$ and η coincide on a finite strand μ beginning at 0 of length exceeding L given by the claim. Thus μ must contain a substrand ξ' with $[\xi'] = [\xi]$. By taking $s > 0$ such that $\xi' - s\omega$ is centered we see that $T^s\eta$ and γ coincide on a central strand of length $||[\xi]||$. We are done by arbitrariness of ξ. □

4 Translation periodicity and aperiodicity

As it turns out, there is a very sharp dichotomy in the complexity of $\mathcal{T}_\phi$ depending on whether any of the ϕ-periodic pointed bi-infinite words are also periodic under translation. Details follow.

We call an infinite strand γ translation periodic (or T-periodic) if it is a concatenation of strands with the same word. This is equivalent to existence of $x \in \mathbb{R}^d \backslash \{0\}$ such that $\gamma = \gamma + x$ in the case of bi-infinite γ and $\gamma \subset \gamma + x$ in the case of forward-infinite γ.

Proposition 4.1 *For any primitive substitution ϕ, the following are equivalent*

(i) some forward-infinite simple Φ-periodic strand is translation periodic;

(ii) some simple Φ-periodic non-welded strand is translation periodic;

Moreover, in (i) and (ii) the word "some" can be replaced by "all".

The proof of the proposition is left as an exercise. We shall call ϕ *translation periodic* iff the conditions of the above proposition are met and *translation aperiodic* otherwise.

Proposition 4.2 *For any primitive substitution ϕ, if ϕ is translation periodic then $\Phi : \mathcal{T}_\phi^{\min} \to \mathcal{T}_\phi^{\min}$ is conjugate to a covering map of a circle and the flow T^t on $\mathcal{T}_\phi^{\min}$ is conjugate to the rigid circle rotation flow.*

Proof: Exercise. □

To convey the complexity of $\mathcal{T}_\phi^{\min}$ for translation aperiodic ϕ, we introduce the *winding map f_ϕ of ϕ*. Let $\mathbb{T}_\phi$ be the quotient by $\mathbb{Z}^d$ of $\{(x_1, \ldots, x_d) : x_i \notin \mathbb{Z}$ for at most one $i\}$, the integer *grid* in $\mathbb{R}^d$. $\mathbb{T}_\phi$ is naturally a wedge of d circles $\mathbb{T}_1, \ldots \mathbb{T}_d$ obtained by identifying the endpoints of $I_1, \ldots, I_d$. For each $i \in \mathcal{A}$, ϕ induces a map $I_i \to \gamma_i$ where γ_i is the simple strand associated to $\phi(i)$; let $f_i : \mathbb{T}_i \to \mathbb{T}_\phi$ be the quotient map. We define $f_\phi : \mathbb{T}_\phi \to \mathbb{T}_\phi$ as the wedge of $f_1, \ldots, f_d$.

Recall that the inverse limit of f_ϕ is the space of the bi-infinite orbits of f_ϕ, $X_{f_\phi} := \{(t_i)_{i \in \mathbb{Z}} : \ f_\phi(t_i) = t_{i+1}, \ i \in \mathbb{Z}\}$, together with the shift map $\overleftarrow{f}_\phi : X_{f_\phi} \to X_{f_\phi}$, $(t_i) \mapsto (t_{i+1})$. In conventional notation, $\overleftarrow{f}_\phi := \lim_{\leftarrow} f_\phi$. Because of primitivity of ϕ, under sufficiently large iterates of f_ϕ, each circle $\mathbb{T}_i$ maps onto all of $\mathbb{T}_\phi$. This implies that X_{f_ϕ} has a complicated structure: it is an indecomposable continuum (as exemplified by *the solenoid*, the inverse limit of the circle map $t \pmod 1) \mapsto 2t \pmod 1)$).

Proposition 4.3 *For any primitive substitution ϕ, if ϕ is translation aperiodic then $\Phi : \mathcal{T}_\phi^{\min} \to \mathcal{T}_\phi^{\min}$ is semi-conjugate to $\overleftarrow{f}_\phi$, the shift map on the inverse limit of f_ϕ.*

Note that, unlike in the periodic case, we make no assertion about the flow; even measure theoretical classification of those is a major open problem. In contrast, the action of Φ is well understood. In particular, "semi-conjugacy" can be easily improved to "conjugacy" at the expense of replacing f_ϕ with a bit more elaborate map on a bouquet of a larger number of circles or a finite graph that retracts to $\mathbb{T}_\phi$ — as is elaborated in Appendix 21. At any rate, it is the invertibility of Φ on $\mathcal{T}_\phi$ that is the pivotal feature of the aperiodic case and the key ingredient of the proof of Proposition 4.3:

Theorem 4.4 (Mossé) *If ϕ is a primitive translation aperiodic substitution then $\Phi : \mathcal{T}_\phi \to \mathcal{T}_\phi$ is a homeomorphism.*

Since $\mathcal{T}_\phi$ is compact and $\Phi : \mathcal{T}_\phi \to \mathcal{T}_\phi$ is manifestly onto, the theorem is equivalent to injectivity of Φ, which — in the parlance of substitution systems — amounts to a property called *recognizability*. Recognizability was established by Mossé in [23] (and later generalized to higher dimensions in [35]). For completeness, we give a proof of the theorem in the appendix.

Sketch of proof of Proposition 4.3: It suffices to construct a map $r : \mathcal{T}_\phi \to \mathbb{T}_\phi$ factoring Φ to f_ϕ. Indeed, then $\lim_\leftarrow \Phi$ factors to $\lim_\leftarrow f_\phi$ via $\lim_\leftarrow r$; and $\lim_\leftarrow \Phi = \Phi$ since Φ is a homeomorphism.

The construction of the factor map $r : \mathcal{T}_\phi \to \mathbb{T}_\phi$ is straightforward. Given an element of $\mathcal{T}_\phi$ represented by a strand γ passing through 0, let I be an edge of γ containing 0. Also, let $x \in \mathbb{R}^d$ be such that $I + x = I_i$ for some $i \in \{1, \ldots, d\}$. Note that $x \in I_i$ and thus it determines a point $p \in \mathbb{T}_\phi$; set $r(\gamma) := p$. We leave it to the reader to check that r is well defined and satisfies $r \circ \Phi = f_\phi \circ r$. □

5 Canonical torus h_ϕ for Pisot substitutions

A matrix is called *Pisot* iff it has a unique (counting with multiplicity) eigenvalue of modulus greater than one and all other eigenvalues are of modulus less than one. A primitive substitution is called *Pisot* iff its matrix A is Pisot, that is the spectrum of $A|_{E^s}$ is contained in the disk $\{|z| < \mu\}$ for some $\mu \in (0, 1)$.

Throughout this section, we assume that ϕ is Pisot. It is convenient to fix *a stable adapted semi-norm* $|\cdot|_s$ on $\mathbb{R}^d$ so that, for $x \in \mathbb{R}^d$,

$$|Ax|_s \leq \mu |x|_s$$

and $|x|_s = 0$ iff $x \in E^u$. (Thus $|\cdot|_s + |\cdot|_u$ is *the adapted norm* on $\mathbb{R}^d$ where $|x|_u := |\mathrm{pr}^u(x)| = |\langle \omega^* | x \rangle|$.) Denote $\mathcal{C}^R := \{p \in \mathbb{R}^d : \ |p|_s \le R\}$ and $\mathcal{F}^R := \{\gamma \in \mathcal{F} : \ \gamma \subset \mathcal{C}^R\}$. From its definition, Φ is a bounded perturbation of A: there is a uniform bound on $|\Phi_J(y) - Ay|$ for all edges J and $y \in J$. We record the following consequence:

Lemma 5.1 *There is $\alpha > 0$ such that for any strand $\gamma \subset \mathcal{C}^R$, we have $\Phi(\gamma) \subset \mathcal{C}^{R'}$ where $R' = \mu R + \alpha$.*

Fix $R_0 > \alpha/(1-\mu)$. The lemma assures that $\Phi^n(\mathcal{F}^{R_0}) \supset \Phi^{n+1}(\mathcal{F}^{R_0})$ for $n \ge 0$; and for any $R > 0$ there is $n \in \mathbb{N}$ such that $\Phi^n(\mathcal{F}^R) \subset \mathcal{F}^{R_0}$.

Definition 5.2 *The* strand space of *ϕ is*

$$\mathcal{F}_\phi := \bigcap_{n \in \mathbb{N}} \Phi^n(\mathcal{F}^{R_0}), \qquad R_0 > \alpha/(1-\mu). \tag{5.1}$$

Observe that $\mathcal{F}_\phi$ does not depend on the choice of R_0 and is compact (because $\mathcal{F}^{R_0}$ already is). Also, $\Phi(\mathcal{F}_\phi) = \mathcal{F}_\phi$ and $T^t(\mathcal{F}_\phi) = \mathcal{F}_\phi$ for $t \in \mathbb{R}$. Remarkably, $\mathcal{F}_\phi$ is just another presentation of $\mathcal{T}_\phi$:

Theorem 5.3 *The restriction $\pi_\phi := \pi|_{\mathcal{F}_\phi}$ of the natural projection $\pi : \mathcal{F} \to \mathcal{T}$ maps $\mathcal{F}_\phi$ homeomorphically onto $\mathcal{T}_\phi$.*

Proof: We have to show that π_ϕ is onto and $1-1$. Since $\Phi \circ \pi = \pi \circ \Phi$, it is clear that $\pi(\mathcal{F}_\phi) \subset \mathcal{T}_\phi$ right from the definitions of $\mathcal{T}_\phi$ and $\mathcal{F}_\phi$. To see $\mathcal{T}_\phi \subset \pi(\mathcal{F}_\phi)$, it suffices to show that given a strand $\gamma \in \mathrm{Per}^\pm(\Phi)$ we have $\gamma \in \mathcal{F}_\phi$; indeed, then $T^t\gamma \pmod{E^s} \subset \pi(\mathcal{F}_\phi)$ for $t \in \mathbb{R}$ and we are done by (iii) of Proposition 3.3. For a proof, recall that $\gamma = \lim_{n\to\infty} \Phi^{nN}(-I_i \cup I_j)$ for some $i, j \in \mathcal{A}$ with $\phi^N_\pm(ij) = ij$ (see Proposition 3.2). Clearly, $-I_j \cup I_i \subset \mathcal{C}^R$ for some $R > R_0$ so that Lemma 5.1 implies $\gamma \subset \mathcal{C}^{R_0}$. Coupled with $\Phi^N(\gamma) = \gamma$, this yields $\gamma \in \mathcal{F}_\phi$.

Now suppose that, for $\gamma, \tilde\gamma \in \mathcal{F}_\phi$, we have $\pi(\gamma) = \pi(\tilde\gamma)$, that is $\gamma \equiv \tilde\gamma \pmod{E^s}$. For $n \in \mathbb{N}$, we can pick $\gamma_{-n}, \tilde\gamma_{-n} \in \mathcal{F}^{R_0}_\phi$ with $\Phi^n(\gamma_{-n}) = \gamma$ and $\Phi^n(\tilde\gamma_{-n}) = \tilde\gamma$. Because $\Phi : \mathcal{T}_\phi \to \mathcal{T}_\phi$ is a homeomorphism, we must have $\gamma_{-n} (\mathrm{mod}\ E^s) = \tilde\gamma_{-n} (\mathrm{mod}\ E^s) = \Phi^{-n}(\gamma (\mathrm{mod}\ E^s))$. Thus $\gamma_{-n} = \tilde\gamma_{-n} + y_n$ for some $y_n \in E^s$; and $|y_n|_s \le 2R_0$ since $\gamma_{-n}, \tilde\gamma_{-n} \subset \mathcal{C}^{R_0}$. Now, $\gamma = \Phi^n(\gamma_{-n}) = \Phi^n(\tilde\gamma_{-n} + y_n) = \Phi^n(\tilde\gamma_{-n}) + A^n y_n = \tilde\gamma + A^n y_n$ where $\lim_{n\to\infty} |A^n y_n| = 0$ by the contracting action of $A|_{E^s}$. Thus $\gamma = \tilde\gamma$, which establishes that π_ϕ is $1-1$. $\Box$

The advantage of $\mathcal{F}$ over $\mathcal{T}$ is that there is a natural map $h : \mathcal{F} \to \mathbb{T}^d$ given by $\gamma \mapsto v \pmod{\mathbb{Z}^d}$ where $v \in \gamma$ is a vertex. By the *canonical torus of* ϕ we understand either of the maps

$$h_\phi := h|_{\mathcal{F}_\phi} : \mathcal{F}_\phi \to \mathbb{T}^d,$$

$$h_\phi \circ \pi_\phi^{-1} : \mathcal{T}_\phi \to \mathbb{T}^d,$$

depending on whether we present the substitution as $\mathcal{F}_\phi$ or $\mathcal{T}_\phi$.

From the construction, we have the following commuting diagrams

$$\begin{array}{ccc} \mathcal{F}_\phi & \xrightarrow{\Phi} & \mathcal{F}_\phi \\ {\scriptstyle h_\phi}\downarrow & & {\scriptstyle h_\phi}\downarrow \\ \mathbb{T}^d & \xrightarrow{A} & \mathbb{T}^d \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{F}_\phi & \xrightarrow{T^t} & \mathcal{F}_\phi \\ {\scriptstyle h_\phi}\downarrow & & {\scriptstyle h_\phi}\downarrow \\ \mathbb{T}^d & \xrightarrow{T^t_\omega} & \mathbb{T}^d \end{array} \tag{5.2}$$

where $A : \mathbb{T}^d \to \mathbb{T}^d$ is the endomorphism induced by the matrix A and $T^t_\omega : \mathbb{T}^d \to \mathbb{T}^d$ is the translation along ω, $x \pmod{\mathbb{Z}^d} \mapsto x + t\omega \pmod{\mathbb{Z}^d}$.

The following standard fact readily implies that if A is non-singular then the Kronecker flow T_ω is minimal and thus h_ϕ is surjective

Fact 5.4 *If A is Pisot and non-singular, then*

(i) E^s contains no rational points, $E^s \cap \mathbb{Q}^d = \{0\}$;

(ii) E^u is non-resonant, i.e., for $\xi \in \mathbb{Q}^d$ and $v \in E^u \setminus \{0\}$, $\langle \xi | v \rangle = 0$ implies $\xi = 0$.

In particular, the components $\omega_1, \dots, \omega_d$ of ω are linearly independent over $\mathbb{Q}$.

Proof: (i): Consider $v \in \mathbb{Z}^d \cap E^s$. By the Pisot hypothesis, $\lim_{n\to\infty} A^n v = 0$ and so $v = 0$ because $A^n v \in \mathbb{Z}^d$. It follows that $E^s \cap \mathbb{Q}^d = \{0\}$.

(ii): Let E^s_T and E^u_T be the stable and unstable spaces of the transpose A^T. By the above argument, also $E^s_T \cap \mathbb{Q}^d = \{0\}$. One is done by observing that the space perpendicular to E^u coincides with E^s_T (since it is invariant under A^T and does not contain E^u_T). □

Corollary 5.5 *If the matrix of a substitution ϕ is Pisot and non-singular, then ϕ is primitive and translation aperiodic.*

Proof: Regarding aperiodicity, note that T_ω has no closed orbits so neither does T. As for primitivity, note that (ii) of the fact forces the Perron eigenvector $\omega \geq 0$ to actually satisfy $\omega > 0$. Thus as soon as the direction of $A^N e_i$, $i = 1, \dots, d$, approximates $E^u = \mathrm{lin}(\omega)$ for some large N, then $A^N > 0$. □

We close this section with a digression explaining the relative irrelevance of any specific selection of the lengths of the basic tiles in the tiling space. In our case, these lengths are given by $\mathrm{pr}^u(e_i) = \omega^*_i$, $i = 1, \dots, d$ (assuming ω is of unit length), but choosing diffrent lengths yields a tiling flow conjugate to a simple time rescaling of our flow T^t on $\mathcal{T}_\phi$. This follows by combining Th. 1.3 and Th. 3.1. in [9] and can also be seen by using $\mathcal{F}_\phi$ as follows. Fix an arbitrary positive vector $l := (l_1, \dots, l_d)$ and let $\Sigma := l^\perp$ be the hyperspace perpendicular to l. Upon scaling l, we may require that $\langle l|\omega\rangle = 1$ so that $\mathrm{pr}^l : x \mapsto \langle x|l\rangle\omega$ is the projection onto E^u along Σ. Let us denote by $\mathcal{T}^l_\phi$ the space of strands of $\mathcal{F}_\phi$ taken modulo translation along Σ. Projecting via pr^l presents $\mathcal{T}_\phi^l$ as the tiling space of the substitution ϕ with the l_i now serving as the tile lengths. (As usual, if some l_i coincide, the tiles of the tiling are distiguished by labeling.) The associated translation flow is $T_l^t : \gamma \pmod{\Sigma} \to \gamma + t\omega \pmod{\Sigma}$.

Proposition 5.6 *The flow T_l on $\mathcal{T}_\phi^l$ is conjugated to the flow T on $\mathcal{T}_\phi$.*

Proof: First let us see that the natural projection $\pi^l : \mathcal{F}_\phi \to \mathcal{T}_\phi^l$ is injective and thus a homeomorphism. Indeed, suppose that $\gamma, \gamma + x \in \mathcal{F}_\phi$ for some $x \in \Sigma$. Upon decomposing $x = x_s + x_u$ where $x_s \in E^s$ and $x_u \in E^u$, we see that $\gamma + x_u, \gamma + x_u + x_s \in \mathcal{F}_\phi$ and thus $\gamma + x_u = \gamma + x_u + x_s$ by Theorem 5.3. Hence, $x_s = 0$ and so $x = 0$ since $x = x_u \in \Sigma \cap E^u = \{0\}$.

Taking π as in Theorem 5.3, we have then a homeomorphism $h := \pi^l \circ \pi^{-1} : \mathcal{T}_\phi \to \mathcal{T}_\phi^l$. That h conjugates the corresponding tiling flows is clear. □

Corollary 5.7 *For an appropriate choice of a constant roof function, the suspension flow over the shift map σ on the substitutive system X_ϕ is conjugated to the tiling flow T^t on $\mathcal{T}_\phi^{\min}$.*

Proof: The substitutive system X_ϕ arises as the translation closure of the bi-infinite words that are periodic under ϕ and thus coinsists of the words of the strands in $\mathcal{T}_\phi^{\min}$ (see Proposition 3.4). Taking l proportional to $e_1 + \cdots + e_d$, the flow T_l restricted to the minimal set of $\mathcal{T}_\phi^l$ is then a realization of a suspension flow of σ (were the roof function equals $\langle e_1 + \cdots + e_d | \omega \rangle^{-1} = (\omega_1 + \cdots + \omega_d)^{-1}$, as dictated by the normalization $\langle l | \omega \rangle = 1$). By the proposition, T_l is conjugated to T on $\mathcal{T}_\phi^{\min}$. □

6 Canonical torus h_ϕ in the unimodular case

If ϕ is Pisot and A is unimodular we say that ϕ is *unimodular Pisot*. The *canonical torus* h_ϕ for unimodular Pisot substitutions coincides with what is typically called the *geometric realization of* ϕ. In the non-unimodular Pisot case *the geometric realization* is the map induced by h_ϕ onto the inverse limit of the toral automorphism A (cf. [8]). Also, one can view h_ϕ as an implementation of the idea of *global shadowing* and thus extend the theory to an arbitrary hyperbolic substitution ϕ. (However, h_ϕ will not be “onto” unless ϕ is Pisot, which is connected with the absence of discrete spectrum in the non-Pisot case, cf. [34].)

For the sake of simplicity, we elucidate the unimodular case only. Keep in mind that any unimodular Pisot ϕ is automatically primitive and aperiodic (via Corollary 5.5).

Theorem 6.1 *Suppose that ϕ is a unimodular Pisot substitution. There is $M \in \mathbb{N}$ such that h_ϕ is at most M–to–1.*

In preparation for the proof of the theorem, for $p \in \mathbb{T}^d$, we shall write

$$\mathcal{F}^{R_0}_p := \{\gamma \in \mathcal{F}^{R_0} : \; p = q \pmod{\mathbb{Z}^d} \text{ for some vertex } q \text{ of } \gamma\} = h^{-1}(p) \cap \mathcal{F}^{R_0}.$$

Clearly, if $p_{-n} := A^{-n}p$, $\Phi^n(\mathcal{F}^{R_0}_{p_{-n}}) \supset \Phi^{n+1}(\mathcal{F}^{R_0}_{p_{-n-1}})$; and we claim that

$$h_\phi^{-1}(p) = h^{-1}(p) \cap \mathcal{F}_\phi = \bigcap_{n \geq 0} \Phi^n(\mathcal{F}^{R_0}_{p_{-n}}). \tag{6.1}$$

Indeed, having fixed $\gamma_0 \in h^{-1}(p) \cap \mathcal{F}_\phi$, $\Phi : \mathcal{F}_\phi \to \mathcal{F}_\phi$ being a homeomorphism yields $\gamma_{-n} := \left(\Phi|_{\mathcal{F}_\phi}\right)^{-n}(\gamma) \in \mathcal{F}_\phi$ for $n \in \mathbb{N}$; and $\gamma \in \Phi^n(\mathcal{F}_{p_{-n}})$ from $h \circ \Phi = A \circ h$, which shows "$\subset$" inclusion. For "$\supset$", it suffices to combine $\mathcal{F}^{R_0}_{p_0} \subset h^{-1}(p_0)$ with $\bigcap_{n\geq 0} \Phi^n(\mathcal{F}^{R_0}_{p_{-n}}) \subset \bigcap_{n \geq 0} \Phi^n(\mathcal{F}^{R_0}) = \mathcal{F}_\phi$.

Proof of Theorem 6.1: Fix $p \in \mathbb{T}^d$. It suffices to find $M \in \mathbb{N}$ (independent of p) such that $\#h^{-1}(p) \cap \mathcal{F}_\phi \leq M$. Consider an arbitrary finite subset $U \subset h^{-1}(p) \cap \mathcal{F}_\phi$. One can pick in each $\gamma_0 \in U$ a finite substrand η_0 so that the collection W of η_0's satisfies $\#W = \#U$ and no two strands in W are substrands of the same strand in $\mathcal{F}$.

Now, let $S_{p_{-n}}$ be the collection of all length three substrands of strands in $\mathcal{F}^{R_0}_{p_{-n}}$ whose middle edge intersects E^s. It is easy to see that there is $M \in \mathbb{N}$ independent of p_{-n} such that $\#S_{p_{-n}} \leq M$. For sufficiently large $n \in \mathbb{N}$, every strand of W is a substrand of $\Phi^n(\xi)$ where $\xi \in S_{p_{-n}}$. Therefore $\#U = \#W \leq \#S_{p_{-n}} \leq M$. By arbitrariness of U, we conclude $\#h^{-1}(p_0) \leq M$. □

7 Coincidence and h_ϕ (Coincidence Theorem)

We continue to consider a unimodular Pisot substitution ϕ. From the previous section, the minimal degree of h_ϕ,

$$m_\phi := \min\{\#h_\phi^{-1}(p) : \; p \in \mathbb{T}^d\}, \tag{7.1}$$

is finite. Our goal is to show that h_ϕ is a.e. m_ϕ-to-1 and to give a combinatorial characterization of m_ϕ. To formulate the results we need several definitions.

By a ***state*** we understand any strand I of length one (an edge) that intersects E^s at a point that is not the terminal vertex $\max I$ of I (where "max" refers to the obvious linear order, the one pulled back from E^u via pr^u). We shall denote by $\mathbb{S}$ the collection of all states and set, for $p \in \mathbb{T}^d$ and $R > 0$,

$$\mathbb{S}_p := \{I \in \mathbb{S} : \;\; p = \max I \pmod{\mathbb{Z}^d}\},$$

$$\mathbb{S}^R := \{I \in \mathbb{S} : \; I \subset \mathcal{C}^R\},$$

$$\mathbb{S}^R_p := \mathbb{S}^R \cap \mathbb{S}_p.$$

We shall refer to states in $\mathbb{S}_0$ as *integer states*.

Any strand γ contains at most one state, which we denote by $\hat{\gamma} \in \mathbb{S}$ (if it exists); and Φ induces $\hat{\Phi} : \mathbb{S} \to \mathbb{S}$ characterized by

$$\hat{\Phi}(\hat{\gamma}) = \Phi(\gamma)^\wedge$$

valid for any strand γ for which $\hat{\gamma}$ exists. Observe that $\hat{\Phi}(\mathbb{S}_p) \subset \mathbb{S}_{Ap}$ for $p \in \mathbb{T}^d$; in particular, integer states are preserved by $\hat{\Phi}$. Also, as a consequence of Lemma 5.1, $\hat{\Phi}(\mathbb{S}^{R_0}) \subset \mathbb{S}^{R_0}$, and for any $R > 0$ there is $n \in \mathbb{N}$ such that $\hat{\Phi}^n(\mathbb{S}^R) \subset \mathbb{S}^{R_0}$.

Definition 7.1 *For two edges I, J, we say that I and J are coincident along $E^s + t\omega$ denoted $I \sim_{t\omega} J$, iff there is $n \in \mathbb{N}$ such that $I - t\omega, J - t\omega \in \mathbb{S}_p$ for some $p \in \mathbb{T}^d$ and $(\Phi^n(I - t\omega))^\wedge = (\Phi^n(J - t\omega))^\wedge$. Also, I and J are coincident, denoted $I \sim J$, iff there is $t \in \mathbb{R}$ such that $I \sim_{t\omega} J$. A family of <u>states</u> $\mathcal{I} \subset \mathbb{S}$ is called non-coincident iff $\mathcal{I} \subset \mathbb{S}_p$ for some $p \in \mathbb{T}^d$ and no two different states in $\mathcal{I}$ are coincident. The coincidence rank of ϕ is the maximal cardinality of a non-coincident family:*

$$cr_\phi := \max\{\#\mathcal{I} : \ \mathcal{I} \subset \mathbb{S} \ \textit{non-coincident}\}. \tag{7.2}$$

We see that $I \sim J$ iff, for some $n \in \mathbb{N}$, the strands $\Phi^n(I)$ and $\Phi^n(J)$ coincide along an edge; and $\sim$ is translation equivariant: if $I, J, I + x, J + x$ are edges and $x \in \mathbb{R}^d$, then

$$I \sim J \quad \Leftrightarrow \quad I + x \quad \sim \quad J + x, \qquad x \in \mathbb{R}^d. \tag{7.3}$$

It is natural then that cr_ϕ can be determined over any $q \in \mathbb{T}^d$:

Fact 7.2 *For any $q \in \mathbb{T}^d$,*

$$cr_\phi = \max\{\#\mathcal{I} : \ \mathcal{I} \subset \mathbb{S}_q^{R_0} \ \textit{is non-coincident}\}.$$

Proof: Let $\mathcal{I} \subset \mathbb{S}_p$ be any non-coincident family. There exists $\epsilon > 0$ so that $\mathcal{I} - t\omega$ is a family of states, and thus also a non-coincident family, for all $0 \le t < \epsilon$; and so is $\mathcal{I} + x - t\omega$ for $x \in E^s$. Now, by density of the projection of E^s in $\mathbb{T}^d$, there is $R_1 > 0$ such that any $z \in \mathbb{T}^d$ can be represented as $z = p - t\omega + x (\text{mod } \mathbb{Z}^d)$ where $0 \le t < \epsilon$ and $x \in E^s$ with $|x| < R_1$. Thus, given $q \in \mathbb{T}^d$, we can produce a non-coincident family $\mathcal{I} + x - t \in \mathbb{S}^R_{A^{-n}q}$ where $R = R_1 + \text{diam}(\mathcal{I})$. If $n \in \mathbb{N}$ is chosen big enough, Lemma 5.1 assures that $\mathcal{J} := \hat{\Phi}^n(\mathcal{I} + x - t) \in \mathbb{S}_q^{R_0}$; and $\mathcal{J}$ is clearly non-coincident with $\#\mathcal{J} = \#\mathcal{I}$. □

As explained in Section 17, cr_ϕ can be readily computed for any given ϕ; and here is why this is a worthwhile task.

