Rigidity and Mapping Class Group for Abstract Tiling Spaces

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Abstract

We study abstract self-affine tiling actions, which are an intrinsically defined class of minimal expansive actions of $\mathbb{R}^d$ on a compact space. They include the translation actions on the compact spaces associated to aperiodic repetitive tilings or Delone sets in $\mathbb{R}^d$. In the self-similar case, we show that existence of a homeomorphism between tiling spaces implies conjugacy of the actions up to a linear rescaling. We also introduce the general linear group of an (abstract) tiling, prove its discreteness, and show that it is naturally isomorphic with the (pointed) mapping class group of the tiling space. To illustrate our theory, we compute the mapping class group for a five-fold symmetric Penrose tiling.

1 Introduction

The goal of this work is three-fold. First, we propose a concept of an *abstract self-affine tiling action*, not as an attempt of generalization, but as a way of axiomatizing the essential properties of the translation actions associated to self-affine tilings or Delone sets. These have a large literature devoted to them (see [Rob04, Sad08, LW03, Sol06] for some entry points) most of which is concerned with dynamical properties that are invariant under conjugacies. An intrinsic characterization of such actions among all $\mathbb{R}^d$ actions on a compact space is a logical step properly bringing this subject into the fold of topological dynamics. (We apologize for violating some entrenched jargon and notations in order to effect this unification.) Second, inspired by a rank-one result in [BS07], we prove topological rigidity for tiling actions of arbitrary rank. (Our axioms provide a natural platform for this development.) Third, we introduce concepts of a (pointed) *mapping class group of a tiling space* and of a *general linear group of a tiling* and use the rigidity to show that the two groups are naturally isomorphic. To give a concrete example, we compute the mapping class group for a Penrose tiling.

Realizing that many readers may care litle for abstract topological dynamics, let us set the general definitions aside for a moment and state the key results in the context of the prevailing definition of tiling actions, which we briefly recall below. (Later we strive to adopt notations that should make it possible for a reader well versed in tiling theory to follow the arguments without consulting the axioms.)

1.1 Results in the context of concrete tiling spaces

A tiling is a covering of $\mathbb{R}^d$ by infinitely many sets with pairwise disjoint interiors and each congruent under translation to a set in a finite family $\{\sigma_1, \ldots, \sigma_M\}$ of so called *prototiles*; each prototile is required to be compact and the closure of its interior. A tiling x can be translated by a vector $t \in \mathbb{R}^d$, and x is called *aperiodic* iff $x + t = x$ implies that $t = 0$. Given a tiling x_0, the closure of the set of all its translates, $X := \mathrm{cl}\{x_0 + t : \ t \in \mathbb{R}^d\}$, is referred to as the *tiling space of* x_0. Here the closure is taken with respect to a suitable topology on the space of all tilings whereby, roughly speaking, two tilings are considered close if upon small translation they can be made to agree on a large neighborhood of the origin (see e.g. [Rob04], cf. [RW92]).

It is desirable that X be compact, which is secured by stipulating *finite local complexity* of x_0, i.e., requiring that there are only finitely many congruency classes of R-patches in x_0. (Here by an R-patch we understand the tiles contained in a ball of radius R.) In such case, X is not only a compact space but also locally a product of $\mathbb{R}^d$ by a Cantor set ([Kel95, KI00, BG03]). On this space, $\mathbb{R}^d$ acts by translations and (in this context) we shall use T^t to denote the mapping $x \mapsto x+t$. The recurrence of patterns in x_0 is reflected by the recurrence properties of the action $T = (T^t)_{t\in\mathbb{R}^d}$ and our interest is in T that are minimal (i.e., every orbit is dense). For a finite local complexity x_0, T is minimal exactly when x_0 is *repetitive*, i.e., for any $R > 0$ there is $R_1 > 0$ so that any R_1-patch of x_0 contains a translated copy of every R-patch of x_0.

The most intensively studied examples of aperiodic tilings have the property of *self-similarity*. In its most restrictive sense this means that the tiling is invariant under an *inflation-substitution rule* (see e.g. [Sol97]) but we will be quite content with the following more general definitions. Given a linear isomorphism $\Lambda : \mathbb{R}^d \to \mathbb{R}^d$ that is expanding (i.e., all its eigenvalues are greater than one in modulus), we say that x_0 is Λ-*self-affine* iff there is a homeomorphism $\Phi : X \to X$ such that $\Phi(x_0) = x_0$ and

$$T^{\Lambda t} \circ \Phi = \Phi \circ T^t \quad (\forall t \in \mathbb{R}^d). \tag{1.1}$$

Furthermore, x_0 is *self-similar* iff it is Λ-self-affine with Λ that is conformal, i.e., $\Lambda = \lambda U$ where $\lambda > 1$ and U is an orthogonal transformation of $\mathbb{R}^d$. Self-affine tilings are necessarily aperiodic.

Topological classification of tilings entails classifying their tiling spaces up to a homeomorphism (or a pointed homeomorphism). Our first result (answering Question 1 in [Cla07]) shows that such classification is not much cruder than the more refined classification of the associated tiling actions up to a topological conjugacy.

Theorem 1.1 (translational rigidity) *(cf. Theorem 5.3) Suppose that x_0 and $\tilde{x}_0$ are repetitive tilings of finite local complexity that are Λ-self-similar and $\tilde{\Lambda}$-self-similar, respectively. Let $(T^t)_{t\in\mathbb{R}^d}$ and $(\tilde{T}^t)_{t\in\mathbb{R}^d}$ be the translation actions on the corresponding tiling spaces X and $\tilde{X}$. If there is a homeomorphism $h_0 : X \to \tilde{X}$, then there is a linear isomorphism $A : \mathbb{R}^d \to \mathbb{R}^d$ and a homeomorphism $h : X \to \tilde{X}$ conjugating $(T^t)_{t\in\mathbb{R}^d}$ to the linear rescaling $(\tilde{T}^{At})_{t\in\mathbb{R}^d}$, i.e., $h \circ T^t = \tilde{T}^{At} \circ h$ $(\forall t \in \mathbb{R}^d)$.*

Keep in mind that, due to the fact that locally tiling spaces are products of a Cantor set by $\mathbb{R}^d$, h_0 must be an orbit equivalence (mapping orbits to orbits). Hence, $\tilde{T}$ is a priori conjugated to a reparametrization (*time change*) of T. However, arbitrary reparametrizations of T are hardly ever conjugated to a linear rescaling of T. This can already be observed in non-Pisot tilings subjected to deformations of tiles (see [CS06] and also [Kel08]). The assumption of self-similarity on both T and $\tilde{T}$ is therefore crucial. One wonders if self-affinity would be enough:

Question: Are there two self-affine tiling $\mathbb{R}^d$-actions for which the tiling spaces are homeomorphic but the actions are not conjugated up to rescaling?

In an attempt to capture "topological symmetries" of a tiling, it is natural to inquire about the group $\mathcal{H}(X, x_0)$ of all *pointed self-homeomorphisms* of (X, x_0), i.e.,

$$\mathcal{H}(X, x_0) := \{h : \ h \text{ is a homeomorphism } X \to X \text{ with } h(x_0) = x_0\}.$$

By using the $\mathbb{R}^d$-action T, we can distinguish among the elements of $\mathcal{H}(X, x_0)$ *linear homeomorphisms* comprising $h \in \mathcal{H}(X, x_0)$ for which there exists a linear isomorphism $A : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$h \circ T^t = T^{At} \circ h \quad (\forall t \in \mathbb{R}^d). \tag{1.2}$$

Note that the homeomorphism h in the above definition is uniquely determined by A because the translation orbit of x_0 is dense in X. (Indeed, for any $x \in X$, we have $x_0 + t_n \to x$ for some $(t_n) \subset \mathbb{R}^d$ and so $h(x) = \lim_{n\to\infty} h(x_0+t_n) = \lim_{n\to\infty} x_0 + At_n$.) It is natural then to associate to x_0 the following subgroup of the group $\mathbb{G}l_d(\mathbb{R})$ of all linear automorphisms of $\mathbb{R}^d$:

$$\mathbb{G}l(T, x_0) := \{A \in \mathbb{G}l_d(\mathbb{R}) : \ \exists_{h\in\mathcal{H}(X,x_0)} \forall_{t\in\mathbb{R}^d} \ h^{At} \circ T = h \circ T^t\}. \tag{1.3}$$

We shall refer to this group as the *general linear group of* x_0 and simply write $\mathbb{G}l(x_0)$ when there is no ambiguity as to the underlying action. The orthogonal elements of $\mathbb{G}l(x_0)$ form a subgroup that is an analogue of the *euclidean point symmetry group* of a crystal but may contain more than just the out-right symmetries of the tiling x_0 as a pattern in $\mathbb{R}^d$. For instance, consider the kite-and-dart Penrose tiling x_0 obtained by repeated inflation and substitution of the regular pentagon made of five kites (Figure 1.1[1]). It is not hard to see that, beside the obvious rotation by $2\pi/5$, $\mathbb{G}l(x_0)$ contains the central symmetry $t \mapsto -t$. (In terms of x_0, this corresponds to the fact that, for any $t, s \in \mathbb{R}^2$, $x_0 + t$ and $x_0 + s$ coincide on a large patch around the origin iff $x_0 - t$ and $x_0 - s$ do so[2].) Thus $\mathbb{G}l(x_0)$ also contains the rotation by $2\pi/10$ and the Penrose tiling x_0 is *topologically 10-fold symmetric.* (Its *statistical 10-fold symmetry* is discussed in [Rad95].) Additionally, $\mathbb{G}l(x_0)$ contains the *golden dilation*, λI, $\lambda := (\sqrt{5} + 1)/2$, which underpins the self-similarity of x_0. Together with the symmetry about the horizontal axis, the rotation by $2\pi/10$ and the golden dilation generate all of $\mathbb{G}l(x_0)$ (Theorem 10.1). Experimentally, $\mathbb{G}l(x_0)$ should manifest itself via *linear symmetries* of the diffraction spectrum of the quasi-crystal modeled by x_0 (cf. Fact 10.2).

Theorem 1.2 (discreteness) *(cf. Lemma 6.1) For any repetitive tiling x_0 of finite local complexity, the group $\mathbb{G}l(x_0)$ is a discrete subgroup of $\mathbb{G}l_d(\mathbb{R})$.*

This theorem is hardly surprising but it records a fundamental property of $\mathbb{G}l(x_0)$. In fact, $\mathbb{G}l(T, x_0)$ is a well defined discrete subgroup of $\mathbb{G}l_d(\mathbb{R})$ for any $\mathbb{R}^d$-action T on a compact space X and $x_0 \in X$ provided T is minimal and such that $x + t \neq x$ for sufficiently small $t \in \mathbb{R}^d$ and $x \in X$. (See the axiom (LF), ahead.) Aperiodicity is not required, and a "baby example" is provided by a $\mathbb{Z}^d$-periodic tiling of $\mathbb{R}^d$ by translates of the unit cube, for which $\mathbb{G}l(x_0)$ is the classical group $\mathbb{G}l_d(\mathbb{Z})$. For

[1] Created by modifying Eric W. Weisstein's *Mathematica* code, available at `mathworld.wolfram.com/PenroseTiles.html`.