Theorem 7.3 (Coincidence Theorem) *Let h_ϕ be the canonical torus map of a unimodular Pisot substitution ϕ. The minimal degree of h_ϕ equals the coincidence rank of ϕ, $m_\phi = cr_\phi$, and there is a full measure G_δ-set $G^u_\phi \subset \mathbb{T}^d$ such that $\#h_\phi^{-1}(p) = cr_\phi$ exactly for $p \in G^u_\phi$. Moreover, the map $p \mapsto h_\phi^{-1}(p)$ is continuous at $p \in G^u_\phi$ (with the Hausdorff topology on closed subsets of $\mathcal{F}_\phi$) and if $h_\phi(\gamma_1) = h_\phi(\gamma_2) \in G^u_\phi$ for $\gamma_1 \neq \gamma_2$, then γ_1 and γ_2 are noncoincident: $(\gamma_1 + t\omega)^\wedge \not\sim (\gamma_2 + t\omega)^\wedge$ for all $t \in \mathbb{R}$.*

The proof of the theorem is relegated to the next section. For future reference let us introduce a special notation for the portion of $\mathcal{F}_\phi$ on which h_ϕ is generic, the E^u*-generic core of* $\mathcal{F}_\phi$:

$$\overset{u}{\mathcal{F}}_\phi := {h_\phi}^{-1}(G^u_\phi). \tag{7.4}$$

The over-stacked u serves to remind us that $\overset{u}{\mathcal{F}}_\phi$ is invariant under T, which is the action of E^u. In Section 18, it will become clear that the complement of the set G^u_ϕ consists of the E^u-translation orbit of the unstable boundary of the Markov partition of $\mathbb{T}^d$ associated to ϕ. The theorem yields the following picture:

Corollary 7.4 *$\overset{u}{\mathcal{F}}_\phi$ is a full measure G_δ subset of $\mathcal{F}^{min}_\phi$ invariant under Φ and the flow T; and the restriction $h_\phi|_{\overset{u}{\mathcal{F}}_\phi} : \overset{u}{\mathcal{F}}_\phi \to G^u_\phi$ is an m_ϕ-1 covering (alas between non-compact spaces).*

In particular, *the local product structure* for the toral automorphism lifts to $\overset{u}{\mathcal{F}}_\phi$, which suggests existence of a nice "stable foliation" in $\mathcal{F}_\phi$. This idea is developed in Section 14.

Proof of Corollary 7.4: Immediately from the theorem, we see that $h_\phi : \overset{u}{\mathcal{F}}_\phi \to G^u_\phi$ has a structure of a covering: it is m_ϕ-1 and there is $\epsilon > 0$ such that for an open subset $U \subset G^u_\phi$ of diameter less than ϵ, $\left(h_\phi|_{\overset{u}{\mathcal{F}}_\phi}\right)^{-1}(U) = U_1 \cup \ldots \cup U_{m_\phi}$ where $h_\phi|_{U_i} : U_i \to U$ is a homeomorphism.

To see that $\overset{u}{\mathcal{F}}_\phi \subset \mathcal{F}^{\min}_\phi$, let $\gamma \in \overset{u}{\mathcal{F}}_\phi$ and $p = h(\gamma) \in G^u_\phi$. Pick $t_k \in \mathbb{R}$, $t_k \to \infty$, such that $\lim_{k\to\infty} p + t_k\omega = p$ in $\mathbb{T}^d$. By continuity of ${h_\phi}^{-1}$, after perhaps passing to a subsequence of t_k, we find $\eta \in {h_\phi}^{-1}(p)$ such that $T^{t_k}(\eta) \to \gamma$. Thus $\gamma \in \mathcal{F}^{\min}_\phi$ (via Proposition 3.4). □

8 Proof of Coincidence Theorem

Lemma 8.1

$$cr_\phi \leq \min\{\#{h_\phi}^{-1}(p)^\wedge : \ p \in \mathbb{T}^d\} \leq m_\phi.$$

For $\gamma \in \mathcal{F}$, we shall use

$$\gamma|^N_{-N} := \gamma \cap (\mathrm{pr}^u)^{-1}([-N\omega, N\omega]). \tag{8.1}$$

For $K \subset \mathcal{F}$, $K|^N_{-N} := \{\gamma|^N_{-N} : \ \gamma \in K\}$. Although, $\gamma|^N_{-N}$ may fail to be a strand, we shall adopt $\Phi(\gamma|^N_{-N}) := \Phi_\gamma(\gamma|^N_{-N})$ as a natural way to extend the action of Φ to such partial strands (cf. 2.2).

Proof of Lemma 8.1: Fix $p \in \mathbb{T}^d$ and $n \in \mathbb{N}$. By Fact 7.2, there is a noncoincident family $\mathcal{I} = \{J_1, ..., J_c\} \subset \mathbb{S}^{R_0}_{A^{-n}p}$ with $cr_\phi = \#\mathcal{I}$. Since $\mathcal{I}$ is noncoincident, $\#\hat{\Phi}^n(\mathcal{I}) = \#\mathcal{I} = cr_\phi$; and we have

$$cr_\phi = \#\hat{\Phi}^n(\mathcal{I}) \leq \#(\Phi^n(\mathcal{F}^{R_0}_{A^{-n}p})^\wedge) = \#\hat{\Phi}^n(\mathbb{S}^{R_0}_{A^{-n}p}), \quad n \in \mathbb{N}, \tag{8.2}$$

where the inequality hinges on the states in $\mathcal{I}$ being substrands of strands in $\mathcal{F}^{R_0}_{A^{-n}p}$. In view of (6.1), taking $n \to \infty$, we obtain

$$cr_\phi \leq \lim_{n\to\infty} \#\Phi^n(\mathcal{F}^{R_0}_{A^{-n}p})^\wedge = \#h_\phi^{-1}(p)^\wedge \leq \#h_\phi^{-1}(p).$$

□

The opposite inequality $m_\phi \leq cr_\phi$ requires more work. We start by making some a priori observations about $p \in \mathbb{T}^d$ with $\#h_\phi^{-1}(p) \leq c$ for some $c \in \mathbb{N}$. Define, for $n, c \in \mathbb{N}$ and $N > 0$,

$$G^n_N(c) := \{p \in \mathbb{T}^d : \ \#\Phi^n(\mathcal{F}^{R_0}_{A^{-n}p})|^N_{-N} \leq c\} \tag{8.3}$$

$$G_N(c) := \bigcup_{n\geq 0} G^n_N(c) \tag{8.4}$$

$$G(c) := \bigcap_{N>0} G_N(c). \tag{8.5}$$

Fact 8.2 *(i)* $G(c) = \{p \in \mathbb{T}^d : \ \#h_\phi^{-1}(p) \leq c\}$.

(ii) $A(G_N(c)) \subset \mathrm{int}(G_N(c))$.

(iii) $|G(c)| = 1$ *iff* $|G(c)| > 0$ *iff* $\forall_N G_N(c) \neq \emptyset$.
(Here $|\cdot|$ *is the Haar measure on* $\mathbb{T}^d$*.)*

Proof: Set $K^n_p := \Phi^n\left(\mathcal{F}^{R_0}_{A^{-n}p}\right)$ so that (6.1) becomes

$$h_\phi^{-1}(p) = \bigcap_{n\geq 0} K^n_p. \tag{8.6}$$

(i): $\#h_\phi^{-1}(p) \leq c$ iff $\forall N \ \#h_\phi^{-1}(p)|^N_{-N} \leq c$ iff $\forall N \exists n \ \#K^n_p|^N_{-N} \leq c$ iff $p \in \bigcap_{N>0}\bigcup_{n\geq 0} G^n_N(c) = G(c)$, where the last but one equivalence uses that $K^n_p \supset K^{n+1}_p$ and that, for every $N \in \mathbb{N}$, the sequence $K^n_p|^N_{-N}$, $n \in \mathbb{N}$, stabilizes on $h_\phi^{-1}(p)|^N_{-N}$.

(ii): Fix $n_0 \in \mathbb{N}$ such that $\Phi^{n_0}(\mathcal{F}^{R_0+0.1}_q) \subset \mathcal{F}^{R_0}_{A^{n_0}q}$ for any $q \in \mathbb{T}^d$. It is enough to see

$$A(G^n_N(c)) \subset \mathrm{int}(G^{n+n_0+1}_N(c)).$$

Consider then $p \in G_N^n(c)$. We have

$$\#\Phi^{n+n_0+1}\left(\mathcal{F}^{R_0+0.1}_{A^{-n-n_0}p}\right)|^{\lambda N}_{-\lambda N} = \#\Phi\left(\Phi^n\left(\Phi^{n_0}\left(\mathcal{F}^{R_0+0.1}_{A^{-n_0}(A^{-n}p)}\right)\right)|^{N}_{-N}\right) \tag{8.7}$$

$$\leq \#\Phi^n\left(\mathcal{F}^{R_0}_{A^{-n}p}\right)|^{N}_{-N} \leq c \tag{8.8}$$

Now, since $R_0 + 0.1 > R_0$ and $\lambda N > N$, the above inequality yields

$$\#\Phi^{n+n_0+1}\left(\mathcal{F}^{R_0}_{A^{-n-n_0}q}\right)|^{N}_{-N} \leq c$$

valid for all q in an open ball U around p. Hence $A(U) \subset \mathrm{int}(G_N^{n+n_0+1}(c))$.

(iii): Clearly, $|G(c)| = 1$ iff $|G_N(c)| = 1$ for all $N \in \mathbb{N}$; and, by invoking ergodicity of the toral automorphism, $|G_N(c)| = 1$ iff $|G_N(c)| > 0$. However, by (ii) $|G_N(c)| > 0$ exactly when $G_N(c) \neq \emptyset$, which happens iff $|G_N^n(c)| > 0$ for some n. □

Now, to guarantee $G_N(c) \neq \emptyset$ we check that $\mathbb{S}_0^{R_0}$ can be *compressed* to at most c states by an iterate of Φ and a translation:

Lemma 8.3 *In order for $G_N(c) \neq \emptyset$ for all $N \in \mathbb{N}$ it suffices that, for some $q \in \mathbb{T}^d$, $n \in \mathbb{N}$ and $t \in \mathbb{R}$, we have that $(\Phi^n(I) - t\omega)^\wedge$ exists for all $I \in \mathbb{S}_q^{R_0}$ and*

$$\#\{(\Phi^n(I) - t\omega)^\wedge : \ I \in \mathbb{S}_q^{R_0}\} = c. \tag{8.9}$$

Proof. Fix $N \in \mathbb{N}$. It is easy to see that (8.9) must hold for a whole open segment of $t \in (t_0 - \epsilon, t_0 + \epsilon)$ for some $\epsilon > 0$. Since

$$\{(\Phi^n(I) - t_0\omega)^\wedge : \ I \in \mathbb{S}_q^{R_0}\} = \{(\Phi^n(I))^\wedge : \ I \in \mathbb{S}^{R_0}_{q-\lambda^{-n}t_0\omega}\},$$

we may as well assume that $t_0 = 0$ (at the expense of replacing q by $q - t_0\omega$). It follows that, for any $m \in \mathbb{N}$,

$$\#\{\Phi^{n+m}(I)|^{\lambda^m\epsilon}_{-\lambda^m\epsilon} : \ I \in \mathbb{S}_q^{R_0}\} \leq c,$$

which yields

$$\#\{\Phi^{n+m}(I)|^{N}_{-N} : \ I \in \mathbb{S}_q^{R_0}\} \leq c$$

provided m is taken large enough. Thus $A^{n+m}q \in G_N^{m+n}(c)$. □

The next logical step is to see that cr_ϕ is the minimum of c appearing in the previous lemma:

Lemma 8.4 *For any $R \geq R_0$,*

$$cr_\phi = \min\left\{\#\{(\Phi^n(I) - t\omega)^\wedge : \ I \in \mathbb{S}_0^R\}\right\} \tag{8.10}$$

where the minimum is taken over all $n \in \mathbb{N}$ and $t \in \mathbb{R}$ such that $(\Phi^n(I) - t\omega)^\wedge$ exists for all $I \in \mathbb{S}_0^R$.

Proof. First observe that $\#\{(\Phi^n(I)-t\omega)^\wedge : I\in\mathbb{S}_0^R\}\geq cr_\phi$. Indeed, pick a non-coincident family of maximal cardinality, $\mathcal{I}\subset\mathbb{S}_0$. Taking t and n as above, the family $(\Phi^n(\mathcal{I})-t\omega)^\wedge$ is well defined, and it must also be of cardinality cr_ϕ or otherwise some two states in $\mathcal{I}$ would be coincident. Hence, $\#\{(\Phi^n(I)-t\omega)^\wedge : I\in\mathbb{S}_0^R\}\geq\#(\Phi^n(\mathcal{I})-t\omega)^\wedge=cr_\phi$.

It is left to show that $c\leq cr_\phi$ where c is the right hand side minimum in (8.10). For $n\in\mathbb{N}$, let T_n be the set of $t\in\mathbb{R}$ realizing the minimum c so that if $t\in T_n$, then $\{\Phi^n(I): I\in\mathbb{S}_0^R\}$ *coalesces* into c edges upon traversing $E^s+t\omega$ (which is to mean that the strands in $\{\Phi^n(I): I\in\mathbb{S}_0^R\}$ contain only c different edges intersecting $E^s+t\omega$ at their interior or initial points). Clearly, $T_n\subset T_{n+1}$ for $n\in\mathbb{N}$, and non-empty T_n's exist and have interior. Thus, for any $r>0$ (to be specified later), by picking $t_0\in\operatorname{int}(\bigcup_{n\geq0}T_n)$ and taking $n\in\mathbb{N}$ sufficiently large, we can assure that the family of strands $\{\Phi^n(I): I\in\mathbb{S}_0^R\}$ coalesces into some c edges upon traversing $E^s+\tau\omega$ for every τ with $|\tau-t_0|<r$. Let $J_1,\ldots,J_c$ be the c edges to which $\{\Phi^n(I): I\in\mathbb{S}_0^R\}$ coalesces upon traversing $E^s+t_0\omega$. We require that r be large enough so that $|\tau-t_0|<r$ whenever $(E^s+\tau\omega)\cap\bigcup_{i=1}^c J_i\neq\emptyset$. That $c\leq cr_\phi$ follows once we show that the family $\{(\Phi^n(I)-t_0\omega)^\wedge : I\in\mathbb{S}_0^R\}=\{J_1',\ldots,J_c'\}$, where $J_i'=J_i-t_0\omega$, is non-coincident.

Suppose $J_i'\sim J_j'$ for some i,j so that $(\Phi^m(J_i')-t'\omega)^\wedge=(\Phi^m(J_j')-t'\omega)^\wedge$ for some $t'\in\mathbb{R}$ and $m\in\mathbb{N}$. Thus $\Phi^m(J_i)$ and $\Phi^m(J_j)$ coalesce upon traversing $E^s+(\lambda^m t_0+t')\omega$, which means that $\{\Phi^{n+m}(I): I\in\mathbb{S}_0^R\}$ coalesces to fewer than c edges upon traversing $E^s+(\lambda^m t_0+t')\omega$ — see Figure 8.1. This contradicts the minimality of c. $\square$

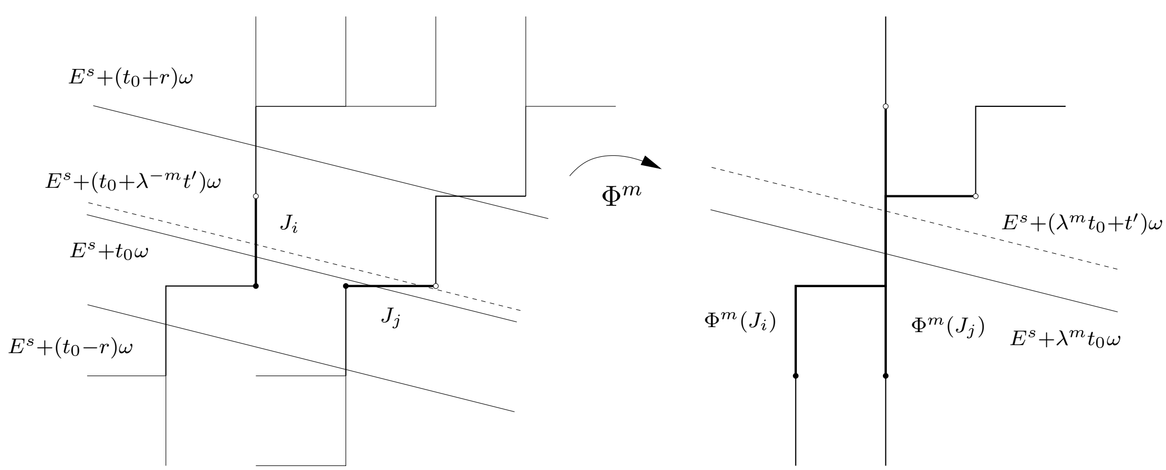

Figure 8.1: $\Phi^n(\mathbb{S}_0^R)$, on the left, further coalesces under some iterate Φ^m if $J_i\sim J_j$.

Conclusion of the proof of Theorem 7.3: By combining the last two lemmas, we see that $G(cr_\phi)$ is of full measure and so non-empty, which yields $m_\phi\leq cr_\phi$. In view of Lemma 8.1, we get $m_\phi=cr_\phi$. Now, from (i) of Fact 8.2, $G_\phi^u:=\{p\in\mathbb{T}^d :$

$\#{h_\phi}^{-1}(p) = m_\phi\} = G(cr_\phi)$. Also, via (ii),

$$\begin{aligned} G_\phi^u =& A(G_\phi^u) = A(G(cr_\phi)) = A\left(\bigcap_{N>0} G_N(cr_\phi)\right) \\ =& \bigcap_{N>0} A(G_N(cr_\phi)) \subset \bigcap_{N>0} \mathrm{int}(G_N(cr_\phi)) \subset G(cr_\phi) = G_\phi^u \end{aligned}$$

so that $G_\phi^u = \bigcap_{N>0} \mathrm{int}(G_N(cr_\phi))$ making G_ϕ^u a G_δ set.

Now, regarding the noncoincidence condition, suppose that $h_\phi(\gamma_1) = h_\phi(\gamma_2) = p \in G_\phi^u$ for $\gamma_1 \neq \gamma_2$ and γ_1 and γ_2 have coincident edges. Then for large enough $n \in \mathbb{N}$ and some $t \in \mathbb{R}$, $\Phi^n(T^t(\gamma_1))^\wedge = \Phi^n(T^t(\gamma_2))^\wedge$. But $\eta_i := \Phi^n(T^t(\gamma_i)) \in {h_\phi}^{-1}(q)$ where $q := A^n(p + t\omega) \in G_\phi^u$ since G_ϕ^u is invariant under both T and A. This violates Lemma 8.1, which guarantees that ${h_\phi}^{-1}(q) \ni \eta \mapsto \hat{\eta} \in {h_\phi}^{-1}(q)^\wedge$ is a bijection.

Now, by compactness, if $p_n \to p$ then for any $\epsilon > 0$ there is n_0 so that $n > n_0$ forces ${h_\phi}^{-1}(p_n) \subset B_\epsilon({h_\phi}^{-1}(p))$. Thus, for the continuity of the inverse ${h_\phi}^{-1}$, one has to show that, for $p \in G_\phi^u$, ${h_\phi}^{-1}(p) \subset B_\epsilon({h_\phi}^{-1}(p_n))$ for sufficiently large n. This follows from the following claim:

Claim 8.5 *Let $p \in G_\phi^u$ and $N > 0$ be given. There is $\epsilon > 0$ so that if $|y| \leq \epsilon$ then*

$${h_\phi}^{-1}(p + y)|_{-N}^{N} = ({h_\phi}^{-1}(p) + y)|_{-N}^{N}$$

Toward the proof of claim, note that, since $A(G_\phi^u) = G_\phi^u$, $p_1 := A^{-1}(p) \in G_\phi^u$ as well. Let then $n \in \mathbb{N}$ be such that $p_1 \in G_N^n(cr_\phi)$. Taking n_0 as in the proof of (ii) of Fact 8.2 and $\delta > 0$ sufficiently small, we get

$$\#\Phi^{n+n_0+1}\left(\mathcal{F}^{R_0+0.01}_{A^{-n-n_0}(p_1+x)}\right)|_{-N}^{N} \leq cr_\phi$$

for all $|x| < \delta$. Thus, by substituting $k := n + n_0 + 1$ and $y := Ax$,

$$\#\Phi^{k}\left(\mathcal{F}^{R_0+0.01}_{A^{-k}(p+y)}\right)|_{-N}^{N} \leq cr_\phi$$

for all $|y| < \epsilon$ provided $\epsilon > 0$ is small. Since ${h_\phi}^{-1}(p + y)|_{-N}^{N} \subset \Phi^k\left(\mathcal{F}^{R_0+0.01}_{A^{-k}(p+y)}\right)|_{-N}^{N}$ satisfies (due to Lemma 8.1)

$$\#{h_\phi}^{-1}(p + y)|_{-N}^{N} \geq \#{h_\phi}^{-1}(p + y)^\wedge \geq cr_\phi,$$

we have, in fact,

$${h_\phi}^{-1}(p + y)|_{-N}^{N} = \Phi^k\left(\mathcal{F}^{R_0+0.01}_{A^{-k}(p+y)}\right)|_{-N}^{N}. \tag{8.11}$$

On the other hand, ${h_\phi}^{-1}(p) \subset \Phi^k\left(\mathcal{F}^{R_0}_{A^{-k}(p)}\right)$ so that, after perhaps some further diminishing of $\epsilon > 0$, $|y| < \epsilon$ guarantees

$${h_\phi}^{-1}(p) + y \subset \Phi^k\left(\mathcal{F}^{R_0}_{A^{-k}(p)}\right) + y = \Phi^k\left(\mathcal{F}^{R_0}_{A^{-k}(p)} + A^{-k}y\right) \subset \Phi^k\left(\mathcal{F}^{R_0+0.01}_{A^{-k}(p+y)}\right).$$

As $p \in G^u_\phi$ forces $\#({h_\phi}^{-1}(p)+y)|_{-N}^{N} = cr_\phi$ (because the m_ϕ-strands in ${h_\phi}^{-1}(p)$ never share an edge) we actually have

$$({h_\phi}^{-1}(p)+y)|_{-N}^{N} = \Phi^k \left(\mathcal{F}^{R_0+0.01}_{A^{-k}(p+y)} \right) |_{-N}^{N}. \tag{8.12}$$

The claim follows by combining (8.11) and (8.12). □

9 Geometric Coincidence Conjecture and Pure Discrete Spectrum

All known examples support the following conjecture.

Conjecture 9.1 (Geometric Coincidence Conjecture) *The coincidence rank of any unimodular Pisot substitution is $cr_\phi = 1$.*

For $d = 2$, the conjecture is true (see Section 19). GCC gains particular importance in connection with the following conjecture, which has been open for some years.

Conjecture 9.2 (Pure Discrete Spectrum Conjecture) *The tiling flow of any unimodular Pisot substitution has pure discrete spectrum.*

Let us explain what that is to mean. First of all, for ϕ that is primitive, the flow T is not only minimal but also uniquely ergodic, i.e., it has a unique invariant probability measure μ_ϕ. In [34], Solomyak gives a proof of this fundamental fact for one-dimensional as well as higher-dimensional tiling spaces. Also, in [11], unique ergodicity of the shift map on *the substitutive system associated with* ϕ is shown. This shift map is naturally conjugate to the return map under T to the subset $\Sigma_\phi := \{\tau \in \mathcal{T}_\phi^{\min} : \ \tau \text{ has a vertex on } E^s\}$. The unique ergodicity of the return map and the flow are of course equivalent. While we refer the reader to the literature for the proofs, let us mention that the crux of the matter lies in the following consequence of the Perron-Frobenius theorem applied to A: each letter $i \in \mathcal{A}$ occurs in any bi-infinite word $[\tau]$ of a tiling $\tau \in \mathcal{T}_\phi$ with a well defined asymptotic frequency f_i independent of τ, and the vector of those frequencies $(f_1, \ldots, f_d)$ is proportional to the eigenvector $\omega = (\omega_1, \ldots, \omega_d)$. (We shall use that fact in Section 12.)

By definition, T has pure discrete spectrum iff its eigenfunctions are linearly dense in the space $L^2(\mathcal{T}_\phi, \mu_\phi)$ of μ_ϕ-square integrable functions. (By Corollary 5.7, T has pure discrete spectrum iff the shift map on the substitutive system X_ϕ has pure discrete spectrum.) A function $g : \mathcal{T}_\phi \to \mathbb{C}$ is an eigenfunction with eigenvalue α iff $g \circ T^t = e^{2\pi i \alpha t} g$ for all $t \in \mathbb{R}$. The classical Halmos-von Neumann theorem (see Ch. 3 in [39]) asserts that pure discrete spectrum for T is equivalent to T being measure theoretically isomorphic to a translation flow on a compact abelian topological group (taken with the Haar measure). The group's characters serve as the eigenfunctions. Although, we cannot prove PDSC at this time, the following theorem computes the spectrum of T and thus identifies the pertinent group as $\mathbb{T}^d$.

Theorem 9.3 (Spectrum Theorem) *If ϕ is a unimodular Pisot substitution, then the spectrum of its translation flow T equals $\{\langle k|\omega\rangle : k \in \mathbb{Z}^d\}$. Any $\alpha = \langle k|\omega\rangle$ is a simple eigenvalue with the corresponding eigenspace generated by the continuous function $\gamma \mapsto \chi_k \circ h_\phi(\gamma)$ where χ_k is a k^{th}-harmonic on $\mathbb{T}^d$: $\chi_k(x(mod\ \mathbb{Z}^d)) := e^{2\pi i\langle k|x\rangle}, x \in \mathbb{R}^d$.*

The theorem readily implies that the toral flow is *the maximal equicontinuous factor* of T (via $h_\phi : \mathcal{F}_\phi \to \mathbb{T}^d$) — as was suggested in [11].