[2] More precisely, for any $(t_n), (s_n) \subset \mathbb{R}^2$, $\mathrm{dist}(x_0+t_n, x_0+s_n) \to 0$ iff $\mathrm{dist}(x_0-t_n, x_0-s_n) \to 0$.

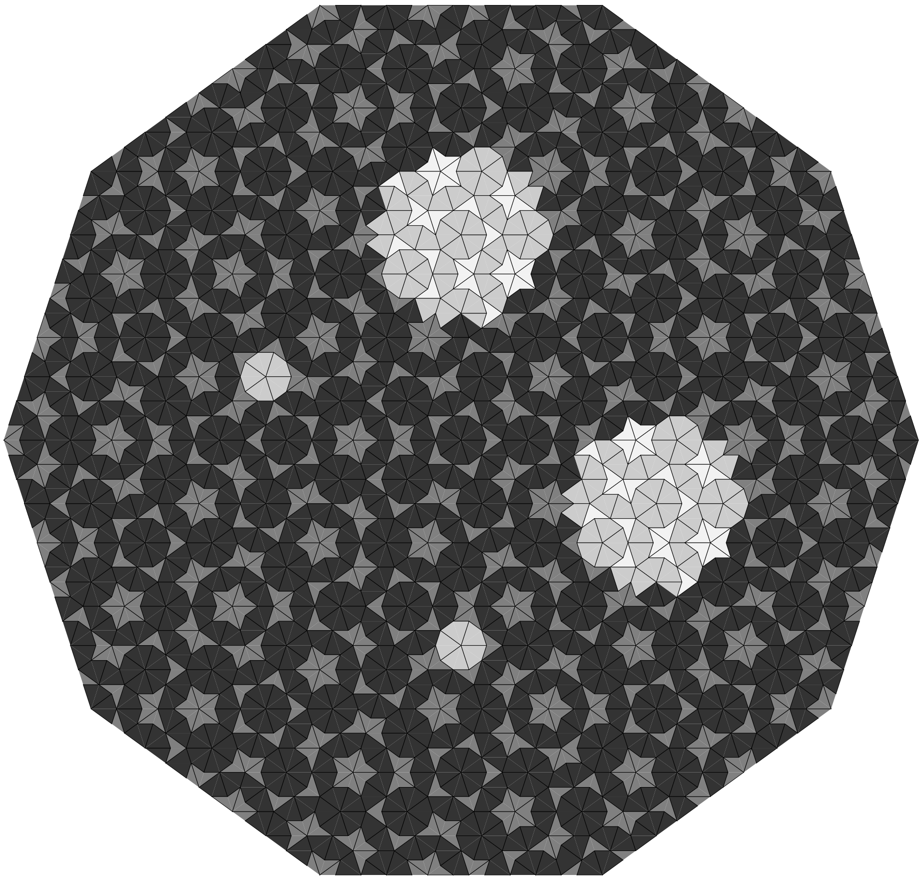

Figure 1.1: The 5-fold symmetric “sun” Penrose tiling is topologically 10-fold symmetric. In particular, it is topologically centrally symmetric. The highlighted patches illustrate this topological central symmetry: occurrence of the same (large) patch at $s, t \in \mathbb{R}^d$ forces coincidence of (maybe somewhat smaller) patches at $-s$ and $-t$.

1-dimensional tilings, $\mathbb{G}l(x_0)$ is a discrete subgroup of $\mathbb{G}l_1(\mathbb{R}) \simeq \mathbb{R}^* := \mathbb{R} \setminus \{0\}$, and it is non-trivial for self-similar x_0 (when its orientation preserving part $\mathbb{G}l^+(x_0)$ is isomorphic to $\mathbb{Z}$). By taking the mth Cartesian power of a self-similar tiling (with the $\mathbb{R}^m$-action $T^{t_1} \times \cdots \times T^{t_m}$) one obtains richer $\mathbb{G}l(x_0)$ containing as a subgroup the semi-direct product of $\mathbb{Z}^m$ by the permutation group Σ_m. However, characterization of possible groups $\mathbb{G}l(x_0)$ or, for that matter, systematic computation of $\mathbb{G}l(x_0)$ in dimension $n > 1$ is not entirely trivial, as evidenced by the following algebraic sub-problem.

Consider the classical Kronecker actions whereby $\mathbb{R}^d$ acts on the torus $X = \mathbb{T}_\Gamma^{d+k} := \mathbb{R}^{d+k}/\Gamma$, $\Gamma \subset \mathbb{Q}^{d+k}$ being a lattice, via $T_{alg}^{(t_1,\dots,t_d)} : x \mapsto x + t_1 v_1 + \cdots t_d v_d$ where $v_1, \dots, v_d$ are fixed vectors with the linear span $E := \mathrm{lin}(v_1, \dots, v_d)$ satisfying $E^\perp \cap \mathbb{Q}^{d+k} = \{0\}$ to secure minimality. Fix $x_0 \in \mathbb{T}_\Gamma^{d+k}$, say $x_0 := 0$. For $k = 0$, we get $\mathbb{G}l(T_{alg}, x_0) = \mathbb{G}l_d(\mathbb{Z})$, as already observed. For $d = 1$, note that $\mathbb{G}l(T_{alg}, x_0) =$

$\{I\}$ unless v_1 is algebraic over $\mathbb{Q}$ in the sense that it is an eigenvector of some rational $(d+k)\times(d+k)$ matrix A. Suppose then that v_1 is algebraic and (for simplicity) of degree $d+k$, i.e., the matrix A does not reduce over $\mathbb{Q}$. Then the algebra $\mathcal{K}$ of rational matrices that commute with A, $\mathcal{K} := \{B \in \mathbb{Q}^{(d+k)\times(d+k)} : BA = AB\}$, is a field and $\mathbb{G}l(T_{alg}, x_0)$ can be seeen to coincide with the group of the algebraic integers that are units of $\mathcal{K}$. It is not hard to see that any finite extension $\mathcal{K}$ of $\mathbb{Q}$ can arise in this way so classification of the groups $\mathbb{G}l(T_{alg}, x_0)$ subsumes a classical theme in algebraic number theory. For $d \geq 2$, many examples can be analyzed but a general theory of $\mathbb{G}l(T_{alg}, x_0)$ for Kronecker actions has yet to be developed.

Our interest in Kronecker actions, or more generally minimal algebraic actions T_{alg} of $\mathbb{R}^d$ on a compact abelian group (to which our discussion readily extends), stems from the construction of the *maximal equicontinuous factor* (see [Ell69]), which associates such an algebraic action to any minimal $\mathbb{R}^d$-action, including minimal tiling actions. The factor map induces a homomorphism of the general linear group of the tiling into the general linear group of the algebraic action (Proposition 10.6).

Returning to our main development, the importance of $\mathbb{G}l(x_0)$ is in its connection with the *pointed mapping class group* $\mathcal{MCG}(X, x_0)$, which we define as the quotient

$$\mathcal{MCG}(X, x_0) := \mathcal{H}(X, x_0)/\mathcal{H}_0(X, x_0) \tag{1.4}$$

where

$$\mathcal{H}_0(X, x_0) := \{f \in \mathcal{H}(X, x_0) : \ \exists_{\psi: X \overset{cont}{\longrightarrow} \mathbb{R}^d} \ \forall_{x\in X} \ f(x) = x + \psi(x)\}. \tag{1.5}$$

Observe that $\mathcal{H}_0(X, x_0)$ consists of all pointed homeomorphisms in $\mathcal{H}(X, x_0)$ that are homotopic to the identity[3]. Indeed, on one hand, the total disconnectedness of X in the direction transversal to the action forces any homotopy starting at the identity to terminate at f of the form $f(x) = x + \psi(x)$ (cf. the lemma in [Kwa10]). On the other hand, any $f \in \mathcal{H}_0(X, x_0)$ homotopes to the identity via $F(t, x) := x + t\psi(x)$, $t \in [0, 1]$. Possibly, such f is also isotopic to the identity and $\mathcal{H}_0(X, x_0)$ is the connected component of the identity in $\mathcal{H}(X, x_0)$, but we do not want to be distracted by this delicate issue. Here, all we need is the simple fact that $\mathcal{H}_0(X, x_0)$ is a normal subgroup[4] of $\mathcal{H}(X, x_0)$, making $\mathcal{MCG}(X, x_0)$ a group, and that we have a natural homomorphism $\mathbb{G}l(x_0) \to \mathcal{MCG}(X, x_0)$ sending A to the coset $[h]$ where h is the unique homeomorphism satisfying (1.2).

Theorem 1.3 (MCG) *For a self-similar repetitive tiling x_0 of finite local complexity, $\mathcal{MCG}(X, x_0)$ is isomorphic to $\mathbb{G}l(x_0)$ via the natural map $\mathbb{G}l(x_0) \to \mathcal{MCG}(X, x_0)$.*

[3] Or merely preserve each $\mathbb{R}^d$-orbit [Kwa10].

[4] To see the group property, let $f(x) = x+\psi(x)$ and $g(x) = x+\phi(x)$; then $g\circ f^{-1}(x) = x+\gamma(x)$ where $\gamma(x) = -\psi \circ f^{-1}(x) + \phi \circ f^{-1}(x)$. Normality is even more clear.

Injectivity of $\mathbb{G}l(x_0) \to \mathcal{MCG}(X, x_0)$ does not use the self-similarity hypothesis and amounts to a simple observation that, given $h, h' \in \mathcal{H}(X, x_0)$, if $h' \circ h^{-1} \in \mathcal{H}_0(X, x_0)$ then $h'(x_0+t) = h(x_0+t)+\psi(x_0+t)$ for some continuous $\psi : X \to \mathbb{R}^d$ and all $t \in \mathbb{R}^d$, which implies $\sup_{t\in\mathbb{R}^d} |A't - At| \leq \sup |\psi| < \infty$, so $A = A'$. Surjectivity of $\mathbb{G}l(x_0) \to \mathcal{MCG}(X, x_0)$ is subtler and presumably fails for some non-self-similar tilings. In our self-similar setting, it is readily obtained by specializing the following addendum to Theorem 1.1 to the case when $\tilde{X} = X$ and applying it to an arbitrary $h_0 \in \mathcal{H}(X, x_0)$.

Theorem 1.4 (global linearization) *In the context of Theorem 1.1, if additionally $h_0(x_0) = \tilde{x}_0$, then h_0 is homotopic to some homeomorphism $h_{lin} : X \to \tilde{X}$ such that $h_{lin}(x_0) = \tilde{x}_0$ and h_{lin} is linear (i.e., for some linear map L, $h_{lin}(x+t) = h_{lin}(x) + Lt, \ \forall t \in \mathbb{R}^d$).*

Before parting with concrete tilings, let us note the following corollary (whose proof is at the end of Section 6, also see Proposition 10.4).

Corollary 1.5 (expanding rigidity) *Under the assumptions of Theorem 1.4, there are $m, n \in \mathbb{N}$ such that Φ^n and $\tilde{\Phi}^m$ are topologically conjugated.*

This should be compared to the main result of [BS07] where the case of one-dimensional tilings was dealt with (by different methods) and which initiated this study. We note that Corollary 1.5 is weaker than the one-dimensional result in that it postulates that the homeomorphism h_0 maps x_0 to $\tilde{x}_0$. Such a priori "pinning down" of h_0 would not have been necessary if we knew more about the inhomogeneities of self-similar tiling spaces (cf. [BD07]).