Corollary 9.4 *GCC is equivalent to PDSC, i.e., for unimodular Pisot ϕ, $cr_\phi = 1$ iff T has pure discrete spectrum.*

Proof of Corollary 9.4 from Theorem 9.3: If $cr_\phi = 1$ then h_ϕ provides the sought after measure theoretical isomorphism to the translation on $\mathbb{T}^d$ (cf. the diagram (5.2)).

For the other implication, suppose that $cr_\phi > 1$ yet T has pure discrete spectrum. To get a contradiction, we shall construct a non-zero element of $L^2(\mathcal{T}_\phi, \mu_\phi)$ orthogonal to all the eigenfunctions.

Fix $p \in G^u_\phi$. By Corollary 7.4, for a small $\epsilon > 0$, $V := B_\epsilon(p) \cap G^u_\phi$ is well covered so that $h_\phi^{-1}(V) = U_1 \cup \ldots U_{cr_\phi}$ where U_j's are disjoint and $h_\phi|_{U_j} : U_j \to V$ is a homeomorphism. Observe that $h_\phi|_{U_j} : U_j \to V$ pushes μ_ϕ to $1/cr_\phi$ of the Haar measure on $\mathbb{T}^d$. (Indeed, by unique ergodicity, the push forward $(h_\phi)_*(\mu_\phi)$ is the Haar measure, and the a.e. defined Jacobian $J_\phi : \mathcal{T}_\phi \to \mathbb{R}$ must be equal to $1/cr_\phi$ because it is invariant under T and of integral one.)

Define a function $g \in L^2(\mathcal{T}_\phi, \mu_\phi)$ by $g|_{U_1} = 1$, $g|_{U_2} = -1$ and $g = 0$ elsewhere. Thus, for $k \in \mathbb{Z}$,

$$\int_{\mathcal{T}_\phi} g \cdot \chi_k \circ h_\phi = \int_{U_1} \chi_k \circ h_\phi - \int_{U_2} \chi_k \circ h_\phi = \frac{1}{cr_\phi} \int_V \chi_k - \frac{1}{cr_\phi} \int_V \chi_k = 0.$$

This contradicts the linear density of $\chi_k \circ h_\phi$'s in $L^2(\mathcal{T}_\phi, \mu_\phi)$ because g is clearly not a.e. 0. □

10 Coincidence classes for generic $\mathbb{S}_p$

In preparation for the proof of Theorem 9.3, we interpret the minimal degree of h_ϕ as the number of equivalence classes of $\sim_0$ on the space of states $\mathbb{S}_p$ over $p \in \mathbb{T}^d$. This is more precise than $m_\phi = cr_\phi$ but holds only for a generic set of $p \in \mathbb{T}^d$, flushed out by the following lemma. (The lemma serves also as the foundation for the development of the definition of the dual tiling in Section 14.)

Lemma 10.1 *There is a full measure dense G_δ set $G^s_\phi \subset \mathbb{T}^d$ invariant under E^s translations and the toral automorphism induced by A such that for $p \in G^s_\phi$ we have* *(i)[cardinality]* $\mathbb{S}_p$ *consists of exactly* m_ϕ *equivalence classes of* $\sim_0$*;*

(ii)[continuity] if $I \sim_0 J$ *for* $I, J \in \mathbb{S}_p$*, then* $I + z \sim_0 J + z$ *for all sufficiently small* $z \in \mathbb{R}^d$*;*

(iii)[relative density] there is an R_1 *such that, for any* $y \in E^s$*, each equivalence class in* $\mathbb{S}_p$ *has a representative in the cylinder* $y + \mathcal{C}^{R_1}$*.*

To complement $\overset{u}{\mathcal{F}}_\phi$, the E^u-generic core defined by (7.4), we introduce

$$\overset{s}{\mathcal{F}}_\phi := \{\gamma \in \mathcal{F}_\phi : \ h_\phi(\gamma) \in G^s_\phi\}, \quad \overset{su}{\mathcal{F}}_\phi := \overset{s}{\mathcal{F}}_\phi \cap \overset{u}{\mathcal{F}}_\phi \tag{10.1}$$

referred to as the E^s-*generic core of* $\mathcal{F}_\phi$ and the *generic core of* $\mathcal{F}_\phi$, respectively. All three $\overset{u}{\mathcal{F}}_\phi$, $\overset{s}{\mathcal{F}}_\phi$, and $\overset{su}{\mathcal{F}}_\phi$ are full measure Φ-invariant G_δ subsets of $\mathcal{F}_\phi$. (If ϕ is, for example, the Fibonacci substitution then the complement of G^s_ϕ is E^s mod $\mathbb{Z}^2$. Generally, it is of the form $C + E^s$ mod $\mathbb{Z}^d$ where $C \subset E^u$ is a zero-dimensional set.)

Proof of Lemma 10.1: We note first that for any $p \in \mathbb{T}^d$ and $R \geq R_1 := R_0 + \sqrt{d}$, $\mathbb{S}^R_p$ has at least m_ϕ equivalence classes of $\sim_0$. Indeed, fix any $q \in G^u_\phi$ (supplied by Theorem 7.3) to get m_ϕ-noncoincident strands $\gamma_1, \dots, \gamma_m \in {h_\phi}^{-1}(q)$. Pick $x \in [0,1]^d$ so that $q + x = p (\text{mod } \mathbb{Z}^d)$. By Theorem 7.3, $(\gamma_1 + x)^\wedge \dots (\gamma_m + x)^\wedge \in \mathbb{S}^R_p$ are mutually non-coincident.

To construct G^s_ϕ, for $R > R_1$, we define

$$D^n_R := \{p \in \mathbb{T}^d : \ \#\hat{\Phi}^n(\mathbb{S}^R_p) \leq m_\phi\}, \quad D_R := \bigcup_{n>0} D^n_R, \quad D := \bigcap_{R>0} \operatorname{int}(D_R). \tag{10.2}$$

Thus, from the definition of $\sim_0$, $p \in D_R$ iff $\mathbb{S}^R_p$ has at most (and thus, for $R > R_1$, exactly) m_ϕ equivalence classes of $\sim_0$; and $p \in D$ iff $\mathbb{S}^R_p$ has exactly m_ϕ such classes "stably" under small perturbation of p for any $R > R_1$. Note that D is E^s-invariant.

From now on we consider $R > R_1$. We claim that D_R is dense. Indeed, otherwise there would be $p \in \mathbb{T}^d$, $\epsilon > 0$ and (since $\mathbb{S}^R_p$ is finite) a single $I \in \mathbb{S}^R_p$ such that $I \not\sim_{t\omega} (\gamma_i + x)^\wedge$ for all $i = 1, \dots, m_\phi$ and all t with $|t| < \epsilon$ (where $\gamma_i + x$ is as before). By applying Φ^n to I and $\gamma_i + x$'s for n large enough (so that, say, $\lambda^n \epsilon > 100$), we would then get $m_\phi + 1$ strands intersecting E^s along pairwise noncoincident states — in contradiction with $m_\phi = cr_\phi$ (see Theorem 7.3).

Moreover, we claim $D_R \subset \overline{\operatorname{int}(D_R)}$. Indeed, $p \in D_R$ means exactly that there is $n \in \mathbb{N}$ such that $\#\hat{\Phi}^n(\mathbb{S}^R_p) = m_\phi$. But then $\#\hat{\Phi}^n(\mathbb{S}^R_{\tilde{p}}) = m_\phi$ for all $\tilde{p} := p - t\omega$ where $0 \leq t < \epsilon$ and $\epsilon > 0$ is sufficiently small. Coupled with E^s-invariance of $\sim_0$ this yields $\#\hat{\Phi}^n(\mathbb{S}^R_{\tilde{p}}) = m_\phi$ for all $\tilde{p}$ in a neighborhood of $p - \frac{\epsilon}{2}\omega$ thus placing $p - \frac{\epsilon}{2}\omega$ in $\operatorname{int}(D_R)$.

So far we know that $\mathrm{int}(D_R)$ is a dense open set. At the same time, for $R > R_0$, $\hat{\Phi}(\mathbb{S}^R_{A^{-1}p}) \subset \mathbb{S}^R_p$ yields $A^{-1}(D_R) \subset D_R$, so $\mathrm{int}(D_R)$ is in fact of full measure by ergodicity of the toral automorphism. Thus D is a full measure dense G_δ invariant under actions of E^s and the toral automorphism; and so is

$$G^s_\phi := D \setminus \left(E^s + \mathbb{Z}^d\right).$$

(i) follows immediately from $G^s_\phi \subset D$ and the construction of D.

(ii) alone can be easily seen to hold for all $p \notin E^s + \mathbb{Z}^d$.

As for (iii), we deal first with the special case of the cylinder centered at $y = 0$. From our initial discussion, we know that $\mathbb{S}^{R_1}_p$ contains representatives of m_ϕ equivalence classes for every p. For $p \in D$, there are no more classes in $\mathbb{S}_p$ and thus all are represented in $\mathbb{S}^{R_1}_p$.

To get (iii) in full generality, we translate along E^s: for $y \in E^s$, all the states of $\mathbb{S}_p$ in $y + \mathcal{C}^{R_1}$ constitute $\mathbb{S}^{R_1}_{p-y} + y$ and $p - y \in D$ whenever $p \in D$. $\Box$

11 Homoclinic returns and Stabilizers

For any compact metric space X with a flow T and a discrete action Φ, one can speak of the *homoclinic return times of a point* $p \in X$:

$$\mathcal{Z}^u_p := \{t \in \mathbb{R} : \ T^t(p) \in W^s(p)\}. \tag{11.1}$$

Here, $W^s(p)$ is the stable set of p with respect to Φ defined in the usual way: $W^s(p) := \{q \in X : \ \lim_{n\to\infty} \mathrm{dist}(\Phi^n(p), \Phi^n(q))\} = 0$.

As a basic example, think of the toral automorphism $A : \mathbb{T}^d \to \mathbb{T}^d$ induced by the matrix A and its unstable (holonomy) flow $x \pmod{\mathbb{Z}^d} \mapsto x + t\omega \pmod{\mathbb{Z}^d}$. By inspecting the situation in the universal cover $\mathbb{R}^d$, one easily concludes that the points of $\mathcal{Z}^u_p$ correspond to the projections of $\mathbb{Z}^d$ onto E^u; namely, $\mathcal{Z}^u_p \cdot \omega = \mathrm{pr}^u(\mathbb{Z}^d) = \{\langle v|\omega^*\rangle : \ v \in \mathbb{Z}^d\}$. Our main interest is of course in dealing with the translation flow T and the inflation-substitution map Φ on $\mathcal{F}$ (associated to a unimodular Pisot ϕ). For $\gamma \in \mathcal{F}$, the following definition supplies the counterpart of the $\mathbb{Z}^d$ in the above toral example.

Definition 11.1 *The stabilizer of $\gamma \in \mathcal{F}$ is*

$$\mathcal{Z}_\gamma := \{v \in \mathbb{Z}^d : \ \hat{\gamma} \sim_0 (\gamma + v)^\wedge\}$$

Inasmuch as $\mathcal{Z}_\gamma$ is not a priori subgroup of $\mathbb{Z}^d$, our terminology is perhaps a bit misleading. Nomenclature aside, $\mathcal{Z}_\gamma$ and $\mathcal{Z}^u_\gamma$ are bijective via pr^u:

Proposition 11.2 *Suppose that $p \notin E^s + \mathbb{Z}^d$ and $\gamma \in \mathcal{F}_p$. Then*

$$\mathcal{Z}^u_\gamma \cdot \omega = \mathrm{pr}^u(\mathcal{Z}_\gamma), \quad i.e., \quad \mathcal{Z}^u_\gamma = \langle \mathcal{Z}_\gamma | \omega^* \rangle.$$

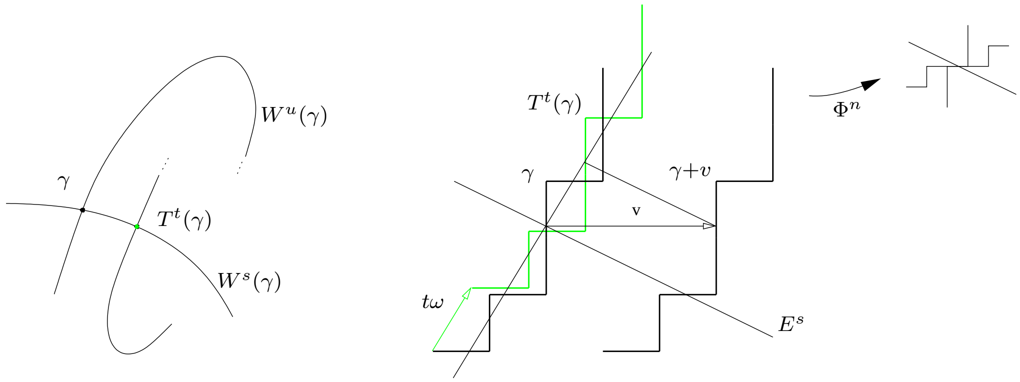

Figure 11.1: Homoclinic return time $t \in \mathcal{Z}_\gamma^u$ and the corresponding $v \in \mathcal{Z}_\gamma$.

Proof: First we show that, for any $p \in \mathbb{T}^d$ and $\gamma \in \mathcal{F}_p$, $t \in \mathcal{Z}_\gamma^u$ implies that $t\omega = \mathrm{pr}^u(v)$ for $v \in \mathcal{Z}_\gamma$.

Suppose that $\eta := \gamma + t\omega \in W^s(\gamma)$. Since h factors Φ to the toral automorphism, $h(\eta) \in W^s(h(\gamma))$ so that $h(\eta) + x = h(\gamma)$ for a (unique) $x \in E^s$. For $\kappa := \eta + x$, we have $h(\kappa) = h(\gamma) = p (\mathrm{mod}\ \mathbb{Z}^d)$ meaning that $\kappa = \gamma + v$ with $v := t\omega + x \in \mathbb{Z}^d$. Hence, $\Phi^n(\kappa), \Phi^n(\gamma) \in \mathcal{F}_{A^n p}$ and $\mathrm{dist}(\Phi^n(\kappa), \Phi^n(\gamma)) \to 0$ by the hypothesis on t. This forces $\Phi^n(\kappa)^\wedge = \Phi^n(\gamma)^\wedge$ for sufficiently large n, which means that $(\gamma+v)^\wedge \sim_0 \hat{\gamma}$ showing that $v \in \mathcal{Z}_\gamma$. Clearly, $t\omega = \mathrm{pr}^u(v)$.

Now, suppose that $v \in \mathcal{Z}_\gamma$ for some $\gamma \in \mathcal{F}_p$ with $p + \mathbb{Z}^d \cap E^s = \emptyset$. Then $\kappa := \gamma + v$ satisfies $I := \Phi^n(\kappa)^\wedge = \Phi^n(\gamma)^\wedge$ for some $n \geq 0$. Since $\Phi^k(I)$ is a common substrand of $\Phi^{n+k}(\kappa)$ and $\Phi^{n+k}(\gamma)$ and I intersects E^s with its interior by the hypothesis on $p = h(\gamma)$, we conclude that $\mathrm{dist}(\Phi^{n+k}(\kappa), \Phi^{n+k}(\gamma)) \to 0$, i.e., $\kappa = \gamma + v \in W^s(\gamma)$. It follows that $\gamma + t\omega \in W^s(\gamma)$ provided $t\omega = \mathrm{pr}^u(v)$ (since then $\gamma + v = \gamma + t\omega$ modulo E^s). $\square$

Here is the key link between the stabilizers and the GCC.

Proposition 11.3 *$cr_\phi = 1$ is equivalent with $\mathcal{Z}_\gamma = \mathbb{Z}^d$ for generic $\gamma \in \mathcal{F}_\phi$. Precisely, we have*

(i) If there is $\gamma \in \mathcal{F}_\phi$ with $\mathcal{Z}_\gamma = \mathbb{Z}^d$ then $cr_\phi = 1$;

(ii) If $cr_\phi = 1$ then $\mathcal{Z}_\gamma = \mathbb{Z}^d$ for all $\gamma \in \overset{s}{\mathcal{F}}_\phi$.

Proof: To begin, observe that any $\gamma \in \mathcal{F}_\phi$ contains edges of all types (by primitivity) so that $\mathbb{S}_p = \{(\gamma + v)^\wedge : v \in \mathbb{Z}^d\}$ for $p = h_\phi(\gamma)$.

(i): If $\mathcal{Z}_\gamma = \mathbb{Z}^d$ then $\mathbb{S}_p$ is a single equivalence class of $\sim_0$; and $cr_\phi = 1$ is immediate via Fact 7.2.

(ii): Suppose that $cr_\phi = 1$ and let $\gamma \in \overset{s}{\mathcal{F}}_\phi$, i.e., $p := h_\phi(\gamma) \in G_\phi^s$. By (i) of Lemma 10.1, $\mathbb{S}_p$ consists of exactly $m_\phi = 1$ equivalence classes of $\sim_0$. By our initial observation, $\mathcal{Z}_\gamma = \mathbb{Z}^d$. $\square$

The reader interested only in the proof of Theorem 9.3 may move on to the next section. We turn to some properties of $\mathcal{Z}_\gamma$ needed for Section 16.

Fact 11.4 *Equip the space of all subsets of $\mathbb{Z}^d$ with the compact-open topology. Fix any $\eta \in \mathcal{F}_\phi$.*
(i) If $\eta \in \overset{su}{\mathcal{F}}_\phi$ then $\eta + \mathrm{pr}^u(v) \in \overset{su}{\mathcal{F}}_\phi$ for all $v \in \mathbb{Z}^d$.
(ii) $\overset{su}{\mathcal{F}}_\phi \subset \overline{\{\eta + \mathrm{pr}^u(v) :\ v \in \mathcal{Z}_\eta\}}$.
(iii) $\mathcal{Z}_{\eta+\mathrm{pr}^u(v)} = \mathcal{Z}_\eta - v, \quad v \in \mathcal{Z}_\eta$;
(iv) The mapping $\mathcal{F}_\phi \ni \gamma \mapsto \mathcal{Z}_\gamma \subset \mathbb{Z}^d$ is continuous at $\gamma \in \overset{su}{\mathcal{F}}_\phi$; and for any $\gamma \in \overset{su}{\mathcal{F}}_\phi$

$$\mathcal{Z}_\gamma \in \overline{\{\mathcal{Z}_\eta - v :\ v \in \mathcal{Z}_\eta\}}$$

Proof: (i): This follows from invariance of G^u_ϕ and G^s_ϕ under the E^u and E^s translations, respectively.

(ii): Fix $\gamma \in \overset{su}{\mathcal{F}}_\phi$. By Corollary 15.5, there are $b, c \in \mathcal{F}^*_\phi$ dual to γ, η, $b \leftrightarrow \eta$ and $c \leftrightarrow \gamma$. Pick $t \in \mathbb{R}$ such that $T^t(\eta)$ is near γ. Pick $x \in E^s$ such that $T^*_x(b)$ is near c, as made possible by minimality of T^*. The idea is very general: near γ we have a *product structure* so one can adjust t, x so that $T^t(\eta)$ is dual to $T^*_x(b)$ while keeping those near γ and c, respectively. Then $T^t(\eta)$ is near γ and $t \in \mathcal{Z}^u_\eta$ — we are done. Since we did not define what is meant by the *product structure*; let us say this again very concretely: $I = (T^t(\eta))^\wedge$ is a state near $J = \hat{\gamma}$, and J also is near some state $K \in T^*_x(b)$ — all three being in particular of the same type. By a small adjustment of t, x, we can achieve that $I = K$, which results in $T^t(\eta)$ being dual to $T^*_x(b)$, as desired.

(iii): The equality is immediate from the definition of $\mathcal{Z}_\eta$.

(iv): Let $\gamma \in \overset{s}{\mathcal{F}}_\phi$, $c = \kappa(\gamma) \in \overset{s}{\mathcal{F}^*}_\phi$ be its dual, and $p = h_\phi(\gamma) = {h_\phi}^*(c) \in \mathbb{T}^d$. Fix any $N, R > 0$. Consider all $\eta \in \mathcal{F}_\phi$ with $\mathrm{dist}(\eta, \gamma) < \delta$ for a small $\delta > 0$. Denoting $y = h_\phi(\eta) - p$, if δ is sufficiently small then the inverse continuity claims for h_ϕ and ${h_\phi}^*$ guarantee that $\eta|^N_{-N} = (\gamma+y)|^N_{-N}$ and $(c+y)^R$ is the full equivalence class of $\hat{\eta}$ in $\mathbb{S}^R_{p+y}$. From the definition of $\mathcal{Z}_\gamma$, it follows that $\mathcal{Z}_\gamma$ and $\mathcal{Z}_\eta$ coincide on a ball about $0 \in \mathbb{Z}^d$ with radius proportional to $\min\{R, N\}$. This establishes the continuity.

The continuity, (i) and (iii) yield the rest of (iv). □

Corollary 11.5

$$\bigcup_{\gamma \in \overset{su}{\mathcal{F}}_\phi} \mathcal{Z}_\gamma = \bigcup_{v \in \mathcal{Z}_\gamma} \mathcal{Z}_\gamma - v, \quad \gamma \in \overset{su}{\mathcal{F}}_\phi,$$

and if $\overset{su}{\mathcal{F}}_\phi \ni \gamma \mapsto \mathcal{Z}_\gamma$ is constant then $\mathcal{Z}_\gamma$ is a subgroup of $\mathbb{Z}^d$ for $\gamma \in \overset{su}{\mathcal{F}}_\phi$.

Proof: The equality follows by putting together (ii), (iii) and (iv). That $\mathcal{Z}_\gamma$ is a subgroup can be seen from (iii) and (i). □

We note that the corollary is also true upon replacing $\overset{su}{\mathcal{F}}_\phi$ by $\mathcal{F}_\phi$ (see (i) of Fact 12.2), and $\mathcal{Z}_\gamma$ being a subgroup of $\mathbb{Z}^d$ for some $\gamma \in \mathcal{F}_\phi$ implies that ϕ satisfies GCC (see Corollary 12.6).

12 Subharmonicity precluded

In this section we study the additive subgroup of $\mathbb{Z}^d$ generated by all of the $\mathcal{Z}_\gamma$'s:

$$\mathcal{H}_\phi := \langle \bigcup_{\gamma \in \mathcal{F}_\phi} \mathcal{Z}_\gamma \rangle,$$

and show the following result

Theorem 12.1 (asubharmonicity) *For a unimodular Pisot substitution ϕ, we have $\mathcal{H}_\phi = \mathbb{Z}^d$.*

As will become increasingly clear, the above theorem says that there is no *geometric realization* onto a torus that is a covering of our canonical torus.

Before demonstrating the theorem, it is convenient to connect $\bigcup_{\gamma \in \mathcal{F}_\phi} \mathcal{Z}_\gamma$ with recurrence times of tiles of a fixed type. Fix for a moment $\gamma \in \mathcal{F}_\phi$ and define

$$\Theta_\phi(i) := \{\min J - \min I : \; J, I \text{ edges of } \gamma \text{ of the same type } i\}, \tag{12.1}$$

$$\Theta_\phi := \bigcup_i \Theta_\phi(i), \tag{12.2}$$

In the literature, one typically encounters the projection $\mathrm{pr}^u(\Theta_\phi)$ of Θ_ϕ to E^u (see e.g. [34]).

Fact 12.2 *The above defined objects are independent of $\gamma \in \mathcal{F}_\phi$ and, for any $i = 1, \ldots, d$, we have*

(i) $\bigcup_{\gamma \in \mathcal{F}_\phi} \mathcal{Z}_\gamma = \bigcup_{\gamma \in \overset{su}{\mathcal{F}}_\phi} \mathcal{Z}_\gamma = \bigcup_{n \in \mathbb{N}} A^{-n} \Theta_\phi(i)$,

(ii) $\mathcal{H}_\phi = \langle \Theta_\phi(i) \rangle$.

Proof: The independence on γ is an easy consequence of primitivity.

(i): Fix $\gamma \in \mathcal{F}_\phi$. If I, J are two edges of γ of the same type $i \in \{1, \ldots, d\}$, then obviously $\min J - \min I \in \mathcal{Z}_{\gamma + t\omega}$ for any $t \in \mathbb{R}$ such that $E^s + t\omega$ intersects I. In particular, there is always such $t = \langle v | \omega^* \rangle$ for some $v \in \mathbb{Z}^d$. Hence,

$$\Theta_\phi \subset \bigcup_{v \in \mathbb{Z}^d} \mathcal{Z}_{\gamma + \mathrm{pr}^u(v)}, \tag{12.3}$$

$$\bigcup_{n\in\mathbb{N}} A^{-n}\Theta_\phi \subset \bigcup_{v\in\mathbb{Z}^d, n\in\mathbb{N}} \mathcal{Z}_{\Phi^{-n}(\gamma)+\mathrm{pr}^u(v)}. \tag{12.4}$$

Now, besides being invariant under A, $G^u_\phi \cap G^s_\phi$ is invariant under $p \mapsto p + \mathrm{pr}^u(v) = p - \mathrm{pr}^s(v)$ for $v \in \mathbb{Z}^d$. Thus, assuming that $\gamma \in \overset{su}{\mathcal{F}}_\phi$, in the last inclusion $\Phi^{-n}(\gamma) + \mathrm{pr}^u(v) \in \overset{su}{\mathcal{F}}_\phi$, and we get

$$\bigcup_{n\in\mathbb{N}} A^{-n}\Theta_\phi \subset \bigcup_{\gamma\in\overset{su}{\mathcal{F}}_\phi} \mathcal{Z}_\gamma. \tag{12.5}$$

On the other hand, for any $\gamma \in \mathcal{F}$, if $v \in \mathcal{Z}_\gamma$, then $\Phi^n(\gamma)$ and $\Phi^n(\gamma + v) = \Phi^n(\gamma) + A^n v$ coincide on an arbitrarily long substrand provided n is large enough. In particular, any type i is represented in that substrand so that $A^n v \in \Theta_\phi(i)$, which proves:

$$\mathcal{Z}_\gamma \subset \bigcup_{n\in\mathbb{N}} A^{-n}\Theta_\phi(i), \quad i = 1, \ldots, d. \tag{12.6}$$

The two above inclusions establish all the equalities.