1.2 Abstract tiling spaces

We turn attention to axioms for abstract self-affine tiling actions which are designed to include actions associated to all finite local complexity self-affine tilings, Delone sets, or expansive $\mathbb{R}^d$-solenoids ([BG03]). The theorems advertised so far in the context of concrete tilings are stated and proven in their definitive form in the sections that follow. We hope that after overcoming the initial hurdle of new definitions the reader will enjoy the clarity of the overall picture and arguments freed from extraneous complications present in any of the concrete settings. Although we embrace abstraction to unify studies incorporating finite local complexity and self-affinity, abandoning any one of these two hypotheses may open up a possibility of abstract tiling actions that are not conjugate to a concrete tiling action. We do not explore this issue here in any detail.

We start by proposing three conditions. The first two are forms of familiar aperiodicity and expansivity hypotheses. The third is a more exotic *phase stability hypothesis* attempting to capture the property in concrete tiling actions whereby, if two large orbit pieces are close to each other and one picks two nearby tilings (one from each piece), then translating the selected tilings leaves them near each other (as long as they still are in their respective orbit pieces).

Let $T = (T^t)_{t\in\mathbb{R}^d}$ be an action of $\mathbb{R}^d$ on a compact metric space X with metric d. Often, $T^t x$ will be simply denoted by $x+t$. The action is called an *abstract tiling action* iff it satisfies the following hypotheses for some positive constants r_0, Δ_0, l_0 with $l_0 < r_0$.

For $R \geq 0$, we set $B_R(A) := \{x : \operatorname{dist}(x, A) < R\}$. We write $B_R(t)$ for $B_R(\{t\})$ and B_R for $B_R(0)$. Also, $B_{-R}(A) := \{s : B_R(s) \subset A\}$.

(LF)$_{r_0}$**:** $\forall_{x\in X,\ t\in\mathbb{R}^d}$ if $x + t = x$ and $|t| < r_0$ then $t = 0$.

(WE)$_{r_0,\Delta_0}$**:** $\forall_{x,y\in X}$ if $\mathsf{d}(x + t, y + t) \leq \Delta_0$ for all $t \in \mathbb{R}^d$, then $x = y + s$ for some $s \in B_{r_0}$.

(PS)$_{r_0,l_0,\Delta_0}$**:** $\forall_{0<\delta<\Delta_0}\ \exists_{\delta_1>0}\ \forall_{x,y\in X,\ U\subset\mathbb{R}^d \text{ connected}}$ if $\inf_{|s|\leq l_0} \mathsf{d}(x + t, y + s + t) < \delta_1$ for all $t \in U$, then there is $s \in B_{r_0}$ such that $\mathsf{d}(x + t, y + s + t) < \delta$ for all $t \in U$.

Looking back, r_0 gives a *scale* up to which the action is free and the *local freedom* axiom (LF) is easily seen to be equivalent to the absence of $x \in X$ fixed by a nontrivial connected subgroup of $\mathbb{R}^d$ (i.e., $\{t \in \mathbb{R}^d : x + t = x\}$ is discrete for all x). Most actions we consider are aperiodic and thus satisfy (LF) but we like singling out (LF) because it is the right hypothesis for the abstract version of Theorem 1.2 (as given by Lemma 6.1). The hypothesis (WE) is, in turn, a weak form of *expansivity* with Δ_0 being the requisite *expansivity constant*. It was considered in [BW72] (see also [KS79, Oka90]); although, it was deemed there unsatisfactory because it allows for uncountably many periodic orbits. Such is the case in the flow rotating, with a constant nonzero linear speed, the concentric circles in a planar annulus. Here (WE) holds thanks to the *shearing* action separating points of nearby circles by increasing their phase (angular) separation. The pivotal hypothesis (PS) stipulates a kind of *phase stability* that precludes such shearing behavior. The role of l_0 is to place an upper bound on (phase) translations, if only because the first infimum would be automatically zero should $l_0 = \infty$ and the action were minimal. (Insisting that l_0 does not exceed the scale of local freedom, $l_0 < r_0$, is natural for a similar reason.) Also, observe that (PS) would be simply a consequence of the continuity of T if not for the fact that the diameter $\operatorname{diam}(U)$ can be arbitrarily large. By itself, (PS) clearly holds for the multitude of (non-expansive) actions by isometries. One can show that the combination of (WE) and (PS) assures that abstract tiling actions are expansive in the stronger sense of [BW72]. We note that to prove our results we do not have to insist on U being an arbitrary connected set: we use (PS) only for U that are (large) solid ellipsoids in $\mathbb{R}^d$.

The following technical remark may be skipped in the first reading.

Remark 1.6 *As soon as (LF)$_{r_0}$ is satisfied for some $r_0 > 0$,*

$$\mu_0(r) := \inf\{\mathsf{d}(x, x+t) : \ x \in X, \ r \leq |t| \leq r_0\} \tag{1.6}$$

is a positive continuous monotonic function of $r \in (0, r_0]$. By considering $t = 0$ in (WE)$_{r_0,\Delta_0}$ and (PS)$_{r_0,l_0,\Delta_0,R_0}$, the shift s existence of which they stipulate must satisfy $|s| \leq l_0 + \mu_0^{-1}(\Delta_0)$, where the right hand side can be made arbitrarily small by taking small l_0 and Δ_0. Thus, if the three axioms can be satisfied, then they can be satisfied with r_0 that is as small as we wish.