(ii): The "$\supset$" inclusion is clear. By primitivity, for sufficiently large $N > 0$, we have $A^N\Theta_\phi(i) \subset \Theta_\phi(j)$ for any i, j. In particular, $A^N\langle\Theta_\phi(i)\rangle \subset \langle\Theta_\phi(i)\rangle$; in fact, $A^N\langle\Theta_\phi(i)\rangle = \langle\Theta_\phi(i)\rangle$ by unimodularity of A. By (i), $v \in \mathcal{H}_\phi$ implies $A^N v \in \langle\Theta_\phi(i)\rangle$ so that $v \in A^{-N}\langle\Theta_\phi(i)\rangle = \langle\Theta_\phi(i)\rangle$. $\square$

We continue to consider an arbitrary fixed $\gamma \in \mathcal{F}_\phi$. To every coset $v + \mathcal{H}_\phi$, where v is a vertex of γ, we associate the types of edges of γ outgoing from the vertices in $v + \mathcal{H}_\phi$:

$$[v + \mathcal{H}_\phi]^+ := \{i : \;\; i \text{ is the type of an edge } I \text{ of } \gamma \text{ with } \min I \in v + \mathcal{H}_\phi\} \tag{12.7}$$

and the types of edges of γ incoming into the vertices in $v + \mathcal{H}_\phi$:

$$[v + \mathcal{H}_\phi]^- := \{i : \;\; i \text{ is the type of an edge } I \text{ of } \gamma \text{ with } \max I \in v + \mathcal{H}_\phi\}. \tag{12.8}$$

Fact 12.3

$$v + \mathcal{H}_\phi \neq u + \mathcal{H}_\phi \implies [v + \mathcal{H}_\phi]^+ \cap [u + \mathcal{H}_\phi]^+ = \emptyset \text{ and } [v + \mathcal{H}_\phi]^- \cap [u + \mathcal{H}_\phi]^- = \emptyset. \tag{12.9}$$

Proof: To see the first equality, suppose $i \in [v + \mathcal{H}_\phi]^+ \cap [u + \mathcal{H}_\phi]^+$. Then $\min I + \mathcal{H}_\phi = v + \mathcal{H}_\phi$ and $\min J + \mathcal{H}_\phi = u + \mathcal{H}_\phi$ for some edges I, J of γ of type i, which puts $v - u \in \Theta_\phi \subset \mathcal{H}_\phi$. The second equality is shown analogously $\square$

Lemma 12.4 *For any vertex $v \in \gamma$, $[v + \mathcal{H}_\phi]^+ = [v + \mathcal{H}_\phi]^-$.*

Proof: Recall (from Section 9) that by unique ergodicity of the T^t flow, each $i \in \{1, \ldots, d\}$ has a well defined frequency f_i of occurrence in γ and that f_i is proportional to ω_i (i.e., $f_i = \omega_i/C$ where $C = \sum_i \omega_i$).

By (12.9), an edge of type in $[v+\mathcal{H}_\phi]^-$ must be followed by an edge of type in $[v+\mathcal{H}_\phi]^+$, and an edge of type in $[v+\mathcal{H}_\phi]^+$ must be preceded by an edge of type in $[v+\mathcal{H}_\phi]^-$. This gives a bijective correspondence between occurrences of types in $[v+\mathcal{H}_\phi]^+$ and occurrences of types in $[v+\mathcal{H}_\phi]^-$. An easy estimate — based on the fact that the distance between the corresponding edges is uniformly bounded from above — shows that the frequency of $[v+\mathcal{H}_\phi]^+$ equals that of $[v+\mathcal{H}_\phi]^-$:

$$\sum_{i\in[v+\mathcal{H}_\phi]^+} \omega_i = \sum_{i\in[v+\mathcal{H}_\phi]^-} \omega_i.$$

That $[v+\mathcal{H}_\phi]^+ = [v+\mathcal{H}_\phi]^-$ follows then by independence of ω_i's over $\mathbb{Z}$ (see Fact 5.4). □

Conclusion of Proof of Theorem 12.1: Fix $i \in \{1,\ldots,d\}$. We shall show that $e_i \in \mathcal{H}_\phi$. By primitivity, there is an edge I of type i in γ. Set $u := I^-$ and $v := I^+$. We have $i \in [u+\mathcal{H}_\phi]^+ = [u+\mathcal{H}_\phi]^-$ and $i \in [v+\mathcal{H}_\phi]^-$. By (12.9), $u+\mathcal{H}_\phi = v+\mathcal{H}_\phi$ so that $e_i = v-u \in \mathcal{H}_\phi$. □

Before leaving this section, we propose the following technical conjecture motivated by a certain esthetic deficiency of the above proof:

Conjecture 12.5 *$\bigcup_{\gamma\in\mathcal{F}_\phi} \mathcal{Z}_\gamma$ is a subgroup of $\mathbb{Z}^d$.*

From Theorem 12.1, this subgroup must, in fact, be $\mathbb{Z}^d$. For comparison we record the following corollary (cf. Corollary 11.5).

Corollary 12.6 *$cr_\phi = 1$ iff $\mathcal{Z}_\eta$ is a subgroup of $\mathbb{Z}^d$ for some $\eta \in \mathcal{F}_\phi$.*

Proof: By Proposition 11.3, if $cr_\phi = 1$ then $\mathcal{Z}_\gamma = \mathbb{Z}^d$ for all $\gamma \in \overset{s}{\mathcal{F}}$. On the other hand, for any $\eta \in \mathcal{F}_\phi$, (i) of Fact 12.2 and Corollary 11.5 yield

$$\bigcup_{\gamma\in\mathcal{F}_\phi} \mathcal{Z}_\gamma = \bigcup_{\gamma\in\overset{su}{\mathcal{F}}_\phi} \mathcal{Z}_\gamma = \bigcup_{v\in\mathcal{Z}_\eta} \mathcal{Z}_\eta - v. \tag{12.10}$$

Thus, if $\mathcal{Z}_\eta$ is a subgroup, then

$$\bigcup_{\gamma\in\mathcal{F}_\phi} \mathcal{Z}_\gamma = \mathcal{Z}_\eta$$

and we conclude that $\mathcal{Z}_\eta = \mathbb{Z}^d$ from Theorem 12.1. $cr_\phi = 1$ follows now from (i) of Proposition 11.3. □

13 Discrete spectrum from stabilizers (proof of Spectrum Theorem)

In this section, we finalize determination of the spectrum for the tiling flow T and thus prove Theorem 9.3.

In preparation for the proof of the theorem, we establish a connection between *the homoclinic return times* and the spectrum valid in a general context of Φ and T that are, respectively, a homeomorphism and a flow on a compact metric space, both measure preserving and satisfying the commutation relation $\Phi \circ T^t = T^{\lambda t} \circ \Phi$. Below, we set $\exp(x) := e^{2\pi i x}$.

Lemma 13.1 (Solomyak) *Suppose that T is ergodic and, for some $t \in \mathbb{R}$ and $A \subset X$ of positive measure, $\lim_{n\to\infty} \operatorname{dist}(\Phi^n \circ T^t(x), \Phi^n(x)) = 0$ for all $x \in A$. Then for any eigenvalue α of T, we must have that $exp(\lambda^n \alpha t) \to 1$.*

The lemma is extracted from Theorem 4.3 in [34], which is analogous to an earlier result by Host for substitutive systems (see [13] and also Theorem VI.20 in [24]).

Proof (Solomyak): Let $g_\alpha \in L^2(X)$ be an eigenfunction corresponding to α, $g_\alpha \circ T^t = \exp(\alpha t) g_\alpha$, normalized so that $|g_\alpha| = 1$ a.e. (via ergodicity of T). For an arbitrary $\epsilon > 0$, one can approximate g_α (in $L^2(X)$) by a continuous g with $\|g - g_\alpha\| \le \epsilon$ and estimate as follows. Below, $\|u\|_B := \sqrt{\int_B |g^2|}$.

$$|\exp(\lambda^n \alpha t) - 1| \cdot \sqrt{|\Phi^n(A)|} = \tag{13.1}$$
$$|\exp(\lambda^n \alpha t) - 1| \cdot \|g_\alpha\|_{\Phi^n(A)} = \tag{13.2}$$
$$\|\exp(\lambda^n \alpha t) g_\alpha - g_\alpha\|_{\Phi^n(A)} = \tag{13.3}$$
$$\|g_\alpha \circ T^{\lambda^n t} - g_\alpha\|_{\Phi^n(A)} = \tag{13.4}$$
$$\|g_\alpha \circ T^{\lambda^n t} \circ \Phi^n - g_\alpha \circ \Phi^n\|_A = \tag{13.5}$$
$$\|g_\alpha \circ \Phi^n \circ T^t - g_\alpha \circ \Phi^n\|_A \le \tag{13.6}$$
$$\|g \circ \Phi^n \circ T^t - g \circ \Phi^n\|_A + \tag{13.7}$$
$$\|g_\alpha \circ \Phi^n \circ T^t - g \circ \Phi^n \circ T^t\|_A + \|g_\alpha \circ \Phi^n - g \circ \Phi^n\|_A \le \tag{13.8}$$
$$\|g \circ \Phi^n \circ T^t - g \circ \Phi^n\|_A + 2\epsilon \tag{13.9}$$

where the norm in the last line converges to 0 by the continuity of g and the hypothesis on A. Thus $|\exp(\lambda^n \alpha t) - 1| \cdot \sqrt{|A|} \to 0$, as desired. □

In the context of $\mathcal{F}_\phi$, the lemma can be reinterpreted as follows.

Corollary 13.2 *For $\gamma \in \overset{su}{\mathcal{F}}_\phi$, $t \in \mathcal{Z}^u_\gamma$, and α an eigenvalue, we have*

$$exp(\lambda^n \alpha t) \to 1. \tag{13.10}$$

Proof: First, we claim that $t \in \mathcal{Z}_\eta^u$ for all $\eta \in A := B_r(\gamma)$ provided $r > 0$ is sufficiently small. Indeed, recall that, by Proposition 11.2, $t \in \mathcal{Z}_\gamma^u$ iff $t = \mathrm{pr}^u(v)$ for $v \in \mathcal{Z}_\gamma$; and $v \in \mathcal{Z}_\eta$ for η sufficiently close to γ by the continuity assertion in (ii) of Fact 11.4.

To see that A has positive measure, observe that $\overset{su}{\mathcal{F}}_\phi \subset \overset{u}{\mathcal{F}}_\phi \subset \mathcal{F}_\phi^{\min}$ where the last set is the support of the invariant measure. □

Our next task is to understand α for which (13.10) can hold. This is an issue in the realm of the theory of Pisot numbers ([28, 6]) and our exposition parallels the development of a refined version of the classical Pisot theorem in [17].

We shall use the concept of *the Fourier dual* of a subset $\mathcal{Z} \subset \mathbb{R}^d$, which is

$$\mathcal{Z}^* := \{k \in \mathbb{R}^d : \ \langle v|k\rangle \in \mathbb{Z} \text{ for all } v \in \mathcal{Z}\}.$$

For $\mathcal{Z}$ that is a lattice (i.e. a discrete subgroup of rank d), $\mathcal{Z}^*$ is its *dual lattice.* Generally, $\mathcal{Z}^*$ is a subgroup of $\mathbb{R}^d$, $\mathcal{Z} \subset (\mathcal{Z}^*)^*$, and $\mathcal{Z}^*$ is a lattice in case $\mathcal{Z}$ is relatively dense in $\mathbb{R}^d$ (i.e. every ball of some fixed radius $R > 0$ contains an element of $\mathcal{Z}$). Also, $\mathcal{Z}^* = \langle \mathcal{Z}\rangle^*$ and $(\mathbb{Z}^d)^* = \mathbb{Z}^d$. Proofs are left as an exercise.

Lemma 13.3 (Pisot theory) *Let $\mathcal{Z} \subset \mathbb{Z}^d$ and, as before, let A be a Pisot unimodular matrix with leading eigenvalue $\lambda > 0$ and ω and ω^* be its right and left eigenvectors: $A\omega = \lambda\omega$ and $A^T\omega^* = \lambda\omega^*$.*
Given $\alpha \in \mathbb{R}$,

$$exp(\lambda^n \alpha t) \to 1 \textit{ for all } t \in \{\langle v|\omega^*\rangle : \ v \in \mathcal{Z}\}$$

implies

$$\alpha = \langle k|\omega\rangle \textit{ for some } k \in \mathcal{Z}^*.$$

Proof: We start with a key preliminary computation. Suppose that $t, \alpha \in \mathbb{R}$ are written in the form $t = \langle v|\omega^*\rangle$ and $\alpha = \langle k|\omega\rangle$ for some $v, k \in \mathbb{R}^d$; and $B = A^T$. We have

$$\lambda^n t\alpha = \langle v|B^n\omega^*\rangle\langle k|\omega\rangle = \langle A^n v|\omega^*\rangle\langle k|\omega\rangle = \langle\langle A^n v|\omega^*\rangle\omega|k\rangle = \langle \mathrm{pr}^u(A^n v)|k\rangle = \langle A^n v|k\rangle - \langle A^n v - \mathrm{pr}^u(A^n v)|k\rangle$$

where the second term converges to 0 (exponentially fast) because $A^n v - \mathrm{pr}^u(A^n v) = A^n(v - \mathrm{pr}^u(v))$ where $v - \mathrm{pr}^u(v) \in E^s$. In this way,

$$\exp(\lambda^n \alpha t) \to 1 \text{ iff } \exp(\langle A^n v|k\rangle) \to 1 \text{ iff } \exp(\langle v|B^n k\rangle) \to 1 \tag{13.11}$$

Now, suppose that (13.11) holds for some $\alpha = \langle k|\omega\rangle$ and any $v \in \mathcal{Z}$. Since $v \in \mathbb{Z}^d$, $\xi_v : x \mapsto \exp(\langle v|x\rangle)$ is a character on $\mathbb{T}^d$ and $G_v := \{x \in \mathbb{T}^d : \ \xi_v(x) = 1\}$ is a closed subgroup of $\mathbb{T}^d$. In terms of $p := k(\mathrm{mod}\, \mathbb{Z}^d) \in \mathbb{T}^d$ and the toral automorphism $f := f_B : \ \mathbb{T}^d \to \mathbb{T}^d$ induced by B, the condition $\exp(\langle v|B^n k\rangle) \to 1$ translates to $\mathrm{dist}(f^n(p), G_v) \to 0$. Observe that, in fact, $\mathrm{dist}(f^n(p), G_{\mathrm{inv}}) \to 0$

where $G_{\text{inv}} := \bigcap_{l\in\mathbb{Z}} f^l(G_v)$. (Indeed, otherwise there would be a subsequence $n_j \to \infty$, $z \in G_v$, and $l \in \mathbb{Z}$ such that $f^{n_j}(p) \to z$ and $f^l(z) \notin G_v$; and we get $f^{n_j+l}(p) \to f^l(z) \notin G_v$ contradicting $\text{dist}(f^n(p), G_v) \to 0$.)

Of course, G_{inv} is a closed subgroup of $\mathbb{T}^d$; we claim that G_{inv} is finite. To show that, consider its *full lift* $\Lambda := \{x \in \mathbb{R}^d : \ x(\text{mod } \mathbb{Z}^d) \in G_{\text{inv}}\}$. From $f(G_{\text{inv}}) = G_{\text{inv}}$ we have $B\Lambda = \Lambda$, and so $A\Lambda^* = \Lambda^*$. Since the Pisot hypothesis forces A to be irreducible over $\mathbb{Q}$, Λ^* must be of rank d. It follows that $(\Lambda^*)^*$ is a lattice in $\mathbb{R}^d$, which makes $\Lambda \subset (\Lambda^*)^*$ also a lattice and shows that $G_{\text{inv}} = \Lambda/\mathbb{Z}^d$ is finite.

Now that we know that G_{inv} is finite, it must just be a finite collection of periodic orbits of f, and we see that $\text{dist}(f^n(p), G_{\text{inv}}) \to 0$ implies that there is a single $p_0 \in G_{\text{inv}}$ such that $\text{dist}(f^n(p), f^n(p_0)) \to 0$. Viewed in the universal cover $\mathbb{R}^d$, this yields a lift $k_0 \in \mathbb{R}^d$ of p_0 such that $k - k_0$ is in the stable space of B so that k and k_0 have equal projections onto the unstable space:

$$\alpha = \langle k|\omega\rangle = \langle k_0|\omega\rangle.$$

What is more, $k_0 \ (\text{mod } \mathbb{Z}^d) = p_0 \in G_v$ translates to $\xi_v(k_0(\text{mod } \mathbb{Z}^d)) = 1$ that is $\langle v|k_0\rangle \in \mathbb{Z}$, and so $k_0 \in \mathcal{Z}^*$ by arbitrariness of $v \in \mathcal{Z}$. □

Conclusion of Proof of Theorem 9.3: Set $\mathcal{Z} := \bigcup_{\gamma\in\mathcal{F}_\phi} \mathcal{Z}_\gamma$. From Theorem 12.1, we have $\mathcal{H}_\phi = \langle\mathcal{Z}\rangle = \mathbb{Z}^d$. Consider an eigenvalue α of T. From Corollary 13.2 (via Proposition 11.2), the hypotheses of Lemma 13.3 are satisfied, and we conclude that $\alpha = \langle k|\omega\rangle$ where $k \in \mathcal{Z}^* = \langle\mathcal{Z}\rangle^* = (\mathbb{Z}^d)^* = \mathbb{Z}^d$.

That α is simple follows from ergodicity of the flow T. That any $\chi_k \circ h_\phi$ where $k \in \mathbb{Z}^d$ is an eigenfunction with eigenvalue $\langle k|\omega\rangle$ is clear since T factors via h_ϕ to the Kronecker flow on $\mathbb{T}^d$. □

14 Dual Tiling Space $\mathcal{F}^*_\phi$

For a unimodular Pisot ϕ, the tiling space flow T has for its orbits the one dimensional unstable sets of Φ thus providing a parametrization of those sets by $E^u \cong \mathbb{R}$. This section is devoted to a proof that there is a natural (measurable) action by $E^s \cong \mathbb{R}^{d-1}$ having the stable sets of Φ as its orbits. We shall not embark on a detailed study of this dual action here, although, in the subsequent section, we shall see that GCC is equivalent to the a.e. commutation of the E^u and E^s actions. Lemma 10.1 is the departure point of the considerations of this section.

Let $\mathcal{F}^*$ be the space that consists of collections of edges such that vertices of any two edges are congruent modulo $\mathbb{Z}^d$, each edge intersects E^s, and no two edges form a strand of length two (this last condition clearly regards only edges intersecting E^s at their endpoint). $\mathcal{F}^*$ is to be taken with the topology of Hausdorff convergence on compact subsets (of $\mathbb{R}^d$), which makes it a compact metrizable space.

Definition 14.1 *The* dual tiling space of a substitution *ϕ is*

$$\mathcal{F}^*_\phi := \overline{\bigcup_{p\in G^s_\phi} \{[I]_0 : I \in \mathbb{S}_p\}}$$

where $[I]_0$ is the equivalence class of $\sim_0$ of I and the closure is taken in $\mathcal{F}^*$. The E^s-generic core of $\mathcal{F}^*_\phi$ *is*

$$\overset{s}{\mathcal{F}^*}_\phi := \bigcup_{p\in G^s_\phi} \{[I]_0 : I \in \mathbb{S}_p\}.$$

The term *tiling space* for $\mathcal{F}^*_\phi$ will be justified by Proposition 18.4.

In this way, $c \in \mathcal{F}^*$ belongs to $\mathcal{F}^*_\phi$ iff for any $R, \epsilon > 0$ there is $t \in \mathbb{R}$ with $|t| < \epsilon$ such that $E^s + t\omega \cap \mathbb{Z}^d = \emptyset$ and the edges in c within R distance from 0,

$$c^R := \{I \in c : \ \operatorname{dist}(I, 0) < R\},$$

are precisely those edges within R distance from 0 that intersect $E^s + t\omega$ and are $\sim_{t\omega}$-equivalent (so that $\hat{\Phi}^n(c^R - t\omega)$ is a single state for some $n \in \mathbb{N}$). Note that every $c \in \mathcal{F}^*$ is relatively dense along E^s in the sense of (iii) of Lemma 10.1; in particular, c is nonempty.

Now, the natural action of E^s on $\mathcal{F}^*$ by translation induces an action, denoted T^*, of E^s on $\mathcal{F}^*_\phi$:

$$T^*_x(c) := c + x, \quad x \in E^s. \tag{14.1}$$

In the next section we show that this action is minimal (see Proposition 15.3). More subtly, if I is an edge intersecting E^s then denote by $\Phi^*(I)$ all the edges J such that I is an edge of $\Phi(J)$; and observe that $\Phi^*(I) \in \mathcal{F}^*$. We have then a natural

$$\Phi^* : \mathcal{F}^* \to \mathcal{F}^*, \quad \Phi^*(c) := \bigcup \{\Phi^*(I) : \ I \in c\}.$$

Since $\Phi^*(c) = \hat{\Phi}^{-1}(c)$ for $c \in \overset{s}{\mathcal{F}^*}_\phi$ and $\sim_0$ is $\hat{\Phi}$-invariant, $\overset{s}{\mathcal{F}^*}_\phi$ and $\mathcal{F}^*_\phi$ map to themselves under Φ^*. Let us record that $(\Phi^*)^{-1}$ is truly a very decent compactification of $\hat{\Phi}$:

Fact 14.2 $\Phi^* : \mathcal{F}^*_\phi \to \mathcal{F}^*_\phi$ *is a homeomorphism.*

Proof: Checking continuity of $\Phi^* : \mathcal{F}^* \to \mathcal{F}^*$ is left as an exercise. We shall prove that Φ^* is 1-1 and onto $\mathcal{F}^*_\phi$. Regarding "onto", for any $c \in \overset{s}{\mathcal{F}^*}_\phi$, we have $b := \{\hat{\Phi}(I) : \ I \in c\} \in \overset{s}{\mathcal{F}^*}_\phi$ and $\Phi^*(b) = c$, which shows that $\overset{s}{\mathcal{F}^*}_\phi$ is in the range, and thus so is its closure $\mathcal{F}^*_\phi$. As for "1-1", we record first that an element $c \in \mathcal{F}^*_\phi$ falls into one of three mutually disjoint categories:
interior: all edges of c intersect E^s at an interior point;
upper: c has some edges with min vertex on E^s;

lower: c has some edges with max vertex on E^s.

Of course all $c \in \overset{s}{\mathcal{F}^*}_\phi$ are *interior*. Also, observe that, for two edges I, J intersecting E^s, $\Phi^*(I) \cap \Phi^*(J) = \emptyset$ unless I and J form a strand through 0. It follows that, if we consider $b, c \in \mathcal{F}^*_\phi$ and suppose that $e := \Phi^*(b) = \Phi^*(c)$, then $b = c$ is immediate unless b is *upper* and c is *lower*, or vice versa.

To fix attention suppose that b is *upper* and thus has an edge $I = I_i + v$ for some $v \in E^s$ and $i \in \{1, \ldots, d\}$. Take j and n_0 so that $\phi^{n_0}(i) = j\ldots$ and $\phi^{n_0}(j) = j\ldots$, which implies that $I_i \sim_{t\omega} I_j$ for all sufficiently small $t > 0$. Because $b \in \mathcal{F}^*_\phi$, there are small $t_n > 0$ such that $t_n \to 0$ and all states among I_k, $k = 1, \ldots, d$, coincident along $E^s + t_n\omega$ belong to b; in particular, $I_j \in b$. Hence, $I_i \in (\Phi^*)^{n_0}(b)$. However, $I_i \notin (\Phi^*)^{n_0}(c)$ because c is *lower*; which shows that $(\Phi^*)^{n_0}(b) \neq (\Phi^*)^{n_0}(c)$ — a contradiction with $\Phi^*(b) = \Phi^*(c)$. □

The natural map $\mathcal{F}^* \to \mathbb{T}^d$ sending c to $p (\mathrm{mod}\ \mathbb{Z}^d)$ where p is a vertex of any of the states in c, restricted to $\mathcal{F}^*_\phi$, yields what we call the ***dual canonical torus*** (or the ***dual geometric realization***),

$$h_\phi{}^* : \mathcal{F}^*_\phi \to \mathbb{T}^d, \quad h_\phi{}^*(c) := \text{vertex of } c \ (\mathrm{mod}\ \mathbb{Z}^d).$$

Theorem 14.3 (Dual Coincidence Theorem) *$h_\phi{}^* : \mathcal{F}^*_\phi \to \mathbb{T}^d$ is at most M-to-1 for some $M \in \mathbb{N}$ with the minimal degree $\min\{\#(h_\phi{}^*)^{-1}(p) : \ p \in \mathbb{T}^d\} = m_\phi$. Moreover, $\#(h_\phi{}^*)^{-1}(p) = m_\phi$ for $p \in G^s_\phi$ and the mapping $p \mapsto (h_\phi{}^*)^{-1}(p)$ is continuous at $p \in G^s_\phi$.*

Proof: We start by noting the following consequence of Lemma 5.1. For every $R > 0$, there is $n \in \mathbb{N}$ such that

$$c^R \subset (\Phi^*)^n \left(c^{R_0}_{-n}\right) \subset c, \quad c \in \mathcal{F}^*, c_{-n} := (\Phi^*)^{-n}(c). \tag{14.2}$$

Indeed, by definition of Φ^*, $c = (\Phi^*)^n(c_{-n})$ implies that every edge $I \in c^R$ has $\Phi^n(I)$ passing through an edge $J \in c_{-n}$. By Lemma 5.1, $J \in c^{R_0}_{-n}$ if n is sufficiently large. Hence, $I \in (\Phi^*)^n \left(c^{R_0}_{-n}\right)$, which proves the first inclusion. The second inclusion is clear.

For a proof that the degree of $h_\phi{}^*$ is uniformly bounded, fix $p \in \mathbb{T}^d$ and let M be the number of different collections each consisting of edges intersecting E^s no furher than R_0 from E^u and with vertices $\mathbb{Z}^d$-congruent to some common $q \in \mathbb{T}^d$. Note that, taking any $R > 0$ and n as before, every $c^{R_0}_{-n} := (\Phi^*)^{-n}(c)^{R_0}$ where $c \in (h_\phi{}^*)^{-1}(p)$ forms such a collection. From (14.2), we conclude that $\#\{c^R : \ c \in (h_\phi{}^*)^{-1}(p)\} \leq M$, and so $\#(h_\phi{}^*)^{-1}(p) \leq M$ by arbitraryness of $R > 0$.