Our final axiom is the self-affinity:

(EXP) there is a continuous $\Phi : X \to X$ and an expanding linear map $\Lambda : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$T^{\Lambda t} \circ \Phi = \Phi \circ T^t \qquad (\forall t \in \mathbb{R}^d). \tag{1.7}$$

In many arguments it will be important that Φ in (EXP) is actually a homeomorphism. This is a priori true for the actions we consider on the force of Theorem 8.1 (which is an incarnation of the recognizability results in [Mos92, Sol98]). As it is well known, (LF) and (EXP) with an invertible Φ imply aperiodicity: $x + t = x$ forces $\Phi^{-n}(x) + \Lambda^{-n}t = \Phi^{-n}(x)$ where $|\Lambda^{-n}t| \to 0$ by the expanding property of Λ and so $\Lambda^{-n}t = 0$ for some n by (LF).

Definition 1.7 *An abstract tiling action T on X satisfying (EXP) is called an* abstract self-affine tiling action*. If additionally X is locally a product of a Cantor set and $\mathbb{R}^d$, the action is* FLC *(for Finite Local Complexity). If the linear map Λ is conformal, the action is* self-similar*.*

Although we left it out of the above definition, we are almost exclusively interested in actions that are minimal and uniquely ergodic. Unique ergodicity is already a consequence of minimality for self-affine abstract tiling actions per Theorem 9.1, cf. [Sol97].

Our main examples of abstract tiling actions are the actions obtained from repetitive tilings with finite local complexity. For reader's convenience, we verify this in an appendix (Proposition 7.1). Similar arguments can be made for repetitive Delone sets of finite type and minimal expansive $\mathbb{R}^d$-solenoids. In the self-affine FLC setting, on which we center our attention, these are the only examples (up to conjugacy, see Theorem 1.9 ahead). Beyond that the situation is not clear. It is also possible that the FLC hypothesis is redundant (cf. [Man79]) as no known tilings without finite local complexity give rise to minimal[5] abstract tiling actions. The *Pinwheel tiling* [Rad94, Sad98] fails (PS) and the tilings in [PFS] fail (WE). In any case, we are not motivated as much by an urge to generalize but rather want

[5]If minimality were no object one could take a suspension of an expansive homeomorphism on a space that is not zero-dimensional, say a torus.

to respond to a natural call for an intrinsic characterization of topological actions arising in tiling theory. The question is as much cultural as it is mathematical: *What types of topological actions are amenable to tiling theory methods?* Our axioms are a first serious attempt of an answer and their main feature (required of any answer) is their purely topological nature: each is invariant under conjugating the action by a homeomorphism, as asserted by the following theorem the proof of which is a routine exercise.

Theorem 1.8 (intrinsicity) *Among all $\mathbb{R}^d$-actions on compact metric spaces, the classes of abstract tiling actions, self-affine abstract tiling actions, and self-similar abstract tiling actions (with or without FLC) are invariant under topological conjugacy.*

What makes possible our "soft" topological take on the self-affine tilings is another rigidity phenomenon: hyperbolicity and the requisite *local product structure* (*Smale space*, [AP98]) typically afforded by the inflation-substitution formulation of self-affinity can be recovered (Section 2) already from purely topological hypotheses and without postulating any geometric structure (like that of a G-solenoid in [BG03]). Furthermore, with the product structure at our disposal, we can invoke the standard results on existence of Markov partitions to strip away the abstractness and reconnect with concrete tilings:

Theorem 1.9 (tiling model) *For any minimal abstract self-affine tiling action T (acting on X), there is a repetitive self-affine tiling with finite local complexity such that its tiling space Y maps to X via a finite-to-one continuous map $\pi : Y \to X$ which factors the translation action on Y to T. The factor map $\pi : Y \to X$ is 1–1 on the complement of a nowhere dense set of measure zero. Moreover, if the abstract tiling action T is additionally FLC, then π is a homeomorphism conjugating T to the translation action on Y.*

Putting together the last two theorems identifies the exact place (within all $\mathbb{R}^d$-actions on compact spaces) occupied by the actions conjugated to those arising from repetitive self-affine tilings of finite local complexity.

Finally, we mention that the prototiles constructed in the proof of the theorem (Section 3) satisfy an exact inflation-substitution rule and thus are typically very fractal and possibly disconnected. If one is content with only an approximate inflation-substitution rule then the results in [Pri00, PS01] concern passing to nice polyhedral tilings.

* * *

The structure of the paper is easily revealed by the section titles. Sections 2 and 4 develop key tools. Sections 3, 5, 6 contain the proofs of the flagship theorems together with their more general and technical formulations. A reader interested only in concrete tilings should skip Section 3 and only skim Section 2 to pick up notations introduced in its beginning while omitting all the proofs. The appendices collect results that are tangential to our main development with the lengthy Appendix 10 offering insights into computation of the mapping class groups together with two examples.

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2 Local Product Structure

For concrete self-affine tiling spaces the key technical device is the *local product structure* whereby the tiling space is locally decomposed into a cartesian product $U \times C$ where U is an open set in $\mathbb{R}^d$ and C is the transversal made of tilings that coincide on a large patch about the origin. In the self-affine case, C is nothing else than the local stable set of the inflation-substitution map Φ and our objective in this section is to construct this set based on purely topological axioms. In particular, we show that abstract self-affine tiling spaces are *Smale spaces* (cf. [AP98]). We note that considering more than one Φ on the same space X is an important facet of our approach.

We assume that X is a self-affine abstract tiling space and our goal is to construct the local stable set of a point $x \in X$ with respect to any Φ that satisfies (EXP). When Φ is unambiguous, for $\Delta \in (0, \Delta_0)$, $E \subset \mathbb{R}^d$ and $x, y \in X$, we use the following notations:

$$x|_E^{(\Delta)} = y|_E^{(\Delta)} \quad \stackrel{def}{\equiv} \quad \mathsf{d}(x+t, y+t) < \Delta \ \text{ for all } t \in E,$$
$$x|_E^{(\Delta)} \stackrel{s}{=} y|_E^{(\Delta)} \quad \stackrel{def}{\equiv} \quad \mathsf{d}(\Phi^m(x+t), \Phi^m(y+t)) < \Delta \ \text{ for all } t \in E,\ m \geq 0.$$

We also adopt the standard definition of the ϵ-*local stable set* of x with respect to Φ,

$$W^s_\epsilon(x, \Phi) := \{y : \ \mathsf{d}(\Phi^n x, \Phi^n y) < \epsilon \ \text{ for all } n \geq 0\}. \tag{2.1}$$

Ostensibly, $x|_E^{(\Delta)} = y|_E^{(\Delta)}$ abstracts the situation of tilings x and y having the same patches over E up to a small translation. It can be used for expressing proximity of x and y because the family of sets

$$U_{R,x} := \left\{ y \in X : \ y|_{B_R}^{(\Delta_0)} = x|_{B_R}^{(\Delta_0)} \right\}, \quad R > 0, \tag{2.2}$$

constitutes a basis of the topology at $x \in X$. This can be deduced from Remark 1.6 and the following consequence of the expansivity axiom (WE).

Fact 2.1 $\forall_{\delta>0}\,\exists_{R_1>0}\,\forall_{x,y\in X}$ *if* $\mathsf{d}(x+t,y+t)<\Delta_0$ *for all* $|t|<R_1$*, then* $\mathsf{d}(x+s,y)<\delta$ *for some* s *with* $|s|\le r_0$*. Moreover,* $|s|<\mu_0^{-1}(\delta)$ *(where* μ_0 *is as in Remark 1.6).*

Proof of Fact 2.1: Suppose the implication fails: there exist $\delta>0$ and sequences (x_n) and (y_n) such that $\mathsf{d}(x_n+t,y_n+t)<\Delta_0$ for all $|t|\le n$ yet $\mathsf{d}(x_n+s,y_n)\ge\delta$ for all $|s|\le r_0$. X being compact, we may assume that the sequences converge to some x and y, respectively. Then $\mathsf{d}(x+t,y+t)\le\Delta_0$ for all $t\in\mathbb{R}^d$ yet $\mathsf{d}(x+s,y)\ge\delta$ if $|s|\le r_0$. This is a contradiction because $(\mathrm{WE})_{r_0,\Delta_0}$ yields $y=x+s$ for some $|s|\le r_0$. That $|s|<\mu_0^{-1}(\delta)$ follows from $\mathsf{d}(x+s,y)<\delta$. □

The relation $x|_E^{(\Delta)}\overset{s}{=}y|_E^{(\Delta)}$ is stronger than $x|_E^{(\Delta)}=y|_E^{(\Delta)}$ and corresponds to x and y having the same patches over E without adjusting by a small translation. It provides the following convenient description of the local stable sets (the proof of which is omitted).

Fact 2.2 *Fix* $\Delta\in(0,\Delta_0]$*. Given a radius* $R>0$ *, if* $\epsilon>0$ *is small enough, then*

$$y\in W^s_\epsilon(x,\Phi)\implies y|_{B_R}^{(\Delta)}\overset{s}{=}x|_{B_R}^{(\Delta)}.$$

Vice versa, given $\epsilon>0$*, if the radius* $R>0$ *is large enough, then*

$$y|_{B_R}^{(\Delta)}\overset{s}{=}x|_{B_R}^{(\Delta)}\implies y\in W^s_\epsilon(x,\Phi).$$

The lemma below is an abstract analogue of the fundamental fact that any tiling y sufficiently near x can be adjusted by a minute translation to coincide with x on a large patch.

Lemma 2.3 (local stable sets) *Fix* $\epsilon>0$*. For any* Φ *that satisfies (EXP), given* $x\in X$*, the set* $W^s_\epsilon(x,\Phi)$ *is a* local cross section *in the sense that: there is* $\epsilon_1>0$ *(independent of* x*) so that if* $\mathsf{d}(y,x)<\epsilon_1$ *then there exists* $y_*\in W^s_\epsilon(x,\Phi)$ *with* $y=y_*+t_*$ *for some* $t_*\in B_{r_0}$*.*

Moreover, given $R>0$*,* $\epsilon_1>0$ *can be taken small enough (depending on* Φ *only) so that*

$$x+\tau|_{B_R}^{(\Delta_0)}\overset{s}{=}y_*+\tau|_{B_R}^{(\Delta_0)}\quad\text{for all }\tau\in B_{r_0}.\tag{2.3}$$

Remark 2.4 *Taking* $\epsilon>0$ *small enough, we can ask that additionally (for all* $x\in X$*) we have* $y_*+\tau\in W^s_\epsilon(x+\tau,\Phi)$ *for all* $\tau\in B_{r_0}$ *and that* y_* *is the unique point of* $(y+B_{r_0})\cap W_\epsilon(x,\Phi)$*.*

Postponing the proofs for a moment, we adopt the traditional bracket notation for the point y_*:

$$[x,y]_\Phi:=y_*.\tag{2.4}$$

Here the necessity of assuring proximity of x and y as stipulated by the lemma and the remark is silently understood. Note that the remark gives that $[x+\tau,y]_\Phi=[x,y]_\Phi+\tau$ for small τ (i.e., the $\mathbb{R}^d$-action permutes stable local sets). Also, taking

$\tau = t_*$ in the remark, we see that $y = [x+t, y_*]_\Phi$. Thus a point y near x can be uniquely described by a pair $(x+t, y_*) \in (x + B_{r_0}) \times W^s_\epsilon(x, \Phi)$. Upon taking $\epsilon > 0$ small enough, we obtain an identification $y \leftrightarrow (x+t, y_*)$ between a neighborhood of x and $(x + B_{r_0}) \times W^s_\epsilon(x, \Phi)$. This identification is called the *local product structure* near $x \in X$. It is effected by a homeomorphism as can be seen with the help of the lemma below. (Below $\mathrm{dist}(\Phi, \Phi') = \sup_{x\in X} \mathsf{d}(\Phi(x), \Phi'(x))$ is the uniform distance.)