That $\#(h_\phi{}^*)^{-1}(p) = m_\phi$ for $p \in G^s_\phi$ is immediate from (i) of Lemma 10.1 provided we know:

Fact 14.4 $(h_\phi{}^*)^{-1}(p) \subset \overset{s}{\mathcal{F}^*}_\phi$ *for* $p \in G^s_\phi$

Proof: Here it suffices to see that, for $p \in G^s_\phi$ and $I_1, I_2 \in \mathbb{S}_p$, if there is a sequence $t_k \to 0$ such that $I_1 \sim_{t_k\omega} I_2$ then $I_1 \sim_0 I_2$. For a proof, let $\gamma_1, \gamma_2 \in \mathcal{F}_\phi$ be such that $J_1 := (\gamma_1 + x)^\wedge \sim_0 I_1$ and $J_2 := (\gamma_2 + x)^\wedge \sim_0 I_2$ — as constructed at the beginning of the proof of Lemma 10.1. By (ii) of Lemma 10.1, $I_i \sim_{t\omega} J_i$ for $i = 1, 2$ and all $|t| < \epsilon$ for some $\epsilon > 0$. In particular, $J_1 \sim_{t_k\omega} I_1 \sim_{t_k\omega} I_2 \sim_{t_k\omega} J_2$ meaning that γ_1 and γ_2 are coincident and thus equal; $I_1 \sim_0 I_2$ follows. □

Since m_ϕ is the cardinality of the fiber over a dense set G^s_ϕ, it must be the minimal fiber cardinality (since fibers are uniformly discrete).

Lastly, we shall establish continuity of the inverse, which is equivalent to the following claim.

Claim 14.5 *Let $p \in G^s_\phi$ and $R > 0$ be given. There is $\delta > 0$ such that if $|y| < \delta$ then*

$$({h_\phi}^*)^{-1}(p + y)^R = \left(({h_\phi}^*)^{-1}(p) + y\right)^R,$$

and the above collection of states is a full $\sim_0$ equivalence class in $\mathbb{S}^R_{p+y}$.

Proof of Claim: Fix $p \in G^s_\phi$. Consider $c \in ({h_\phi}^*)^{-1}(p)$. By (ii) of Lemma 10.1, for any two states $I, J \in c$, we have $I + y \sim_0 J + y$ provided $y \in \mathbb{R}^d$ is small. Since there are finitely many states in the R-ball around 0, there is then a common $\delta > 0$ such that if $|y| < \delta$, then $(c+y)^R$ is contained in a single equivalence class of $\sim_0$ for all $c \in ({h_\phi}^*)^{-1}(p)$. From the definition of $\mathcal{F}^*_\phi$, it follows that, any such $(c + y)^R$ is contained in some $b \in ({h_\phi}^*)^{-1}(p + y)$; hence, $({h_\phi}^*)^{-1}(p + y)^R \supset \left(({h_\phi}^*)^{-1}(p) + y\right)^R$. The other inclusion, $({h_\phi}^*)^{-1}(p + y)^R \subset \left(({h_\phi}^*)^{-1}(p) + y\right)^R$ is a general consequence of compactness and continuity, which imply that $({h_\phi}^*)^{-1}(p + y)$ is contained in the ϵ-nbhd of $({h_\phi}^*)^{-1}(p)$ for sufficiently small $\epsilon > 0$. □

The theorem is shown. □

For future reference we record the following commutation relations:

$$ {h_\phi}^* \circ \Phi^* = A^{-1} \circ {h_\phi}^*, \quad \Phi^* \circ T^*_x = T^*_{A^{-1}x} \circ \Phi^*. \tag{14.3}$$

15 Duality Isomorphism between $\mathcal{F}^*_\phi$ and $\mathcal{F}_\phi$

As the final step, we relate the tiling space and its dual. It is convenient to consider together with the already defined $\overset{s}{\mathcal{F}}_\phi, \overset{u}{\mathcal{F}}_\phi, \overset{su}{\mathcal{F}}_\phi, \overset{s}{\mathcal{F}^*}_\phi$ the following:

$$\overset{u}{\mathcal{F}^*}_\phi := \{\gamma \in \mathcal{F}^*_\phi : \ {h_\phi}^*(\gamma) \in G^u_\phi\} \tag{15.1}$$

$$\overset{su}{\mathcal{F}^*}_\phi := \overset{s}{\mathcal{F}^*}_\phi \cap \overset{u}{\mathcal{F}^*}_\phi. \tag{15.2}$$

From the construction, all these are full measure G_δ subsets of their ambient spaces invariant under the Φ and Φ^* actions, whichever applies. (Later, in Section 18, we shall see that these sets are complements of the E^u and/or E^s orbits of the stable/unstable boundaries of the natural Markov partitions into so called Rauzy fractals.)

Definition 15.1 *$c \in \mathcal{F}^*$ is* **dual** *to $\gamma \in \mathcal{F}$, denoted $c \leftrightarrow \gamma$, iff γ and c share an edge.*

Duality is easily seen to be a closed relation i.e. its graph $\mathcal{D} := \{(\gamma, c) : \gamma \cap c \neq \emptyset\}$ is closed in $\mathcal{F} \times \mathcal{F}^*$. Thus the restriction to $\mathcal{F}_\phi \times \mathcal{F}^*_\phi$,

$$\mathcal{D}_\phi := \mathcal{D} \cap \left(\mathcal{F}_\phi \times \mathcal{F}^*_\phi\right)$$

is also closed in $\mathcal{F}_\phi \times \mathcal{F}^*_\phi$.

Proposition 15.2 *The appropriate restrictions of $\mathcal{D}$ yield continuous maps $\kappa : \overset{s}{\mathcal{F}}_\phi \to \overset{s}{\mathcal{F}^*}_\phi$ and $\kappa^* : \overset{u}{\mathcal{F}^*}_\phi \to \overset{u}{\mathcal{F}}_\phi$ where $\kappa^* \circ \kappa = \mathrm{Id}$ and $\kappa \circ \kappa^* = \mathrm{Id}$ on their natural domains. In particular, duality induces a homeomorphism $\overset{su}{\mathcal{F}}_\phi \to \overset{su}{\mathcal{F}^*}_\phi$.*

Proof: The key is to realize that Theorem 7.3 and Lemma 10.1 assure that the restriction of duality to $\overset{su}{\mathcal{F}}_\phi \times \overset{su}{\mathcal{F}^*}_\phi$ yields (is a graph of) a bijection $\overset{su}{\mathcal{F}}_\phi \leftrightarrow \overset{su}{\mathcal{F}^*}_\phi$ (where $\gamma \leftrightarrow c$ amounts to $c := [\hat{\gamma}]_0$). Now, observe that any $\gamma \in \overset{s}{\mathcal{F}}_\phi$ has a dual and any $c \in \overset{u}{\mathcal{F}^*}_\phi$ has a dual; after all, this is so for $\gamma \in \overset{su}{\mathcal{F}}_\phi$, $c \in \overset{su}{\mathcal{F}^*}_\phi$, and $\overset{s}{\mathcal{F}}_\phi \subset \overline{\overset{su}{\mathcal{F}}_\phi}$, $\overset{u}{\mathcal{F}^*}_\phi \subset \overline{\overset{su}{\mathcal{F}^*}_\phi}$, and $\mathcal{D}_\phi$ is closed. Since generally, any two duals γ_1, γ_2 of the same c are coincident, there can be only one dual if $c \in \overset{u}{\mathcal{F}^*}_\phi$. Also, any two duals c_1, c_2 of $\gamma \in \overset{s}{\mathcal{F}}_\phi$, being equivalence classes of $\sim_0$ in $\mathbb{S}_p$ where $p = h_\phi(\gamma)$, must coincide since they share common state $\hat{\gamma}$. Continuity of κ and κ^* is immediate from $\mathcal{D}$ being closed. All other assertions also follow easily. □

In what follows we shall often silently use the identification of $\overset{su}{\mathcal{F}}_\phi$ and $\overset{su}{\mathcal{F}^*}_\phi$ via the duality. In particular, we can speak about a.e. defined actions: of E^s on $\mathcal{F}_\phi$ and E^u on $\mathcal{F}^*_\phi$. Note that $\overset{s}{\mathcal{F}^*}_\phi$ and $\overset{s}{\mathcal{F}}_\phi$ are invariant under the E^s-action and $\overset{u}{\mathcal{F}^*}_\phi$ and $\overset{u}{\mathcal{F}}_\phi$ are invariant under the E^u-action.

Proposition 15.3 (E^s minimality) *The natural E^s-action T^* on $\mathcal{F}^*_\phi$ is minimal.*

Proof: First of all notice that, among edges I of fixed type i, $\Phi^*(I)$ is determined by i uniquely up to a translation, and the number of edges of type j in $\Phi^*(I)$ is a_{ji}. Since A is primitive so is A^T and there is $n_0 > 0$ with $(\Phi^*)^{n_0}(I)$ containing edges of all types for each edge I. It follows that there is $R_2 > 0$ such that any $b \in \mathcal{F}^*_\phi$ has edges of all types in b^{R_2}. (Indeed, (iii) of Lemma 10.1 guarantees an edge $I \in \left((\Phi^*)^{-n_0}(b)\right)^{R_1}$, and so $(\Phi^*)^{n_0}(I) \subset b^{R_2}$ where R_2 depends on R_1 and n_0.)

Minimality follows readily from the following claim.

Claim 15.4 *Given* $c \in \overset{s}{\mathcal{F}^*}_\phi$ *and* $\epsilon > 0$ *there is* $C > 0$ *such that if* $b \in \overset{s}{\mathcal{F}^*}_\phi$ *then* $\operatorname{dist}(b + x, c) < \epsilon$ *for some* $x \in E^s$ *with* $|x| < C$.

(The density of $b + E^s$ for all $b \in \mathcal{F}^*_\phi$ can be seen by approximating with $b \in \overset{s}{\mathcal{F}^*}_\phi$, which is viable since C does not depend on b.)

It is left to prove the claim. Fix $c \in \overset{s}{\mathcal{F}^*}_\phi$, set $p := h_\phi{}^*(c)$, and take an arbitrary $\epsilon > 0$. Fix $R > 0$ large enough so that if $c_1, c_2 \in \mathcal{F}^*_\phi$ are such that $c_1^R = c_2^R + t\omega$ with $|t| < R^{-1}$, then $\operatorname{dist}(c_1, c_2) < \epsilon$. Since $p \in G^s_\phi$, there is $n \in \mathbb{N}$ such that $\hat{\Phi}^n(\mathbb{S}^R_p)$ consists of m_ϕ non-coincident states. This has an important consequence that there is $\delta > 0$ such that, for $K, L \in \mathbb{S}^R_p$ and $|t| < \delta$, we have

$$K \sim_0 L \text{ iff } K \sim_{t\omega} L. \tag{15.3}$$

By further increasing n, we may also require that $E^s - t\lambda^{-n}\omega$ interior intersects all the states in $\mathbb{S}^R_p$ and $\sqrt{d}\lambda^{-n} < \min\{\delta, R^{-1}\}$.

Set $I = \hat{\Phi}^n(c^R) \in \mathbb{S}^R_{A^n p}$. Find a state $J \in b_{-n} := (\Phi^*)^{-n}(b)$ such that $J \in b^{R_2}_{-n}$ and J is of the same type as I. Translate J to I: pick $x \in E^s$ and $t \in [-\sqrt{d}, \sqrt{d}]$ so that $I = J + x + t\omega$; clearly,

$$(\Phi^*)^n(I) = (\Phi^*)^n(J) + y + \lambda^{-n}t\omega, \quad y = A^{-n}x. \tag{15.4}$$

Note that $(\Phi^*)^n(I)$ is "a big patch" of c: $c^R \subset (\Phi^*)^n(I) \subset c$. If we also knew that $(\Phi^*)^n(J) + y$ is a "big patch" of $b + y$, i.e.

$$(b + y)^R \subset (\Phi^*)^n(J) + y \subset b + y, \tag{15.5}$$

then we would conclude that $\operatorname{dist}((\Phi^*)^n(I), (\Phi^*)^n(J) + y) < \epsilon$ and be done.

In (15.5), only the left inclusion poses a challenge: we have to show that any $K \in (b + y + t\lambda^{-n}\omega)^R \subset \mathbb{S}^R_p$ belongs to $(\Phi^*)^n(J) + y + t\lambda^{-n}\omega = (\Phi^*)^n(I)$. Pick then any state $L \in (\Phi^*)^n(I) \subset \mathbb{S}^R_p$. Since $L - t\lambda^{-n}\omega, K - t\lambda^{-n}\omega \in b + y \in \overset{s}{\mathcal{F}^*}_\phi$, we have that $L - t\lambda^{-n}\omega \sim_0 K - t\lambda^{-n}\omega$, i.e. $L \sim_{\lambda^{-n}t} K$. The choice of δ and n assures $L \sim_0 K$ and thus yields $K \in c^R \subset (\Phi^*)^n(I)$. □

For future reference we record the following corollary

Corollary 15.5 *Every* $c \in \mathcal{F}^*_\phi$ *has a dual* $\gamma \in \mathcal{F}_\phi$*, and every* $\gamma \in \mathcal{F}^{\min}_\phi$ *has a dual* $c \in \mathcal{F}^*_\phi$.

Proof: The natural projections of $\mathcal{D}_\phi$ into $\mathcal{F}_\phi$ and $\mathcal{F}^*_\phi$ are closed sets invariant under the actions of E^u and E^s, respectively. Thus our claim follows by minimality of these actions. □

16 Commutation of E^u and E^s actions and GCC

Having identified $\overset{su}{\mathcal{F}}{}^*$ and $\overset{su}{\mathcal{F}}$ via duality $\gamma \leftrightarrow c$ we can meaningfully talk about E^u and E^s acting (measurably) on the same space, (be that $\overset{su}{\mathcal{F}}{}^*$ or $\overset{su}{\mathcal{F}}$). In this section, we show that GCC is equivalent with commutation, or merely partial commutation, of these two actions T and T^*. This will be our departure point for development of algorithms for verifying GCC in the next section.

Proposition 16.1 *$cr_\phi = 1$ iff the E^u and E^s actions commute.*

Proof: If $cr_\phi = 1$ then the E^u and E^s actions are conjugated via h_ϕ to the translation (Kronecker) actions on $\mathbb{T}^d$ and they manifestly commute.

Assume that T and T^* commute. Consider an arbitrary $\gamma, T^r(\gamma) \in \overset{su}{\mathcal{F}}_\phi$, $r \in \mathbb{R}$. Commutation amounts to each of the two actions permuting the orbits of the other. In particular, since $W^s(\gamma)$ (restricted to $\overset{su}{\mathcal{F}}_\phi$) is the E^s-orbit of γ, The commutation implies that $T^r(W^s(\gamma) = W^s(T^r(\gamma))$. Now, $v \in \mathcal{Z}_\gamma$ iff $t := \mathrm{pr}^u(v) \in \mathcal{Z}^u_\gamma$ (see proposition 11.2) iff $T^t(\gamma) \in W^s(\gamma)$ iff $T^{t+r}(\gamma) \in T^r(W^s(\gamma)) = W^s(T^r(\gamma))$ iff $v \in \mathcal{Z}_{T^r(\gamma)}$; hence, $\mathcal{Z}_\gamma = \mathcal{Z}_{T^r(\gamma)}$. We conclude that $\mathcal{Z}_\gamma$ is constant on $\overset{su}{\mathcal{F}}_\phi$ by density of $T^r(\gamma)$, $r \in \mathbb{R}$, in $\overset{su}{\mathcal{F}}_\phi$ (see Fact 11.4).

To finish, Corollary 11.5 implies that $\mathcal{Z}_\gamma = \bigcup_{\gamma \in \overset{su}{\mathcal{F}}_\phi} \mathcal{Z}_\gamma$ is a subgroup of $\mathbb{Z}^d$, which must then be equal to $\mathbb{Z}^d$ by Theorem 12.1 (and (i) Fact 12.2). Proposition 11.3 yields $cr_\phi = 1$. □

In order to improve on the above proposition, let us weaken the condition on non-constancy of $\gamma \mapsto \mathcal{Z}_\gamma$ used above for for detecting failure of GCC.

Lemma 16.2 *If $cr_\phi > 1$ then*

$$\bigcap_{\gamma \in \overset{su}{\mathcal{F}}_\phi} \mathcal{Z}_\gamma = \{0\}.$$

Proof: By using (i) and (iii) of Fact 11.4, one sees that $\Gamma := \bigcap_{\gamma \in \overset{su}{\mathcal{F}}_\phi} \mathcal{Z}_\gamma$ is a subgroup of $\mathbb{Z}^d$. Being invariant under A, Γ is a co-compact lattice as soon as it is non-zero, which we assume. Now, observe that, for any $\gamma \in \overset{su}{\mathcal{F}}_\phi$, $\mathcal{Z}_\gamma$ is a union of cosets of Γ in $\mathbb{Z}^d$. Indeed, $v \in \mathcal{Z}_\gamma$ allows us to write $\Gamma \subset \mathcal{Z}_{\gamma + \mathrm{pr}^u(v)} = \mathcal{Z}_\gamma - v$, so $v + \Gamma \subset \mathcal{Z}_\gamma$. Consider then the function $g : \gamma \mapsto \mathcal{Z}_\gamma/\Gamma \subset \mathbb{Z}^d/\Gamma$. Because $g \circ \Phi = A \circ g$ and there is $n_0 \in \mathbb{N}$ with A inducing identity on $\mathbb{Z}^d/\Gamma$, we see that $g \circ \Phi^{n_0} = g$. By ergodicity of Φ^{n_0}, g is a.e. constant. By continuity of $\overset{su}{\mathcal{F}}_\phi \ni \gamma \mapsto \mathcal{Z}_\gamma$, g is constant on all of $\overset{su}{\mathcal{F}}_\phi$. This implies $cr_\phi = 1$ (as in the conclusion of the previous proof). □

Theorem 16.3 (partial commutation) *Suppose that $cr_\phi > 1$. For any $\epsilon > 0$, there is $D > 0$ such that if $K, L \in \mathbb{S}_p$ are two edges with $K \sim_{t\omega} L$ for all $t \in [-\epsilon, \epsilon]$ with $p + t\omega \in G^s_\phi$, then $\operatorname{dist}(K, L) < D$.*

Proof: Suppose $cr_\phi > 1$ yet the assertion of the theorem fails. We claim that there are then $\epsilon > 0$, $p \in \mathbb{T}^d$ and an infinite unbounded family of states in $\mathbb{S}_p$, $J_1, J_2, \ldots$, such that $J_i \sim_{t\omega} J_j$ for all $i, j \in \mathbb{N}$ and all $t \in [-\epsilon, \epsilon]$ with $p + t\omega \in G^s_\phi$. Indeed, by our hypothesis, there exist $\epsilon > 0$ and $K_n, L_n \in \mathbb{S}_{p_n}$, $n \in \mathbb{N}$, such that $\operatorname{dist}(K_n, L_n) > n$ and $K_n \sim_{t\omega} L_n$ for all $t \in [-2\epsilon, 2\epsilon]$ with $p_n + t\omega \in G^s_\phi$ and with all K_n of the same type. By compactness, one can arrange that the p_n converge to some $p \in \mathbb{T}^d$. Taking $v_n \in \mathbb{R}^d$ so that $p_n + v_n = p$ and $v_n \to 0$, one readily sees that $J_1 := K_n + v_n$, $J_2 := L_n + v_n$, $J_3 := L_{n+1} + v_{n+1}$, $J_4 := L_{n+2} + v_{n+2}$, ... are as desired provided n is large enough.

To finish, we use the claim to show $\bigcap_{\gamma \in \mathcal{F}^{su}_\phi} \mathcal{Z}_\gamma \neq \{0\}$, which yields the contradiction $cr_\phi = 1$, via the previous lemma.

To do that, for every $k \in \mathbb{N}$, pick from among the partial strands $\Phi^k(J_1)|^{\lambda^k\epsilon}_{-\lambda^k\epsilon}, \Phi^k(J_2)|^{\lambda^k\epsilon}_{-\lambda^k\epsilon}, \ldots$ two, call them α_k and β_k, that are disjoint and determine the same word $a_k := [\alpha_k] = [\beta_k]$, and intersect E^s at points x_k and y_k that are further than $100R_0$ apart. Here R_0 is as in Lemma 5.1; in particular, $\alpha_k \subset x_k + \mathcal{C}^{2R_0}$ and $\beta_k \subset y_k + \mathcal{C}^{2R_0}$. What is more, by replacing α_k and β_k with $\Phi^l(\alpha_k)|^{\lambda^k\epsilon}_{-\lambda^k\epsilon}$ and $\Phi^l(\beta_k)|^{\lambda^k\epsilon}_{-\lambda^k\epsilon}$ for some large $l > 0$, we may require as well that $\operatorname{dist}(x_k, y_k) < 200\lambda R_0$. Finally, let us translate α_k and β_k by a common vector in E^s so that $\alpha_k, \beta_k \subset \mathcal{C}^{200\lambda R_0 + 4R_0}$.

By passing to a subsequence if necessary, we have $a_k \to a$, $\alpha_k \to \alpha$, $\beta_k \to \beta$ for some bi-infinite word a and bi-infinite strands α, β. By construction, $\alpha(\text{mod } E^s), \beta(\text{mod } E^s) \in \mathcal{T}_\phi$ and $\alpha \sim_{t\omega} \beta$ for all $t \in \mathbb{R}$ for which $q + t\omega \in G^s_\phi$ where $q = h(\alpha)$. (Above we use that G^s_ϕ is invariant under the toral automorphism.)

From $\alpha(\text{mod } E^s) \in \mathcal{T}_\phi$, there is $x \in E^s$ so that $\gamma := \alpha + x \in \mathcal{F}_\phi$. Also, $\beta + x = \gamma + v$ for some $v \in \mathbb{R}^d \setminus \{0\}$. Note that $v \in \mathbb{Z}^d \setminus \{0\}$ because the vertices of α_k and β_k are in the same coset of $\mathbb{Z}^d$ and thus the same is true for α and β. We have $\gamma \sim_{t\omega} \gamma + v$ and thus $v \in \mathcal{Z}_{T^t(\gamma)}$ for all $t \in \mathbb{R}$ with $h_\phi(\gamma) + t\omega \in G^s_\phi$. This implies (via Fact 11.4) that $v \in \bigcap_{\gamma \in \mathcal{F}^{su}_\phi} \mathcal{Z}_\gamma$. □

Remark 16.4 *In the formulation of theorem, one can replace "for all $t \in [-\epsilon, \epsilon]$ with $p + t\omega \in G^s_\phi$" by "for a dense set of $t \in [-\epsilon, \epsilon]$".*

Proof: This is immediate from the following observation. From Lemma 10.1, for t such that $p + t\omega \in G^s_\phi$, if $I \sim_{t\omega} J$ then $I \sim_{s\omega} J$ for all s sufficiently close to t. □

17 Algorithms

Although GCC is open it is easy to algorithmically decide it for a particular substitution. Let us indicate how this can be done. For more a comprehensive picture see [34, 32, 30, 31].

Two $(I,J),(K,L) \in \mathbb{S}_0^{R_0} \times \mathbb{S}_0^{R_0}$ are considered translation equivalent iff $(I,J) = (K+v, L+v)$ for some $v \in \mathbb{Z}^d$. We shall refer to the translation equivalence classes in $\mathbb{S}_0^{R_0} \times \mathbb{S}_0^{R_0}$ as *configurations*; let V be the set of all such classes. One naturally constructs the *configuration graph of* ϕ as the directed graph $G_\phi^{(2)}$ with V serving as the set of vertices by putting an edge from $[(I,J)]$ to $[(K,L)]$ iff there is $w \in \mathbb{Z}^d$ such that $K + w \subset \Phi(I)$ and $L + w \subset \Phi(J)$. For specific ϕ, GCC can be decided by checking a simple connectivity property in $G_\phi^{(2)}$:

Proposition 17.1 (configuration graph) *$cr_\phi = 1$ iff from any vertex in V there is a path in the graph $G_\phi^{(2)}$ to a vertex in the diagonal $V^{diag} := \{[(I,I)] : \ I \in \mathbb{S}_0^{R_0}\}$.*

Proof: Simple induction on n yields from the definition of $G_\phi^{(2)}$ the following:

Claim: There is a path of length n from $[(I,J)]$ to $[(K,L)]$ iff there is $w \in \mathbb{Z}^d$ such that $K + w \subset \Phi^n(I)$ and $L + w \subset \Phi^n(J)$.

It follows that a finite path from $[(I,J)]$ to a vertex in the diagonal V^{diag} exists iff $I \sim J$. Thus if every vertex can be connected to the diagonal then $cr_\phi = 1$ (via Fact 7.2), and vice versa. □

The graph $G_\phi^{(2)}$ may be quite big and its practical construction is made awkward by the necessity to compute R_0 appearing in Lemma 5.1 (which requires *the adapted norm of* A). The inconvenience can be overcome by employing the E^s action.

Let Γ be the 1-skeleton of the unit cube in $\mathbb{R}^d$ (i.e. the union of all edges of $[0,1]^d$).

Proposition 17.2 (flipping) *$cr_\phi = 1$ iff, for a dense set of $t \in \mathbb{R}$, if $I, J \subset \Gamma$ are two edges intersecting $E^s + t\omega$ then $I \sim_{t\omega} J$.*

Proof: Let T_0 be the set of $t \in \mathbb{R}$ such that if $I, J \subset \Gamma$ are two edges intersecting $E^s + t\omega$ then $I \sim_{t\omega} J$. (By this definition, $t \in T_0$ if one of the two edges is not intersecting $E^s + t\omega$.)

If $cr_\phi = 1$ then Lemma 10.1 assures that $\mathbb{S}_p$ is a single equivalence class of $\sim_0$ for a dense G_δ E^s-invariant set of $p \in G_\phi^s \subset \mathbb{T}^d$. Since $I \sim_{t\omega} J$ iff $I - t\omega \sim_0 J - t\omega$, we conclude that $t \in T_0$ whenever $0 - t\omega \in G_\phi^s$. This makes T_0 a dense G_δ.

To prove the other implication, suppose that T_0 is dense. From the definition of $\sim_{t\omega}$, if $I \sim_{t\omega} J$ for some two edges and $t \in \mathbb{R}$, then $I \sim_{s\omega} J$ for all s is an open segment ending at t. Thus the interior of T_0, $\text{int}(T_0)$, is dense in $\mathbb{R}$ as well. By Baire category, $T_\infty := \bigcap_{v \in \mathbb{Z}^d} T_0 + \text{pr}^u(v)$ is a dense G_δ. Let $G \subset \mathbb{T}^d$ be the projection to the torus of $\bigcup_{t \in T_\infty} t\omega + E^s$; G is a dense G_δ.

Consider $p \in G$ and suppose that $K_1, \ldots, K_N \in \mathbb{S}_p$ are such that, for every $i = 1, \ldots, N-1$, K_i and K_{i+1} are in the skeleton in the same unit cube (i.e. $K_i, K_{i+1} \subset \Gamma + y$ for some $y \in \mathbb{R}^d$). By construction of G, $K_i \sim_0 K_{i+1}$ for every $i = 1, \ldots, N-1$. By transitivity, $K_1 \sim_0 K_N$. Since every two states in $\mathbb{S}_p$ can be obtained as K_1 and K_N above, $\mathbb{S}_p$ is a single equivalence class of $\sim_0$. Since G is dense in $\mathbb{T}^d$, Lemma 10.1 assures that $cr_\phi = 1$. □

By using Proposition 16.3, one can make do above by using any single face of Γ above. Indeed, fix any $i, j \in \mathcal{A}$, $i \neq j$, and let Γ_{ij} be the boundary of the unit square with vertices 0, e_i, e_j, $e_i + e_j$.

Proposition 17.3 (less flipping) *Fix any $i, j \in \mathcal{A}$, $i \neq j$. $cr_\phi = 1$ iff for a dense set of $t \in \mathbb{R}$, if $I, J \subset \Gamma_{ij}$ are two edges intersecting $E^s + t\omega$ then $I \sim_{t\omega} J$.*

Proof. As before, let T_0 be the of $t \in \mathbb{R}$ such that if $I, J \subset \Gamma_{ij}$ are two edges intersecting $E^s + t\omega$ then $I \sim_{t\omega} J$.

We already know that if $cr_\phi = 1$ then T_0 is dense. Suppose then that T_0 is dense. By imitating the previous argument, we get a dense G_δ set of $t \in \mathbb{R}$ such that for any two states K and L intersecting $E^s + t\omega$ and contained in the two dimensional *grid*

$$\bigcup_{k,l \in \mathbb{Z}} \Gamma_{ij} + ke_i + le_j$$

we have $K \sim_{t\omega} L$.

Now, for a fixed small $\epsilon > 0$, one easily obtains such K,L that are arbitrarily distant yet intersect $E^s + t\omega$ for a dense G_δ set of $t \in (-\epsilon, \epsilon)$. Thus $K \sim_{t\omega} L$ for $t \in (-\epsilon, \epsilon)$ and Proposition 16.3 forces $cr_\phi = 1$. □

Let us forge Proposition 17.3 into an algorithm. A pair of finite strands $B = (\alpha, \beta)$ with common endpoints will be called a ***bubble***. The bubble B is simple iff α and β have no common edges and B is ***coincident*** iff $\Phi^n(\alpha)$ and $\Phi^n(\beta)$ share an edge for some $n \in \mathbb{N}$, i.e. $\Phi^n(B) := (\Phi^n(\alpha), \Phi^n(\beta))$ fails to be simple for some $n \in \mathbb{N}$. If $\Phi^n(\alpha)$ and $\Phi^n(\beta)$ share an edge I for some $n \in \mathbb{N}$, then we can write $\Phi^n(\alpha) = \alpha_1 \cup I \cup \alpha_2$ and $\Phi^n(\beta) = \beta_1 \cup I \cup \beta_2$. Bubbles (α_1, β_1) and (α_2, β_2) obtained in this way are called ***offspring of*** (α, β), as is $\Phi^n(B)$ itself. By a ***balanced pair*** we understand a class of bubbles congruent by a translation in $\mathbb{R}^d$. The balanced pair $[B]$ associated to bubble B is of course uniquely determined by the words $[\alpha]$ and $[\beta]$; and the concepts of ***simplicity***, ***coincidence***, and an ***offspring*** extend to balanced pairs in an obvious way.

Proposition 17.4 (balanced pairs) *Fix arbitrarily any two letters $i \neq j \in \mathcal{A}$. $cr_\phi = 1$ iff the balanced pair (ij, ji) has finitely many simple offspring.*

Proof: ($\Leftarrow$): Suppose that (ij, ji) has finitely many offspring. Fix $C > 0$ so that offspring of (ij, ji) have length not exceeding C. Then any substrand of

$\Phi(I_i \cup (I_j + e_i))$ of length exceeding C shares an edge with $\Phi(I_j \cup (I_i + e_j))$. This means that if $I, J \subset \Gamma_{ij}$ are two edges intersecting $E^s + t\omega$ then $I \sim_{s\omega} J$ for $s \in \mathbb{R}$ with $|s - t| < C_1 \lambda^{-n}$ for some constant C_1 independent of n. We are done by arbitrariness of n and Proposition 17.2.

($\Rightarrow$): From Proposition 17.2, if $cr_\phi > 1$, then there is an open segment $U \subset \mathbb{R}$ and edges $I, J \subset \Gamma_{ij}$ such that $E^s + t\omega$ intersects I and J and $I \not\sim_{t\omega} J$ for all $t \in U$. It follows that the balanced pair $(\phi^n(ij), \phi^n(ji))$ contains an offspring of (ij, ji) of length exceeding $C_1|U|\lambda^n$ for some constant C_1 independent of n. Thus there are infinitely many simple offspring of (ij, ji). $\square$

The obvious implementation of the proposition — where one recursively tests for new offspring until none are generated — has the deficiency that it would not stop if GCC failed. This can be overcome by using Proposition 17.3 more indirectly to limit the number of vertices one has to consider in the configuration graph $G_\phi^{(2)}$ appearing in Proposition 17.1:

Remark 17.5 (essential configuration graph) *Fix arbitrarily two letters $i \neq j \in \mathcal{A}$. Let V_{ij} consist of all configurations in V that can be obtained as an end of a path in $G_\phi^{(2)}$ that starts at a configuration $[(I, J)]$ with $I, J \subset \Gamma_{ij}$. $cr_\phi = 1$ iff from any vertex in V_{ij} there is a path in the graph $G_\phi^{(2)}$ to a vertex in the diagonal $V^{diag} := \{[(I, I)] : \ I \in \mathbb{S}_0^{R_0}\}$*

Proof: One can adapt the proof of Proposition 17.1 to the current situation. We leave the details to the reader. $\square$

18 Cylinder Model, Rauzy fractals, and IFS's

The geometric realization map h_ϕ can be viewed as a composition $\mathcal{F}_\phi^{\min} \ni \gamma \mapsto \hat{\gamma} \mapsto \min \hat{\gamma} \pmod{\mathbb{Z}^d} \in \mathbb{T}^d$. Unlike h_ϕ, the map $\mathcal{F}_\phi^{\min} \ni \gamma \mapsto \hat{\gamma} \in \mathbb{S}$ is known to be a.e. 1-1 (Theorem 7.3) and — as long as GCC remains open — may serve as *a partial geometric realization*. Its image consists of fractal domains (*Rauzy sets*) constituting a Markov partition for $\hat{\Phi}$ and engaged in a *domain exchange transformation* giving rise to a flow isomorphic to T^t. Details follow. (As before, ϕ is assumed to be unimodular Pisot.)

Each box of the canonical Markov partition for Φ,

$$R_i := \{\gamma \in \mathcal{F}_\phi^{\min} : \ \gamma \text{ has an edge of type } i \text{ intersecting } E^s\}, \tag{18.1}$$

determines a subset of $\mathbb{R}^d$ given by

$$\Omega_i := \{\min(I) : \ I \text{ is an edge of } \gamma \text{ of type } i \text{ intersecting } E^s, \gamma \in R_i\}. \tag{18.2}$$

Note that Ω_i is *a rectangle* with respect to the product structure $\mathbb{R}^d = E^s \times E^u$; precisely, having set $\Omega_i^s := \Omega_i \cap E^s$ and $\Omega_i^u := \text{pr}^u(I_i - e_i)$ we have

$$\Omega_i = \bigcup_{t \in [0,1]} \Omega_i^s - t\text{pr}^u(e_i) = \Omega_i^s \times \Omega_i^u.$$

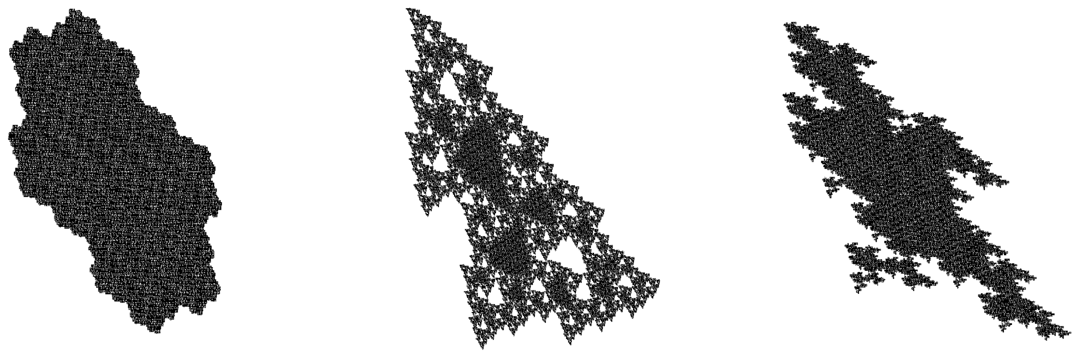

Figure 18.1: Rauzy Fractals for Tribonacci: $1 \mapsto 12$, $2 \mapsto 13$, $3 \mapsto 1$, and $1 \mapsto 12$, $2 \mapsto 31$, $3 \mapsto 1$, and $1 \mapsto 3211121$, $2 \mapsto 231121$, $3 \mapsto 3121$.

The sets Ω_i^s are called *Rauzy fractals* and their structure can be understood in the context of the dynamics induced on their union by Φ. Consider a disjoint union $\Omega := \coprod_{i=1}^d \Omega_i$ as a subset of $B := \coprod_{i=1}^d B_i$ where

$$B_i := \bigcup_{t \in [0,1]} E^s - te_i. \tag{18.3}$$

To avoid dealing with discontinuous maps, for each $y \in E^s$, identify the $2d$ points of the form $y - te_i$ where $t = 0, 1$ and $i = 1, \dots, d$, Denote by $\sim$ the resulting equivalence relation on B. Of course, $B/\sim$ is simply *a bouquet* of infinite Euclidean cylinders, each isomorphic to $E^s \times \mathbb{T}^1$ and all d meeting along a common copy of E^s. By associating to $\gamma \in \mathcal{F}$ the min vertex of an edge $I \subset \gamma$ intersecting E^s we get a well defined *partial geometric realization map*

$$\tau : \mathcal{F} \to B/\sim \qquad \text{with} \qquad \tau(\mathcal{F}_\phi^{\min}) = \Omega/\sim .$$

The inflation substitution map $\Phi : \mathcal{F} \to \mathcal{F}$ factors through τ to a map $F : B/\sim \to B/\sim$. To give an explicit description of F, consider *the transition graph $\mathcal{G}_\phi$ of the canonical Markov partition (into R_i's)*; $\mathcal{A}$ is the vertex set and there is a transition (a directed edge) e from i to j for each occurrence of j in $\phi(i)$. Let us also place over the edge e a vector weight $v_e \in \mathbb{Z}^d$ so that the pertaining edge J of type j in $\Phi(I_i)$ has $\min J = v_e$. One easily identifies then a subset $B_e \subset B_i$ of the form $E^s \times I_e^u$ where I_e^u is a segment in E^u so that

$$F(y) = Ay + v_e, \quad y \in B_e. \tag{18.4}$$

In fact, by projecting B_i onto $I_i - e_i$ along E^s, one can factor $F : B/\sim \to B/\sim$ to the winding map $f_\phi : \mathbb{T}_\phi \to \mathbb{T}_\phi$ from Section 4. The set B_e is the preimage of the subarc in $\mathbb{T}_\phi$ corresponding to the transition e for the Markov partition of $\mathbb{T}_\phi$ into $I_1, \dots, I_d$.

Below, any mention of a measure on $B/\sim$ refers to the natural measure induced by the Lebesgue volume in $\mathbb{R}^d$.

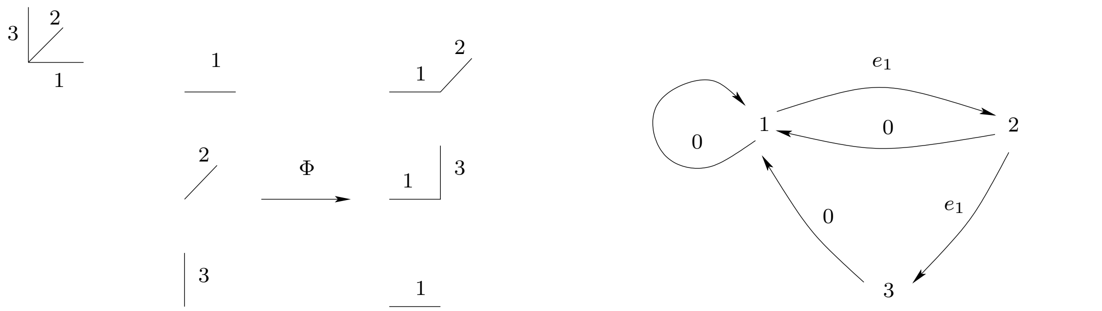

Figure 18.2: Transition graph for the Tribonacci substitution, $\phi : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$.

Proposition 18.1 (cylinder model)

(1-1) The restriction $\tau_\phi := \tau|_{\mathcal{F}_\phi^{\min}}$ *is 1-1 on a full measure dense* G_δ.

(dynamical 1-1) F *maps* $\Omega/\sim$ *onto itself and is a.e. 1-1 and measure preserving on* $\Omega/\sim$

(topreg) Ω_i *is topologically regular,* $\mathrm{cl}(\mathrm{int}(\Omega_i)) = \Omega_i$.

(measreg) the boundary of Ω_i *has zero measure,* $|\partial(\Omega_i)| = 0$.

(attractor) $\Omega/\sim$ *a.e. coincides with the global attractor of* F*; precisely, for sufficiently large* $R > 0$,

$$\bigcap_{n \geq 0} F^n \left(\{(i, y) \in B/\sim : \ |y| < R\} \right)$$

equals $\tau(\mathcal{F}_\phi)$.

Proof of Proposition 18.1: (1-1): From Theorem 7.3, $\overset{u}{\mathcal{F}}_\phi$ is a dense G_δ in $\mathcal{F}_\phi^{\min}$ and the mapping $\overset{u}{\mathcal{F}}_\phi \ni \gamma \mapsto \hat{\gamma} \in \mathbb{S}$ is 1-1. It follows that τ is $1-1$ on $\overset{u}{\mathcal{F}}_\phi$ with strands having a vertex on E^s removed.

(topreg): Let $\tilde{\Omega}_i := \{y \in \Omega_i : \ y (\mathrm{mod}\ \mathbb{Z}^d) \in G_\phi^u\}$. By continuity of the mapping $G_\phi^u \ni p \mapsto h^{-1}(p)$ as provided by Theorem 7.3, we see that $\tilde{\Omega}_i \subset \mathrm{int}(\Omega_i)$. On the other hand, $\overset{u}{\mathcal{F}}_\phi = {h_\phi}^{-1}(G_\phi^u)$ (as defined by (7.4)) is T invariant and thus dense in $\mathcal{F}_\phi^{\min}$ so that any $\tau(\gamma) \in \Omega_i$ can be approximated by $\tau(\eta)$ where $\eta \in \overset{u}{\mathcal{F}}_\phi$. Since $\tau(\eta) \in \tilde{\Omega}_i$ if the approximation is close enough, we have shown $\Omega_i \subset \mathrm{cl}(\tilde{\Omega}_i)$.

(dynamical 1-1): That $F(\Omega/\sim) = \Omega/\sim$ follows immediately from surjectivity of $\Phi : \mathcal{F}_\phi^{\min} \to \mathcal{F}_\phi^{\min}$. As for a.e. 1-1, suppose that $y \in \Omega_i$ and $z \in \Omega_j$ are such that $F(y) = F(z)$. Then the states $I := y + I_i$ and $J := z + I_j$ are coincident, $\hat{\Phi}(I) = \hat{\Phi}(J)$, which is only possible when $I, J \in \mathbb{S}_p$ for the measure zero set of $p \notin G_\phi^u$ (see Theorem 7.3). Finally, from (18.4), the Jacobian of F is $\det(A) = 1$ making the restriction of F to $\Omega/\sim$ measure preserving.

(attractor): Definition 5.2 of $\mathcal{F}_\phi$ readily implies that $\tau(\mathcal{F}_\phi) \subset \Lambda := \bigcap_{n\geq 0} F^n(B^R/\sim)$. By *growing states into strands*, it is not hard to see the opposite inclusion. Concretely, given a point $z \in \Lambda$, we have a sequence of states J_{-n} such that $\min J_0 = z$, $\hat{\Phi}(J_{-n-1}) = J_{-n}$, and $\min J_{-n}$ is uniformly bounded for all $n \geq 0$. The limit $\gamma := \lim_{n\to\infty} \Phi^n(J_{-n})$ is an infinite strand. If γ is bi-infinite, then $\gamma \in \mathcal{F}_\phi^{\min}$; otherwise, one can extend γ to a bi-infinite strand in $\mathcal{F}_\phi$. In any case, $\tau(\gamma) = z$.

(measreg): This is a general property of *Markov Boxes*. The boundary of $\Omega_i = \Omega_i^s \times \Omega_i^u$ consists of the stable boundary $\partial\Omega_i^s \times \Omega_i^u$ and the unstable boundary $\Omega_i^s \times \partial\Omega_i^u$. The later is manifestly of zero measure. Also, the union of the stable boundaries of $\Omega_1, \ldots, \Omega_d$ is invariant under $F|_{\Omega/\sim}^{-1}$ and thus is of measure zero. Here we used that, being a.e. conjugate to Φ, $F : \Omega/\sim \to \Omega/\sim$ is ergodic. □

For a more detailed study of the boundary of Ω_i, see [37] and the references therein.

We saw that by collapsing each B_i along E^s, the cylinder model $F : B/\sim \to B/\sim$ yields $f_\phi : \mathbb{T}_\phi \to \mathbb{T}_\phi$. ($\tau(\mathcal{F}_\phi)$ coincides with the preimage of the inverse limit of f_ϕ under the natural map from B to the bouquet of the circles.) One can also attempt to collapse each B_i along E^u onto its base E^s. As a result one obtains not a single map but an *Iterated Function System* of the substitution ϕ, denoted IFS_ϕ. Precisely, the IFS_ϕ consists of a disjoint union $E_1^s \cup \ldots E_d^s$ of d copies of E^s together with a system of contractions obtained by associating to every edge e of $\mathcal{G}_\phi$ the map $F_e : E^s \to E^s$ given by $y \mapsto Ay + \mathrm{pr}^s(v_e)$ where $\mathrm{pr}^s : \mathbb{R}^d \to E^s$ is the projection along E^u — cf. (18.4). We leave it to the reader to reinterpret Proposition 18.1 to get the following:

Remark 18.2 *$\Omega_1^s \cup \ldots \cup \Omega_d^s$ is the fixed point of IFS_ϕ. In particular, $\Omega_i^s \subset E_i^s$ is a union of translates of $A\Omega_j^s$ corresponding to the edges of $\mathcal{G}_\phi$ incoming into i.*

Note that the tautological map $\Omega/\sim \to \mathbb{T}^d$ is a.e. cr_ϕ-to-1 so that GCC holds for ϕ exactly when $\Omega_1, \ldots, \Omega_d$ (mod $\mathbb{Z}^d$) form a Markov Partition of $\mathbb{T}^d$. In fact, $|\Omega/\sim| = cr_\phi$ and the normalization $\sum_{i=1}^d \omega_i\omega_i^* = 1$ forces that

$$|\Omega_i| = cr_\phi \omega_i^* \omega_i. \tag{18.5}$$

(That $|\Omega_i|$ is proportional to $\omega_i\omega_i^*$ follows from Ω_i's being a Markov partition of $F : \Omega/\sim \to \Omega/\sim$ with transition matrix A^T and of F^{-1} with transition matrix A.) By projecting via $\mathrm{pr}^s : \mathbb{R}^d \to E^s$, these observations can be translated in terms of Ω_i^s's:

Remark 18.3 *The $d-1$-dimensional volume of Ω_i^s in E^s is $|\Omega_i^s| = cr_\phi\omega_i|\omega^*|$ where $|\omega^*|$ is the Euclidean norm of ω^*.*

Finally, the following proposition explains why $\mathcal{F}_\phi^*$ can be thought of as a space of tilings of E^s with tiles Ω_i^s, $i = 1, \ldots, d$.

Proposition 18.4 *For $c \in \mathcal{F}_\phi^*$ consider the family of sets*

$$\Omega(c) := \{-\Omega_i + \min I : \text{ where } i = [I] \text{ is the type (letter) of } I,\ I \in c\}.$$

The sets in $\Omega(c)$ have pairwise disjoint interiors and their union $\Omega(c)$ contains E^s. Thus to every $c \in \mathcal{F}_\phi^*$ we have associated a tiling of E^s into sets congruent to Ω_i^s's. The action of Φ^* on $\mathcal{F}_\phi^*$ corresponds to the inflation-and-substitution map on the space of thus obtained tilings of E^s. The transpose A^T of $A = (a_{ij})$ is the matrix of that map in the sense that each tile $A^{-1}(\Omega_i^s)$ naturally subdivides into a_{ji} of the Ω_j^s's.

Proof: (disjointness:) Suppose that $I, J \in c$, $i := [I]$, $j := [J]$, $a := \min I$, $b := \min J$, and disjointness fails: there is $q \in \operatorname{int}(-\Omega_i + a) \cap \operatorname{int}(-\Omega_j + b)$. Note that $I \sim J$ by definition of $\mathcal{F}_\phi^*$. Our goal is to contradict that and show $I \not\sim J$. Let K be an edge of type i with $\max K = q$ and L be an edge of type j with $\max L = q$. Now, if we transform $-\Omega_i + a$ and K via $x \mapsto -x + a$ and $-\Omega_j + b$ and L via $x \mapsto -x + b$, we obtain $\tilde{K}$ with $\min \tilde{K} = a - q \in \Omega_i$ and $\tilde{L}$ with $\min \tilde{L} = b - q \in \Omega_j$. By definition of Ω_i's, there are strands $\gamma, \eta \in \mathcal{F}_\phi^{\min}$ such that $\tilde{K}$ is an edge of γ and $\tilde{L}$ is an edge of η. By a small perturbation of q, we can assure that $h_\phi(\eta) = b - q \pmod{\mathbb{Z}^d} = a - q \pmod{\mathbb{Z}^d} = h_\phi(\gamma) \in G_\phi^u$. From Theorem 7.3, since $h_\phi(\eta) = b - q \pmod{\mathbb{Z}^d}$ and $h_\phi(\gamma) = a - q \pmod{\mathbb{Z}^d}$, we have that γ and η are then noncoincident; in particular, $\tilde{K} \not\sim \tilde{L}$. Thus $I \not\sim J$ because the pair (I, J) is congruent by translation to $(\tilde{K}, \tilde{L})$.

(covering): Fix $i, j \in \mathcal{A}$ and $n \in \mathbb{N}$. Let $\mathcal{E}_j$ be the set of all edges joining j to i in $\mathcal{G}_{\phi^n}$, the graph associated with ϕ^n. (Such edges are in natural bijective correspondence with paths from j to i of length n in $\mathcal{G}_\phi$, as well as with words a such that $\phi^n(j) = ais$ for some word s.) In view of $F^n(\Omega/\sim) = \Omega/\sim$, the formula (18.4) (applied to ϕ^n), and the Markov Property, we have

$$\Omega_i \subset \bigcup_{j=1}^{d} \bigcup_{e \in \mathcal{E}_j} \left(A^n \Omega_j + v_e\right).$$

Given a state I of type $[I] = i$ and with $\min I \in \Omega_i$, let $c \in \mathcal{F}_\phi^*$ be such that $\hat{\Phi}^{-n}(I) \subset c$. For each $J \in \hat{\Phi}^{-n}(I)$ with $[J] = j$, $A^n(\min J) + v_e = \min I$ for some $e \in \mathcal{E}_j$. It follows that

$$A^n \left(\bigcup_{J \in \hat{\Phi}^{-n}(I)} \left(\min J - \Omega_{[J]}\right) \right) = \min I - \bigcup_{j=1}^{d} \bigcup_{e \in \mathcal{E}_j} \left(A^n \Omega_j + v_e\right) \supset \min I - \Omega_i. \quad (18.6)$$

Thus

$$\bigcup_{J \in \hat{\Phi}^{-n}(I)} \left(\min J - \Omega_{[J]}\right) \supset A^{-n}(\min I - \Omega_i), \quad (18.7)$$

which means that the tiling associated with c covers $A^{-n}(\min I - \Omega_i) \cap E^s$. Since $(\min I - \Omega_i) \cap E^s$ contains a ball of positive radius $r > 0$ and $n \geq 0$ is arbitrary, we see that the tiling of c covers balls of arbitrarily large radius. By using minimality of T^*, one readily shows that the tiling associated to any alement of $\mathcal{F}_\phi^*$ covers all of E^s. □

By using (18.5), one readily sees that $cr_\phi = 1$ iff the union $\bigcup_{i=1}^d \Omega_i$ is essentially disjoint and forms a fundamental domain for $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ (i.e. tiles $\mathbb{R}^d$). Similar statements can be made in terms of tiling properties of Ω_i^s and one can augment Remark 18.3 as follows.

Remark 18.5 *$cr_\phi = 1$ iff the union $\bigcup_{i=1}^d \Omega_i^s$ is essentially disjoint and forms a fundamental domain for the sublattice generated by $\mathrm{pr}^s(e_i) - \mathrm{pr}^s(e_j)$, $i, j = 1, \dots, d$. Also, for $1 \leq i \leq d$, $cr_\phi = 1$ iff Ω_i^s is a fundamental domain of the sublattice of E^s generated by $\mathrm{pr}^s(e_j)$, $j \in \{1, \dots, d\} \setminus \{i\}$.*

Let us stress that this is certainly not new and a proof from an alternative point of view can be found in [30, 2], for instance.

Proof: That any of the assertions about Ω_i implies $cr_\phi = 1$ is clear by considering the measure of Ω_i per Remark 18.3. Suppose now $cr_\phi = 1$. Since $\bigcup_{i=1}^d \Omega_i$ is essentially disjoint and forms a fundamental domain for $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, we have $\mathbb{R}^d$ tiled by the family $\{-\Omega_k + v : \ v \in \mathbb{Z}^d,\ k \in \mathcal{A}\}$. Fix $i \in \mathcal{A}$. To show that Ω_i^s is the fundamental domain of the sublattice of E^s generated by $\mathrm{pr}^s(e_j)$, $j \in \{1, \dots, d\} \setminus \{i\}$, it suffices to show that the translates of Ω_i^s by the vectors of the sublattice are essentially disjoint. Suppose this is not so, which is to say that there are $u, v \in \mathbb{Z}^d$ with $u - v \in \mathbb{Z}_i^{d-1} := \{w \in \mathbb{Z}^d : \ \langle w | e_i \rangle = 0\}$ such that $\mathrm{pr}^s(-\Omega_i + v) \cap \mathrm{pr}^s(-\Omega_i + u)$ has nonempty interior in E^s. As in Proposition 18.4, $A^{-n}(-\Omega_i + v)$ passes through a certain family of Markov boxes labeled by the edges of $(\Phi^*)^n(I_i)$; precisely, the family is

$$C_v^n := \left\{-\Omega_{[J]} + \min J : J \in (\Phi^*)^n(I_i + v)\right\}.$$

Let us denote by C_u^n the analogous family for $A^{-n}(-\Omega_i + u)$. Observe that $C_u^n \cap C_v^n = \emptyset$ because $u - v \in \mathbb{Z}_i^{d-1}$ precludes existence of an edge K such that $\Phi^n(K)$ contains both $I_i + v$ and $I_i + u$. On the other hand, by our hypothesis, as $n \to \infty$, $A^{-n}(-\Omega_i + v)$ and $A^{-n}(-\Omega_i + u)$ are exponentially close (i.e. like $\mathrm{Const} \cdot \lambda^{-n}$) to each other inside a ball of exponentially growing radius. As a consequence, they both have to pass through the same Markov box at some point; that is, there must be $w \in \mathbb{Z}^d$ and $k \in \mathcal{A}$ such that they both pass through $-\Omega_k + w$. This means that $w + I_k \in C_u^n \cap C_v^n$ — a contradiction.

That $\bigcup_{i=1}^d \Omega_i^s$ is the fundamental domain for the lattice generated by $\mathrm{pr}^s(e_i) - \mathrm{pr}^s(e_j)$, $i, j = 1, \dots, d$, is shown in an analogous way based on the fact that $A^{-n}(\bigcup_{i=1}^d \Omega_i)$ passes through the Markov boxes corresponding to the edges of $\bigcup_{i \in \mathcal{A}} (\Phi^*)^n(I_i)$ and that if $u - v \in \{w \in \mathbb{Z}^d : \ \langle w | (1, \dots, 1) \rangle = 0\}$ then $\Phi^n(K)$ cannot have edges in both $\bigcup_{i \in \mathcal{A}} I_i + v$ and $\bigcup_{i \in \mathcal{A}} I_i + u$. □

Finally, let us briefly connect with the theory of substitution Delone sets which are the endpoints of the tiles of the tilings in $\mathcal{T}_\phi^{\min}$. (See [22] for background on model sets.) Delone sets that are *regular model sets* are known to generate a translation action with pure discrete spectrum (see [29]). The opposite implication has been established for lattice substitution sets in [19] and appears in the recent preprint [18] for general substitution Delone sets. In our (narrower) context, this is taken care of by the following easy remark.

Remark 18.6 *For a unimodular Pisot substitution ϕ, if the tiling flow T has pure discrete spectrum then, for any $\gamma \in \overset{u}{\mathcal{F}}_\phi$, the projection via pr^u of the vertices of γ is a regular model set in E^u, i.e., a generic tiling in $\mathcal{T}_\phi$ generates a regular model set.*

Proof: Pure discrete spectrum is equivalent to $cr_\phi = 1$ by Corollary 9.4. By Proposition 18.1, Ω^s is the closure of its interior with zero measure boundary (as a subset of E^s). Thus, for any $p \in G_\phi^u$, the set $\Lambda_p := \mathrm{pr}^u\left((\mathrm{pr}^s)^{-1}(\Omega^s) \cap (p + \mathbb{Z}^d)\right)$ is a regular model set (with Ω^s serving as the window). Now, given $u, v \in (\mathrm{pr}^s)^{-1}(\Omega^s) \cap (p + \mathbb{Z}^d)$, from the definition of the sets Ω_i, there are $\gamma, \eta \in \mathcal{F}_\phi$ such that u is a vertex of γ and v is a vertex of η. Since $cr_\phi = 1$ and $p \in G_\phi^u$ and $\gamma, \eta \in {h_\phi}^{-1}(p)$, we conclude that $\gamma = \eta$ (via Theorem 7.3). This shows that the points of $(\mathrm{pr}^s)^{-1}(\Omega^s) \cap (p + \mathbb{Z}^d)$ are exactly the vertices of a single $\gamma \in \overset{u}{\mathcal{F}}_\phi$. Thus Λ_p consists of the endpoints of the tiles of the tiling of γ. □

19 GCC in the case $d = 2$

In [5], it is shown that there is always a pair of basic edges that are coincident. For the case of alphabet with two letters ($d = 2$), this means that $I_1 \sim I_2$, which is used in [12, 32] to confirm that T has pure discrete spectrum. At this point it is easy to give an alternative proof of this fact. Let us retrace the argument of [5] and augment it to show that (GCC) holds for $d = 2$. As usual, we assume that ϕ is a unimodular Pisot substitution.

Among cardinality m subsets of $\mathbb{S}_p$ where p ranges over $\mathbb{T}^d$ consider the relation of congruence by translation and call the resulting equivalence classes m-*configurations*. Given a subset $C = \{J_1, \ldots, J_m\} \subset \mathbb{S}_p$ let $\min C := \max_{k=1}^m \min J_k$ and $\max C := \min_{l=1}^m \max J_l$ be the *innermost* min/max vertices of C. Within each m-configuration c, as a consequence of irrationality of E^s (see (i) of Fact 5.4), there is a unique representative set c_- with $\min c_- = 0$ and c_+ with $\max c_+ = 0$.

Let $m := cr_\phi$. Any point $p \in G_\phi^u$ determines a configuration $\hat{p}$ represented by the set $\{\hat{\gamma}_1, \ldots, \hat{\gamma}_m\}$ where ${h_\phi}^{-1}(p) = \{\gamma_1, \ldots, \gamma_m\}$. Denote by $\hat{\mathcal{A}}$ the (finite) set of all thus obtained configurations. By translating p along E^u, we can associate with p a parametrized bi-infinite word $[p] = (c_i)_{i \in \mathbb{Z}}$ in the alphabet $\hat{\mathcal{A}}$. To be precise, index the vertices of $\bigcup_{j=1}^m \gamma_j$ as a sequence $(v_i)_{i \in \mathbb{Z}}$ so that $\mathrm{pr}^u(v_i) = t_i\omega$

with $t_i < t_{i+1}$ for all $i \in \mathbb{Z}$, then $c_i := (p + t_i\omega)^\wedge$. (Thus p determines a tiling of $\mathbb{R}$ into segments $[t_i, t_{i+1}]$, labeled by c_i, which belongs to the tiling space of a certain substitution $\hat{\phi} : \hat{\mathcal{A}} \to \hat{\mathcal{A}}^*$ induced by ϕ. The underlying idea is that, unlike for ϕ, the tiling space of $\hat{\phi}$ is a priori measure theoretically isomorphic to the toral flow.) Note that $\bigcup_{j=1}^m \gamma_j$ can be assembled from the c_i in the sense that

$$\bigcup_{j=1}^{m} \gamma_j = \bigcup_{i\in\mathbb{Z}} (c_i)_- + v_i, \quad v_{i+1} - v_i = \max c_i - \min c_i. \tag{19.1}$$

Below, we say that configuration a follows b iff ab is a subword of some $[p]$ for $p \in G^u_\phi$.

Fact 19.1 *If a and b are two different m-configurations each of which follows c, then the symmetric difference of a_- and b_- consists of basic states, i.e.,*

$$a_- \triangle b_- \subset \{I_1, \dots, I_d\},$$

and, for every $t \in [0, \epsilon)$ with $0 - t\omega \in G^s_\phi$, there is a bijection $\sigma : a_- \setminus (a_- \cap b_-) \to b_- \setminus (a_- \cap b_-)$ such that if $K = \sigma(J)$ then $J \sim_{t\omega} K$.

Proof: The first assertion is immediate from the definitions. Pick $\epsilon > 0$ small enough so that $E^s + \epsilon\omega$ intersects all of the edges in a_- and b_-. Below we shall consider $t \in [0, \epsilon)$. By Theorem 7.3, the strands in ${h_\phi}^{-1}(p)$ are non-coincident so that no two different edges in a_- or b_- are equivalent with respect to $\sim_{t\omega}$. On the other hand, as long as $0 - t\omega \in G^s_\phi$, the equivalence relation $\sim_{t\omega}$ on $a_- \cup b_-$ can have at most (and so exactly) m different equivalence classes (see Lemma 10.1). It follows that every $J \in a_- \setminus (a_- \cap b_-)$ is $\sim_{t\omega}$-equivalent to some $K \in b_- \setminus (a_- \cap b_-)$ and such K is unique. The same argument applied with the roles of a_- and b_- reversed shows that $J \mapsto K$ is a bijection. $\Box$

Corollary 19.2 ([5]) *There exist configurations a, b, c satisfying the hypotheses of Fact 19.1. In particular, $I_i \sim I_j$ for some $i \neq j$.*

Proof: For a proof by contradiction, assume that every configuration $c \in \hat{\mathcal{A}}$ has only one configuration $b \in \hat{\mathcal{A}}$ that can follow c. Consider $p \in G^u_\phi$ and let $[p] = (c_i)_{i\in\mathbb{Z}}$. By our hypothesis, c_{i+1} is determined by c_i, which implies that the sequence $(c_i)_{i\in\mathbb{N}}$ is eventually periodic. Taking into account (19.1) and that c_i's and v_0 already determine all v_i's, we see that a large $M > 0$ and a positive vector $u \in \mathbb{Z}^d$ exist such that $C := \bigcup_{i\geq M} (c_i)_- + v_i \subset \bigcup_{j=1}^d \gamma_j \subset \mathcal{C}^{R_0}$ satisfies $T_u C \subset C$. In this way $T_{ku} C \subset \mathcal{C}^{R_0}$ for all $k \geq 0$, which is impossible because $u \notin E^u$. $\Box$

Proposition 19.3 *If $d = 2$ then $cr_\phi = 1$.*

Proof: Suppose $cr_\phi > 1$. Fix $p \in G^u_\phi$ and two different γ_1 and γ_2 in ${h_\phi}^{-1}(p)$. By the corollary $I_1 \sim I_2$ so that γ_1 and γ_2 do not intersect and are separated by some minimal E^s-distance $\delta_1 > 0$; namely, $\delta_1 := \min_{t\in\mathbb{R}}\{\mathrm{diam}((E^s + t\omega) \cap (\gamma_1 \cup \gamma_2))\}$. By minimality of T, we pick $t_n \to \infty$ such that $\gamma_1 + t_n\omega \to \gamma_2$. Upon passing to a subsequence, we can require that $\gamma_2 + t_n\omega \to \eta_1$ for some $\eta_1 \in \mathcal{F}_\phi$. By construction, $\min_{t\in\mathbb{R}}\{\mathrm{diam}((E^s+t\omega)\cap(\gamma_2\cup\eta_1))\} = \delta_1$. More importantly, because E^s is 1-dimensional and γ_1 and η_1 must lie on opposite sides of γ_2, we have

$$\min_{t\in\mathbb{R}}\{\mathrm{diam}((E^s + t\omega) \cap (\gamma_1 \cup \eta_1))\} \geq \delta_1 + \delta_1 = 2\delta_1.$$

Repeating the construction with γ_2 replaced by η_1 yields $\eta_2 \in \mathcal{F}_\phi$ with

$$\min_{t\in\mathbb{R}}\{\mathrm{diam}((E^s + t\omega) \cap (\gamma_1 \cup \eta_2))\} \geq 2\delta_1 + 2\delta_1 = 4\delta_1.$$

Continuation of this process results in a sequence of $\eta_k \in \mathcal{F}_\phi$, $k \in \mathbb{N}$, with

$$\min_{t\in\mathbb{R}}\{\mathrm{diam}((E^s + t\omega) \cap (\gamma_1 \cup \eta_k))\} \geq 2^k\delta_1.$$

A contradiction arises as soon as $2^k\delta_1 > 2R_0$ because $\gamma_1, \eta_k \in \mathcal{F}_\phi \subset \mathcal{C}^{R_0}$. □

Corollary 19.4 ([12]) *A unimodular Pisot substitution over two letters has pure discrete spectrum.*

20 Appendix: Recognizability

Our goal is to show the following theorem, which immediately implies Theorem 4.4.

Theorem 20.1 *If ϕ is a primitive translation aperiodic substitution, then there is $M \in \mathbb{N}$ such that $\Phi : \Phi^M(\mathcal{T}) \to \Phi^{M+1}(\mathcal{T})$ is injective.*

This hinges on the following a priori bound. For $a, b \in \mathcal{A}^*$, we shall write $a \ldots = b$ iff a is a prefix of b and $a \ldots = b \ldots$ iff a and b have the same prefix of length $\min\{|a|, |b|\}$. Denote by $\mathrm{Per}^+(\phi)$ the forward infinite ϕ-periodic words, i.e., $\mathrm{Per}^+(\phi) := \{[\gamma] : \gamma \in \mathrm{Per}^+(\Phi)$. Also, let $N = N_\phi$ be the stabilizing iterate as defined in Section 3.

Lemma 20.2 (a priori estimate) *For primitive and translation aperiodic ϕ, there is an $R > 0$ such that if $P \in \mathrm{Per}^+(\phi)$ and $P = a^r \ldots$ for $a \in \mathcal{A}^*$ then $r \leq R$.*

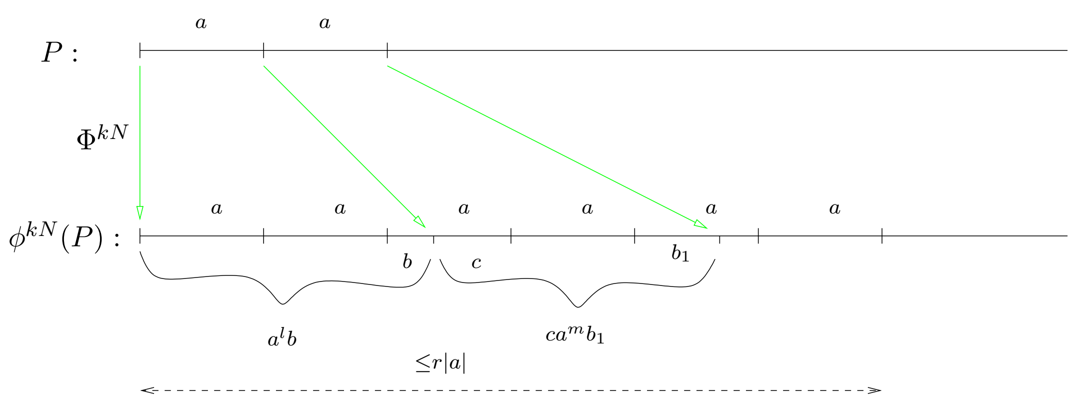

Figure 20.1: A priori estimate (Lemma 20.2).

Proof: To see that $R = 6\lambda^N$ works, we shall derive a contradiction from $P = a^r \ldots$ with $r \geq 6\lambda^N$. We may assume that a is not a power, i.e. $a = \tilde{a}^l$ implies $\tilde{a} = a$. Let $k := \lfloor \frac{\ln r - \ln 2}{N \ln \lambda} \rfloor \in \mathbb{N}$ so that $\lambda^{-N} r/2 \leq \lambda^{kN} \leq r/2$. Since $\lambda^{kN}|a|_u \leq r|a|_u$, $\phi^{kN}(P) = \phi^{kN}(a) \ldots = P = a^r \ldots$ implies $\phi^{kN}(a) = a^l b$ for some $b \in \mathcal{A}^*$ with $|b| < |a|$ and $l \geq \lambda^{kN} - 1 \geq \lambda^{-N} r/2 - 1 \geq 2$. (Here $|b| > 0$ as otherwise $\phi^{kN}(a) = a^l$ in violation of aperiodicity of ϕ.) Similarly, since $2\lambda^{kN}|a|_u \leq r|a|_u$, $\phi^{kN}(P) = \phi^{kN}(aa) \ldots = \phi^{kN}(a)\phi^{kN}(a) \ldots = a^l b \phi^{kN}(a) \ldots = P = a^r \ldots$ guarantees (by canceling the prefix $a^l b$) that $\phi^{kN}(a) = ca^m b_1$ for some $c, b_1 \in \mathcal{A}^*$ with $bc = a$, $|b_1| < |a|$ and $m \geq \lambda^{kN} - 1 \geq \lambda^{-N} r/2 - 2 \geq 1$. Therefore, $a^l b = ca^m b_1$, that is $(bc)^l b = c(bc)^m b_1$ and so $bc = cb$ forcing $a = bc$ to be a power — a contradiction. □

Let R_ϕ be the minimal R as in Lemma 20.2. The definition of the stabilizing iterate $N \in \mathbb{N}$ and Fact 3.1 easily yield $C_1 > 0$ such that, if $i \in \mathcal{A}$ and $m \geq N$ then $\phi^m(i) = p \ldots$ where $p \ldots = P \in \mathrm{Per}^+(\phi)$ and $|p|_u \geq C_1 \lambda^m$. (In fact, $C_1 = \lambda^{-N} \min\{|l|_u : l \in \mathcal{A}\}$.) This leads to the following:

Corollary 20.3 *There is* $0 < \epsilon = \epsilon_\phi$ *such that if* $a\phi^m(i)\ldots = \phi^m(j)\ldots$ *for* $m \in \mathbb{N}$, $a \in \mathcal{A}^*$ *and* $i, j \in \mathcal{A}$, *then* $|a|_u \geq \epsilon\lambda^m$.

This is to say that $|a|_u$ is *comparable* to $|\phi^m(j)|_u$: $|a|_u/|\phi^m(j)|_u \geq \frac{\lambda^m\epsilon}{\lambda^m|j|_u} = \frac{\epsilon}{|j|_u}$.

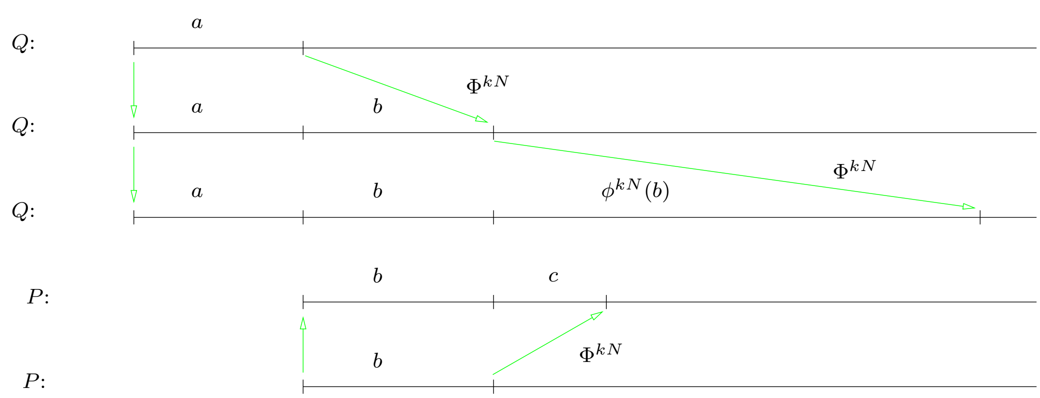

Figure 20.2: Proof of Corollary 20.3).

Proof: We may assume $m \geq N$. Let $\phi^m(i) = p\ldots$ and $\phi^m(j) = q\ldots$ where $P = p\ldots, Q = q\ldots \in \mathrm{Per}^+(\phi)$, and $|p|_u, |q|_u \geq C_1\lambda^m$.

Assume $k \geq 1$ is such that $|a|_u\lambda^{2kN} \leq C_1\lambda^m$. Since $|a|_u \leq C_1\lambda^m \leq |q|_u$, we have $a\ldots = q$ so that $\phi^{kN}(a)\ldots = \phi^{kN}(q) = q\ldots = a\ldots$, which implies $\phi^{kN}(a) = ab$ for some $b \in \mathcal{A}^*$. Applying ϕ^{kN} again yields (via $|a|_u\lambda^{2kN} \leq C_1\lambda^m$)

$$\phi^{2kN}(a) = ab\phi^{kN}(b) = q\ldots. \tag{20.1}$$

At the same time, $ab\ldots = q\ldots = ap\ldots$ and $|b|_u \leq \lambda^{kN}|a|_u \leq C_1\lambda^m \leq |p|_u$ yield $b\ldots = p$ so that $\phi^{kN}(b)\ldots = \phi^{kN}(p)\ldots = p\ldots = b\ldots$, which implies

$$\phi^{kN}(b) = bc = p\ldots \tag{20.2}$$

for some $c \in \mathcal{A}^*$ (see Figure 20). By plugging (20.1) and (20.2) into $q\ldots = ap\ldots$ we arrive at $ab\phi^{kN}(b)\ldots = abc\ldots$ so that $\phi^{kN}(b)\ldots = c\ldots$, which is to say that $bc = c\ldots$. We conclude that $c = b^r\ldots$ where (taking into account $p = b^{r+1}\ldots$ and the lemma)

$$R_\phi \geq r + 1 \geq |c|/|b| \geq C^{-2}|c|_u/|b|_u = C^{-2}(\lambda^{kN}|b|_u - |b|_u)/|b|_u = C^{-2}(\lambda^{kN} - 1),$$

where $C > 0$ is as in (2.3). Thus we have shown that: $|a|_u\lambda^{2kN} \leq C_1\lambda^m \Rightarrow \lambda^{kN} \leq C^2R_\phi + 1$, $k \in \mathbb{N}$. The corollary follows easily. (To be concrete, $|a|_u \geq C_1\lambda^m(C^2R_\phi + 1)^{-2}$ so that $\epsilon := C_1(C^2R_\phi + 1)^{-2}$.) $\square$

Given an edge I of a strand, a subsegment $K \subset I$ is called ***sup-peripheral*** (with respect to I) if K contains $\max I$, K is called ***inf-peripheral*** if it contains $\min I$, and K is called ***full*** if it coincides with I.

For $K \subset I_i$ and $K' \subset I_j$, $i, j \in \mathcal{A}$, we write $K \overset{n}{\sim} K'$ iff $K \neq K'$, $\Phi^n_{I_i}(K)$ and $\Phi^n_{I_j}(K')$ are two strands with the same word, $|K|_u, |K'|_u \geq \epsilon_\phi$, and either one of K and K' is sup- and the other is inf-peripheral or one of K and K' is full. (Clearly, "$\overset{n}{\sim}$" is symmetric but not an equivalence relation.

Given a strand μ and a (non-degenerate) subsegment $K \subset \mu$, there is a unique edge I such that $K \subset I$ and we associate with K a segment $K_\mu := K - \min I \subset \bigcup_{i \in \mathcal{A}} I_i$. Although we have defined $\overset{n}{\sim}$ only on subsegments of $\bigcup_{i\in\mathcal{A}} I_i$, we shall often abuse the notation and let $K \overset{n}{\sim} K'$ stand for $K_\mu \overset{n}{\sim} K'_{\mu'}$.

In order to link "$\overset{n}{\sim}$" with Φ^n's failure to be injective, we shall talk about partitions of strands. By a partition we understand a covering by subarcs any two of which intersect in at most one point. Clearly, a partition Q is determined by ∂Q, the set of endpoints of all its the subarcs. By $P \vee Q$ we denote the common refinement of P and Q; $\partial(P \vee Q) = \partial P \cup \partial Q$. Every strand η has a canonical partition P_η into its edges.

Fact 20.4 *Suppose that $\mu, \mu' \in \Phi^{-n}(\gamma)$. If $M \in \Phi^n_\mu(P_\mu) \vee \Phi^n_{\mu'}(P_{\mu'})$ then $K := (\Phi^n_\mu)^{-1}(M) \subset \mu$ and $K' := (\Phi^n_{\mu'})^{-1}(M) \subset \mu'$ are such that $\mathrm{pr}^u(K) = \mathrm{pr}^u(K')$ and either $K = K'$ or $K \overset{n}{\sim} K'$.*

Proof: From $\Phi^n_\mu(K) = \Phi^n_{\mu'}(K') = M$, we have $\mathrm{pr}^u(K) = \mathrm{pr}^u(K')$ (and $|K|_u = |K'|_u = |M|_u/\lambda^n$). Let I and I' be the unique edges of μ and μ' such that $K \subset I$ and $K' \subset I'$. Observe that $\Phi^n(I) \cap \Phi^n(I') = M$ and depending on whether $\Phi^n(I), \Phi^n(I') \subset \gamma$ are contained in one another or not, one of the two possibilities is realized:

(i) $K = I$ or $K' = I'$;

(ii) $K \subset I$ and $K' \subset I'$ are a pair of sup- and inf-peripheral segments.

Thus, unless $K = K'$, $K \overset{n}{\sim} K'$ as long as we can show $|K|_u = |K'|_u \geq \epsilon_\phi$. Under (i), the inequality is immediate. Assume then (ii). To fix attention, let $K \subset I$ be sup-peripheral and $K' \subset I'$ be inf-peripheral.

Let J be the edge of μ following I. We have $[\Phi^n_\mu(K)][\Phi^n(J)] \ldots = [\Phi^n(I')] \ldots$ and Corollary 20.3 yields $|\Phi^n_\mu(K)|_u \geq \epsilon_\phi \lambda^n$ so that $|K|_u \geq \epsilon_\phi$. □

Set $\mathcal{R}^m := \{(K, K') : \ K \overset{m}{\sim} K', \ K, K' \in \bigcup_{i\in\mathcal{A}} I_i\}$ and $\mathcal{R} := \bigcup_{m\in\mathbb{N}} \mathcal{R}^m$. Note that $\mathcal{R}^m \subset \mathcal{R}^{m+1}$.

Fact 20.5 *$\mathcal{R}$ is finite so that there is $M_\phi \in \mathbb{N}$ with $\mathcal{R} = \mathcal{R}^m$ for $m \geq M_\phi$.*

We first deduce the theorem and then give a proof of the fact.

Proof of Theorem 20.1: Let $m = M_\phi$. Suppose that $\gamma \in \Phi^{m+1}(\mathcal{T})$ and $\eta, \eta' \in \Phi^{-1}(\gamma) \cap \Phi^m(\mathcal{T})$. We have to show that $\eta = \eta'$. Fix $\mu, \mu' \in \mathcal{T}$ so that $\eta = \Phi^m(\mu)$ and $\eta' = \Phi^m(\mu')$. Let $\{M_i\}_{i\in\mathbb{Z}}$ be the elements of the the partition $Q_\gamma := \Phi^{m+1}_\mu(P_\mu) \vee$

$\Phi_{\mu'}^{m+1}(P_{\mu'})$ ordered so that M_i follows M_{i+1} (on γ). Set $K_i := (\Phi_{\mu}^{m+1})^{-1}(M_i)$ and $K_i' := (\Phi_{\mu'}^{m+1})^{-1}(M_i)$. From Fact 20.4, $\mathrm{pr}^u(K_i) = \mathrm{pr}^u(K_i')$ and either $K_i = K_i'$ or $K_i \overset{m+1}{\sim} K_i'$. In the latter case, $K_i \overset{m}{\sim} K_i'$ by definition of $m = M_\phi$. Hence, in any case $[\Phi_{\mu}^{m}(K_i)] = [\Phi_{\mu'}^{m}(K_i')]$. Since also $\mathrm{pr}^u(\Phi_{\mu}^{m}(K_i)) = \mathrm{pr}^u(\Phi_{\mu'}^{m}(K_i'))$ (from $\mathrm{pr}^u(K_i) = \mathrm{pr}^u(K_i')$), we get that $\Phi_{\mu}^{m}(K_i) = \Phi_{\mu'}^{m}(K_i')$ modulo translation along E^s. This implies $\eta = \eta'$ modulo E^s and so $\eta = \eta'$. □

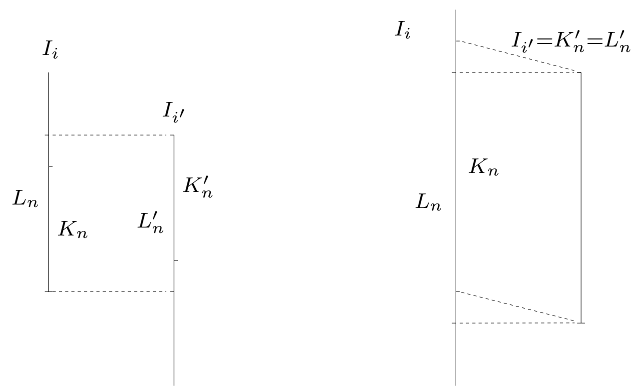

Figure 20.3: Proof of Fact 20.5.(Left: (i) the peripheral case; Right: (ii) full case.)

Proof of Fact 20.5: Suppose that $\mathcal{R}$ is infinite. Then $\mathcal{R}$ has accumulation points in the compact space $X \times X$ where $X = \{K : \ K \text{ compact subset of } \bigcup_{i\in\mathcal{A}} I_i\}$ is taken with the Hausdorff distance between compact sets. Hence, for each $n \in \mathbb{N}$, there is $m_n \in \mathbb{N}$ such that $\mathcal{R}^{m_n}$ contains $(K_n, K_n') \neq (L_n, L_n')$ and $\mathrm{dist}\,((K_n, K_n'), (L_n, L_n')) \to 0$. Because $\mathcal{R}$ is symmetric, we may assume $K_n \neq L_n$. Also, because $\mathcal{A}$ is finite, (after perhaps passing to a subsequence) we may assume $K_n, L_n \subset I_i$ and $K_n', L_n' \subset I_{i'}$, for some $i, i' \in \mathcal{A}$ and all $n \in \mathbb{N}$. We claim that, for each $n \in \mathbb{N}$, there are two (nonexclusive) possibilities:

(i) K_n, K_n', L_n, L_n' are all peripheral

(ii) $K_n' = L_n' = I_{i'}$.

Indeed, suppose (i) fails. To fix attention, assume that this is K_n that is not peripheral. Then $K_n' = I_{i'}$ by definition of $\mathcal{R}$ so that $L_n' \subset K_n'$, which implies $L_n \subset K_n$ (by considering the projections to E^u). Hence, L_n is not peripheral and $L_n' = I_{i'}$ again by definition of $\mathcal{R}$, i.e. (ii) holds.

Suppose that (i) holds for infinitely many n. By passing to a subsequence, we may assume that K_n and L_n contain the same endpoint of I_i and likewise for K_n' and L_n'; to fix attention, suppose $\min I_i \in K_n \cap L_n$ and $\max I_{i'} \in K_n' \cap L_n'$. After possibly swapping K_n with L_n, we may also take that $K_n \subset L_n$. We have

$$[\Phi_{\mu}^{m_n}(L_n)] = [\Phi_{\mu'}^{m_n}(L_n')] = [\Phi_{\mu'}^{m_n}(L_n' \setminus K_n')][\Phi_{\mu'}^{m_n}(K_n')] = [\Phi_{\mu'}^{m_n}(L_n' \setminus K_n')][\Phi_{\mu}^{m_n}(K_n)]. \tag{20.3}$$

Since $|L_n|_u, |K_n|_u \geq \epsilon_\phi$, $[\Phi_\mu^{m_n}(L_n)]$ and $[\Phi_\mu^{m_n}(K_n)]$, both prefixes in $\phi^{m_n}(i)$, are of *comparable length* to $\phi^{m_n}(i)$. Precisely, if we take j to be the first letter of $\phi^{C_2}(i)$ for $C_2 \in \mathbb{N}$ large enough (namely, $C_2 \geq \ln(\max\{|i|_u :\ i \in \mathcal{A}\}/\epsilon_\phi) / \ln\lambda$), then $[\Phi_\mu^{m_n}(L_n)] = \phi^{l_n}(j)\ldots$ and $[\Phi_\mu^{m_n}(K_n)] = \phi^{l_n}(j)\ldots$ for $l_n := m_n - C_2$. Thus, taking $a := [\Phi_{\mu'}^{m_n}(L_n' \setminus K_n')]$, we see that $\phi^{l_n}(j)\ldots = a\phi^{l_n}(j)\ldots$. Corollary 20.3 yields then $|\Phi_{\mu'}^{m_n}(L_n' \setminus K_n')|_u = |a|_u \geq \epsilon_\phi \lambda^{l_n}$ so that $|L_n' \setminus K_n'|_u \geq \epsilon_\phi\lambda^{l_n}/\lambda^{-m_n} \geq \epsilon_\phi \lambda^{-C_2} \not\to 0$ — a contradiction.

Suppose that (ii) holds for infinitely many n. To fix attention, assume $\min L_n < \min K_n$. For large enough n, $|L_n \setminus K_n|_u \leq \epsilon_\phi/4$ so that $|L_n \cap K_n|_u \geq 3/4\epsilon_\phi$ because $|L_n|_u = |K_n|_u \geq \epsilon_\phi$. Hence, $[\Phi_\mu^{m_n}(L_n \cap K_n)]$ is a prefix of *comparable length* in $[\Phi_\mu^{m_n}(K_n)] = [\Phi_\mu^{m_n}(I_{i'})] = \phi^{m_n}(i')$ and we can find $C_3 \in \mathbb{N}$ for which $j' := \phi_+^{C_3}(i')$ satisfies $[\Phi_\mu^{m_n}(L_n \cap K_n)] = \phi^{l_n}(j')\ldots$ for $l_n = m_n - C_3$. Now, taking $a := [\Phi_\mu^{m_n}(L_n \setminus K_n)]$, we have $\phi^{m_n}(i') = [\Phi_{\mu'}^{m_n}(I_{i'})] = [\Phi_\mu^{m_n}(L_n)] = [\Phi_\mu^{m_n}(L_n \setminus K_n)][\Phi_\mu^{m_n}(L_n \cap K_n)] = a\phi^{l_n}(j')\ldots$. Again by Corollary 20.3, $|\Phi_\mu^{m_n}(L_n \setminus K_n)|_u = |a|_u \geq \epsilon_\phi\lambda^{l_n}$ so that $|L_n \setminus K_n|_u \geq \epsilon_\phi\lambda^{l_n}/\lambda^{-m_n} \geq \epsilon_\phi\lambda^{-C_3} \not\to 0$ — a contradiction. $\square$

21 Appendix: Tiling spaces as inverse limits

In Section 4, we introduced the winding map, $f_\phi : \mathcal{T}_\phi \to \mathbb{T}_\phi$, defined on the wedge of d circles $\mathbb{T}_\phi := \{(x_1, \ldots, x_d) :\ x_i \notin \mathbb{Z} \text{ for at most one } i\}/\mathbb{Z}^d$. Now, given $\gamma \in \mathcal{T}_\phi^{\min} \subset \{\gamma \in \mathcal{F} :\ 0 \in \gamma\}$, let $I_i - se_i$ be the edge of γ containing 0. Then $r(\gamma) := se_i \ (\mod \mathbb{Z}^d)$ defines a continuous map so that

$$\begin{array}{ccc} \mathcal{T}_\phi^{\min} & \xrightarrow{\Phi} & \mathcal{T}_\phi^{\min} \\ r\big\downarrow & & r\big\downarrow \\ \mathbb{T}_\phi & \xrightarrow{f_\phi} & \mathbb{T}_\phi \end{array} \tag{21.1}$$

commutes. In case ϕ is primitive and translation aperiodic, this diagram induces a semi-conjugacy (Proposition 4.3)

$$\begin{array}{ccc} \mathcal{T}_\phi^{\min} & \xrightarrow{\Phi} & \mathcal{T}_\phi^{\min} \\ \hat{r}\big\downarrow & & \hat{r}\big\downarrow \\ X_{f_\phi} & \xrightarrow{\overleftarrow{f}_\phi} & X_{f_\phi} \end{array} \tag{21.2}$$

on the inverse limit level defined by $\hat{r}(\gamma) = (r(\gamma), r(\Phi^{-1}(\gamma)), r(\Phi^{-2}(\gamma)), \ldots)$, made possible by recognizability of ϕ (Theorem 4.4). We will show that if, in addition, ϕ is *proper* then $\hat{r}$ is actually a conjugacy. Using the notation of Section 3, with N the stabilizing iterate of ϕ, ϕ is said to be ***proper*** provided $\phi_+^N(\mathcal{A}) = \{i\}$ and $\phi_-^N(\mathcal{A}) = \{j\}$ are singletons. That is, for each k, the word $\phi^N(k)$ starts with i and ends with j.

Proposition 21.1 *If ϕ is primitive, translation aperiodic, and proper, then $\hat{r}$ is a homeomorphism.*

Proof: Surjectivity of $\hat{r}$ follows from that of r. To prove that $\hat{r}$ is injective, suppose that $r(\Phi^{-n}(\gamma)) = r(\Phi^{-n}(\gamma'))$ for some $\gamma, \gamma' \in \mathcal{T}_\phi^{\min}$ and all $n \in \mathbb{Z}^+$. This means that, for every n, either

(i_n) $\Phi^{-n}(\gamma)$ and $\Phi^{-n}(\gamma')$ have a vertex at 0, or

(ii_n) $\Phi^{-n}(\gamma)$ and $\Phi^{-n}(\gamma')$ contain a common edge $I_{i_n} - s_n e_i$ with $0 < s_n < 1$.

If (i_n) occurs for all n, then $\Phi^N(\Phi^{-(k+1)N}(\gamma))$ and $\Phi^N(\Phi^{-(k+1)N}(\gamma'))$ both contain the edges $I_j - e_j$ and I_i (where i, j, N are as in the definition of *proper*). Then $\gamma = \gamma' = \bigcup_{k\geq 0} \Phi^{kN}(I_j - e_j \cup I_i)$.

If (i_M) fails for some M, then (ii_n) must occur for all $n \geq M$, and we consider two subcases

$(ii)'$ there is a subsequence n_k so that s_{n_k} is bounded away from $\{0, 1\}$;

$(ii)''$ $\limsup_{n\to\infty} s_n \subset \{0, 1\}$.

In case $(ii)'$, we have $\gamma = \bigcup_{k\geq 0} \Phi^{n_k}(I_{i_{n_k}} - s_{n_k} e_{i_{n_k}}) = \gamma'$. In case $(ii)''$, it must either happen that $s_n \to 0$ or $s_n \to 1$ as $n \to \infty$; let's assume the former. It must then be that $i_n = i$ and $s_{n+1} = s_n/\lambda$ for all sufficiently large n (by properness). Let $I_{k_n} - s_n e_i - e_{k_n}$ be the edge of $\Phi^{-n}(\gamma)$ preceding $I_i - s_n e_i$. Then $\Phi^N(I_{i_{k_n}+N} - s_{n+N} e_i - e_{k_{n+N}}) = \Phi^N(I_{i_{k_n}+N} - s_n/\lambda^N e_i - e_{k_{n+N}})$ contains [ends with] the edge $I_j - s_n e_i - e_j$, by properness, so that $k_n = j$. The same is true for γ'. Thus, taking $K > 0$ large enough, $\gamma = \bigcup_{k\geq K} \Phi^{kN}((I_j - s_k e_i - e_j) \cup (I_i - s_{kN} e_i)) = \gamma'$. □

In case when ϕ is not proper, we may "rewrite" ϕ, or some power of ϕ, to obtain a proper substitution. If ϕ has an allowed fixed word, that is, if for some i and j, $\phi(i) = i \ldots$, $\phi(j) = \ldots j$, and ji is allowed, then we may ***rewrite ϕ using starting rule i and stopping rule j*** as follows. (A word u is allowed if it is a subword of $\phi^n(k)$ for some $k \in \mathcal{A}$ and $n \in \mathbb{N}$. Subwords of strands in $\mathcal{T}_\phi^{\min}$ are allowed, see (ii) of Proposition 3.4.)

Let $w = \ldots j.i \ldots$ be a bi-infinite fixed word of ϕ. We may factor w uniquely, up to translation, as a product of words $w = \ldots w_{-1}.w_0 w_1 \ldots$ with the property that each w_k begins with i and ends with j, and contains no occurrence of the two letter word ji. Since ji occurs with bounded gap in w (via Proposition 3.5) the collection $\{w_k : k \in \mathbb{Z}\}$ is actually finite, say $\{a_1, \ldots, a_m\} = \{w_k : k \in \mathbb{Z}\}$. Let $\tilde{\mathcal{A}} = \{1, \ldots, m\}$ and define $\tilde{\phi} : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}^*$ by $\tilde{\phi}(l) = l_1 \ldots l_k$ iff $\phi(a_l) = a_{l_1} \ldots a_{l_k}$. The rewriting $\tilde{\phi}$ is primitive and translation aperiodic (if ϕ is) and is proper. (In case ϕ has no allowed bi-infinite fixed words, we may replace ϕ by ϕ^n for some n, $1 < n \leq d$, and then carry out the above rewriting.)

Proposition 21.2 *If ϕ is primitive and translation aperiodic and $\tilde{\phi}$ is a rewriting of ϕ (as above) then there is a homeomorphism $\beta : \mathcal{T}_{\tilde{\phi}}^{\min} \to \mathcal{T}_{\phi}^{\min}$ that conjugates $\tilde{\Phi}$ with Φ and the translation flow on $\mathcal{T}_{\tilde{\phi}}^{\min}$ with that on $\mathcal{T}_{\phi}^{\min}$. (If $\tilde{\phi}$ is obtained from the rewriting of ϕ^n, then β conjugates $\tilde{\Phi}$ with Φ^n and $\beta \circ T_{\tilde{\phi}}^t = T_{\phi}^{\lambda^n t} \circ \beta$.)*

Proof: Let $\beta : \tilde{\mathcal{A}}^* \to \mathcal{A}^*$ be the morphism such that $\beta(k) = a_k$ and let $B = (b_{kl})$ be the $d \times m$ incidence matrix of β: b_{kl} is the number of k's in the word a_l. Then $\phi \circ \beta = \beta \circ \tilde{\phi}$ and $AB = B\tilde{A}$, where A and $\tilde{A}$ are the matrices of ϕ and $\tilde{\phi}$. It follows that B takes the spaces $\tilde{E}^s$ and $\tilde{E}^u$ (for $\tilde{A}$) into and onto the spaces E^s and E^u (for A). This allows us to define $\boldsymbol{\beta} : \tilde{\mathcal{T}} \to \mathcal{T}$ so that $\boldsymbol{\beta} \circ \tilde{\Phi} = \Phi \circ \boldsymbol{\beta}$. From this we see that $\boldsymbol{\beta}(\mathcal{T}_{\tilde{\phi}}) \subset \mathcal{T}_{\phi}$. It is clear that $\boldsymbol{\beta} \circ T_{\tilde{\phi}}^t = T_{\phi}^t \circ \boldsymbol{\beta}$, and from this it follows that $\boldsymbol{\beta}(\mathcal{T}_{\tilde{\phi}}^{\min}) = \mathcal{T}_{\phi}^{\min}$. It remains to show that $\boldsymbol{\beta}$ is injective on $\mathcal{T}_{\tilde{\phi}}^{\min}$. The formula $\boldsymbol{\beta} \circ T_{\tilde{\phi}}^t = T_{\phi}^t \circ \boldsymbol{\beta}$ implies that $\boldsymbol{\beta}$ is injective on the arc component $\{T_{\tilde{\phi}}^t(\gamma) : t \in \mathbb{R}\}$ for each $\gamma \in \mathcal{T}_{\tilde{\phi}}^{\min}$ (since ϕ is translation aperiodic). Now, if $\boldsymbol{\beta}(\gamma) = \boldsymbol{\beta}(\gamma')$ for some $\gamma, \gamma' \in \mathcal{T}_{\tilde{\phi}}^{\min}$ and if x and x' are the bi-infinite words in the alphabet $\tilde{\mathcal{A}}$ corresponding to γ and γ', then $\beta(x) = \beta(x')$ are bi-infinite words corresponding to the same strand $\boldsymbol{\beta}(\gamma) = \boldsymbol{\beta}(\gamma')$. That is, x and x' are rewritings of the same bi-infinite allowed word for ϕ. Since rewriting a word is unique, up to translation, $\gamma' = T_{\tilde{\phi}}^t(\gamma)$ for some t. From injectivity on arc-components, $\gamma = \gamma'$. □

Corollary 21.3 *If ϕ is primitive and translation aperiodic then there is a map f on a wedge of circles and a homeomorphism from $\mathcal{T}_{\phi}^{\min}$ that conjugates Φ^n with the shift homeomorphism $\overleftarrow{f}$. In case ϕ has an allowed fixed bi-infinite word, we may take $n = 1$; otherwise, n is the period of an allowed b-infinite periodic word for ϕ.*

Notes: In the terminology of Williams ([40]), the corollary provides an "elementary presentation" of the dynamical system $\Phi^n : \mathcal{T}_{\phi}^{\min} \to \mathcal{T}_{\phi}^{\min}$. The properness of the substitution ϕ translates to the "flattening axiom" for f_{ϕ}.

If the substitution does not have an allowed bi-infinite fixed word and an elementary presentation of Φ is desired, it is necessary (and sufficient) that the map f_{ϕ} has a fixed point p other than the branch point $0 \in \mathbb{T}_{\phi}$. The procedure then is to split the symbol, say a, corresponding to the petal I_a (mod $\mathbb{Z}^d$) in which p occurs, into initial, a_1, and terminal, a_2, symbols. One obtains a new substitution on $d+1$ letters $(\mathcal{A} \setminus \{a\}) \cup \{a_1, a_2\}$ that has an allowed bi-infinite word $\ldots a_1.a_2 \ldots$. (The definition of the substitution on a_1 and a_2 is determined by the precise location of p in I_a.) If this substitution is then rewritten using starting rule a_2 and stopping rule a_1 to obtain the substitution $\tilde{\phi}$, $f_{\tilde{\phi}}$ on $X_{f_{\tilde{\phi}}}$ provides an elementary presentation of Φ on $\mathcal{T}_{\phi}^{\min}$.

In case the map f_{ϕ} has no fixed points, Anderson and Putnam ([1]) describe a procedure for presenting Φ as a shift $\overleftarrow{f}$ on an inverse limit where $f : X \to X$ is a map on a finite graph determined by "collaring" ϕ as outlined below.

Given ϕ (primitive and translation aperiodic), let $\{w_1, \ldots, w_m\}$ be the collection of all allowed three letter words for ϕ. Let $\tilde{\mathcal{A}} = \{1, \ldots, m\}$ and define $\tilde{\phi} : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}^*$ by $\tilde{\phi}(i) = j_1 \ldots j_{n-2}$ provided $w_i = abc$ with $\phi(a) = \ldots i_1$, $\phi(b) = i_2 \ldots i_{n-1}$, $\phi(c) = i_n \ldots$, and $w_{j_1} = i_1 i_2 i_3$, $w_{j_2} = i_2 i_3 i_4$, $\ldots$, $w_{j_{n-2}} = i_{n-2} i_{n-1} i_n$. Now, let X be the topological graph with vertices labeled by the allowed two letter words (for ϕ) and with a directed edge, labeled i, from vertex ab to vertex bc provided $w_i = abc$, and let this edge have length ω_b, the bth entry in the right Perron-Frobenius eigenvector for $A = A_\phi$. Let $f : X \to X$ be the map that follows the pattern $\tilde{\phi}$ and is uniformly stretching by the factor λ (the dominant eigenvalue of A). A conjugacy between Φ and $\overleftarrow{f}$ can be then constructed along the lines of the proof of Proposition 21.1.

References

[1] J. E. Anderson and I. F. Putnam. Topological invariants for substitution tilings and their associated C^*-algebras. *Ergodic Theory Dynam. Systems*, 18(3):509–537, 1998.

[2] P. Arnoux and S. Ito. Pisot substitutions and Rauzy fractals. *Bull. Belg. Math. Soc. Simon Stevin*, 8(2):181–207, 2001. Journées Montoises (Marne-la-Vallée, 2000).

[3] P. Arnoux, S. Ito, and Y. Sano. Higher dimensional extensions of substitutions and their dual maps. *J. Anal. Math.*, 83:183–206, 2001.

[4] V. Baker, M. Barge, and J. Kwapisz. Geometric realization and coincidence for reducible non-unimodular pisot tiling spaces with an application to beta-shifts. Oct 2005 manuscript. submitted to Annales de l'Institut Fourierr.

[5] M. Barge and B. Diamond. Coincidence for substitutions of Pisot type. *Bull. Soc. Math. France*, 130(4):619–626, 2002.

[6] M.-J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse, J.-P. Schreiber, W. A. Adkins, and S. H. Weintraub. *Pisot and Salem Numbers.* Birkhauser Verlag, Basel, 1992. (With a preface by David W. Boyd).

[7] V. Canterini and A. Siegel. Automate des préfixes-suffixes associé à une substitution primitive. *J. Théor. Nombres Bordeaux*, 13(2):353–369, 2001.

[8] V. Canterini and A. Siegel. Geometric representation of substitutions of Pisot type. *Trans. Amer. Math. Soc.*, 353(12):5121–5144, 2001.

[9] A. Clark and L. Sadun. When shape matters: deformations of tiling spaces. arXiv preprint math.DS/0306214, June 2003.

[10] F. M. Dekking. The spectrum of dynamical systems arising from substitutions of constant length. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 41(3):221–239, 1977/78.

[11] N. Pytheas Fogg. *Substitutions in Dynamics, Arithmetics and Combinatorics.* Springer-Verlag, Berlin, 2002. Lecture Notes in Mathematics, Vol. 1794.

[12] M. Hollander and B. Solomyak. Two-symbol Pisot substitutions have pure discrete spectrum. Preprint, 2001.

[13] B. Host. Valeurs propres des systèmes dynamiques définis par des substitutions de longueur variable. *Ergodic Theory Dynam. Systems*, 6(4):529–540, 1986.

[14] R. Kenyon. *Self-similar tilings.* PhD thesis, Princeton University, 1990.

[15] R. Kenyon. The construction of self-similar tilings. *Geom. Funct. Anal.*, 6(3):471–488, 1996.

[16] R. Kenyon and A. Vershik. Arithmetic construction of sofic partitions of hyperbolic toral automorphisms. *Ergodic Theory Dynam. Systems*, 18(2):357–372, 1998.

[17] J. Kwapisz. Dynamical proof of Pisot's theorem. *Canad. Math. Bull.*, 49(1):108–112, 2006. available at www.math.montana.edu/˜jarek.

[18] J.-Y. Lee. Pure point diffractive sets are model sets in substitutions. arXiv:math.MG/0510425, 20 October 2005.

[19] J-Y. Lee, R.V. Moody, and B. Solomyak. Consequences of pure point diffraction spectra for multiset substitution systems. *Discrete Comput. Geom.*, 29(4):525–560, 2003.

[20] A. N. Livshits. On the spectra of adic transformations of Markov compact sets. *Uspekhi Mat. Nauk*, 42(3(255)):189–190, 1987. English translation: *Russian Math. Surveys* 42(3): 222–223, 1987.

[21] A. Manning. A markov partition that reflects the geometry of a hyperbolic toral automorphism. *Trans. Amer. Math. Soc.*, 354(7):2849–2863, 2002.

[22] R.V. Moody. Model sets: a survey. In F. Axel, F. Dénoyer, and J.P. Gazeau, editors, *From quasicrystals to more complex systems*, pages 145–166. Springer, Berlin, 2000. arXiv:math.MG/0002020.

[23] B. Mossé. Puissances de mots et reconnaissabilité des points fixes d'une substitution. *Theoret. Comput. Sci.*, 99(2):327–334, 1992.

[24] M. Queffélec. *Substitution dynamical systems—spectral analysis.* Springer-Verlag, Berlin, 1987. Lecture Notes in Mathematics, Vol. 1294.

[25] G. Rauzy. Nombres algébriques et substitutions. *Bull. Soc. Math. France*, 110(2):147–178, 1982.

[26] Géard Rauzy. Rotations sur les groupes, nombres algébriques, et substitutions. In *Séminaire de Théorie des Nombres (Talence, 1987–1988).* Univ. Bordeaux I, 1988. Exp. No. 21.

[27] D. J. Rudolph. Markov tilings of $\mathbb{R}^n$ and representations of $\mathbb{R}^n$ actions. In *Measure and measurable dynamics*, volume 94 of *Contemp. Math.*, pages 271–290, Providence, RI, 1989. Amer. Math. Soc.

[28] R. Salem. *Algebraic numbers and Fourier analysis*. D. C. Heath and Co., Bostyon, Mass., 1963.

[29] M. Schlottmann. Generalized model sets and dynamical systems. In F. Axel, F. Dénoyer, and J.P. Gazeau, editors, *Directions in mathematical quasicrystals*, CRM Monogr. Ser., 13, pages 143–159. Amer. Math. Soc., Providence, RI, 2000.

[30] A. Siegel. *Représentation géométrique, combinatoire et arithmétique des substitutions de type Pisot*. PhD thesis, Université de la Méditerranée, 2000.

[31] A. Siegel. Pure discrete spectrum dynamical system and periodic tiling associated with a substitution. *Ann. Inst. Fourier*, 2(54):288–299, 2004.

[32] V. F. Sirvent and B. Solomyak. Pure discrete spectrum for one-dimensional substitution systems of pisot type. *Canad. Math. Bull.*, 45(4):697–710, 2002.

[33] V. F. Sirvent and Y. Wang. Self-affine tiling via substitution dynamical systems and Rauzy fractals. To appear in *Pacific J. Math*, 2002.

[34] B. Solomyak. Dynamics of self-similar tilings. *Ergodic Theory Dynam. Systems*, 17(3):695–738, 1997.

[35] B. Solomyak. Nonperiodicity implies unique composition for self-similar translationally finite tilings. *Discrete Comput. Geometry*, 20(2):265–279, 1998.

[36] W. P. Thurston. *Groups, tilings and finite state automata*. Lectures notes distributed in conjunction with the Colloquium Series, in *AMS Colloquium lectures*, 1989.

[37] J. M. Thuswaldner. Unimodular Pisot substitutions and their associated tiles. to appear in J. Théor. Nombres Bordeaux, 2005.

[38] A. Vince. Digit tiling of Euclidean space. In *Directions in mathematical quasicrystals*, CRM Monogr. Ser. 13, pages 329–370, Providence, RI, 2000. Amer. Math. Soc.

[39] P. Walters. *An introduction to ergodic theory*. Springer-Verlag, New York, 1982.

[40] R. F. Williams. Classification of one-dimensional attractors. *Proc. Symp. Pure Math*, 14:341–361, 1970.