Lemma 2.5 (continuity of local product structure) *The $[x, y]_\Phi$ is a continuous function of the quadruple (x, y, Φ, Λ). Precisely, taking $\epsilon, \epsilon_1 > 0$ as in Lemma 2.3 and Remark 2.4 , for any $\eta > 0$ there is $\mu > 0$ so that, if $\mathsf{d}(y, y') < \mu$ and $\mathsf{d}(x, x') < \mu$ and Φ and Φ' are two maps satisfying (EXP) with $\|\Lambda' - \Lambda\| < \mu$ and $\mathrm{dist}(\Phi, \Phi') < \mu$, then $W^s_\epsilon(x, \Phi)$ and $W^s_\epsilon(x', \Phi')$ are both local cross-sections (in the sense of Lemma 2.3) and $\mathsf{d}([x, y]_\Phi, [x', y']_{\Phi'}) < \eta$.*

Proof of Lemma 2.3: The plan is to construct y_* and establish (2.3) for an arbitrarily given $R > 0$. Setting $\tau = 0$ in (2.3) already gives $y_* \in W^s_\epsilon(x, \Phi)$ by Fact 2.2.

We fix $R > 0$ and start with a tedious selection of a number of auxilliary constants. Fix $\kappa_* \in (0, 1)$ at will. Suppose that $\Phi \circ T^t = T^{\Lambda t} \circ \Phi$, $t \in \mathbb{R}^d$, where the linear map $\Lambda : \mathbb{R}^d \to \mathbb{R}^d$ is expanding so that there are $\lambda > 1$ and $C_1 > 0$ with $|\Lambda^m t| \geq C_1 \lambda^m |t|$ for all $m \in \mathbb{N}$, $t \in \mathbb{R}^d$. One can pick $\kappa_j \in (0, 1)$, $j = 1, 2, 3, \ldots$, so that

$$\inf_{m\geq 1} \kappa_1 \cdots \kappa_m (1 - \kappa_{m+1}) \lambda^{m+1} < +\infty \quad \text{and} \quad \prod_{j=1}^{\infty} \kappa_j \in (\kappa_*, 1). \tag{2.5}$$

(Indeed, taking $\lambda_1 \in (1, \lambda)$ and setting $\kappa_j := (1 - c\lambda_1^{-j})$ does the job provided $c > 0$ is small.)

By using continuity of Φ, pick $\delta \in (0, \Delta_0/2)$ so that

$$\mathsf{d}(y, z) < \delta \implies \mathsf{d}(\Phi(z), \Phi(y)) < \Delta_0/2 \quad (\forall y, z \in X).$$

Shrink δ if necessary so that $\mu_0^{-1}(\delta) < l_0$ and

$$\sup \left\{ \mathsf{d}(z + t, z) : \ z \in X, \ |t| \leq \mu_0^{-1}(\delta) \cdot \sum_{k=1}^{\infty} \|\Lambda^{-k}\| \right\} < \min\{\Delta_0/2, r_0\}. \tag{2.6}$$

Let δ_1 in $(\mathrm{PS})_{r_0, l_0, \Delta_0}$ be chosen for this δ, and R_1 be chosen for the δ_1 as in Fact 2.1. Take L_0 large enough so that

$$R \leq \kappa_* L_0 - r_0 \tag{2.7}$$

and

$$R_1 \leq C_1 \kappa_1 \cdots \kappa_m (1 - \kappa_{m+1}) \lambda^{m+1} L_0 \quad (\forall m \in \mathbb{N}). \tag{2.8}$$

We claim that this secures

$$B_{-R_1}(\kappa_m \cdots \kappa_1 \Lambda^{m+1} B_{L_0}) \supset \kappa_{m+1} \cdots \kappa_1 \Lambda^{m+1} B_{L_0} \quad (\forall m \in \mathbb{N}). \tag{2.9}$$

(Recall, $B_{-R}(D) := \{x : B_R(x) \subset D\}$.) To verify the claim, take $x \in \kappa_{m+1} \cdots \kappa_1 \Lambda^{m+1} B_{L_0}$, and observe that

$$\kappa_1 \cdots \kappa_m (1 - \kappa_{m+1}) L_0 + \kappa_1 \cdots \kappa_{m+1} L_0 = \kappa_1 \cdots \kappa_m L_0$$

guarantees that, in the string of the following inclusions (based on (2.8))

$$B_{R_1}(x) \subset B_{C_1 \lambda^{m+1} \kappa_1 \cdots \kappa_m (1-\kappa_{m+1}) L_0}(x) \subset \Lambda^{m+1} B_{\kappa_1 \cdots \kappa_m (1-\kappa_{m+1}) L_0}(x),$$

the largest set is contained in $\Lambda^{m+1} B_{\kappa_m \cdots \kappa_1 L_0}$. That is $B_{R_1}(x) \subset \Lambda^{m+1} B_{\kappa_m \cdots \kappa_1 L_0}$, affirming (2.9).

Finally, let $\epsilon_1 > 0$ be small enough so that

$$\mathsf{d}(x, y) < \epsilon_1 \quad \Longrightarrow \quad \mathsf{d}(x + t, y + t) < \delta \text{ for all } t \in B_{L_0}. \tag{2.10}$$

To begin the main construction, fix $x \in X$ and take $y \in X$ with $\mathsf{d}(x, y) < \epsilon_1$ so that

$$\mathsf{d}(x + t, y + t) < \delta \text{ for all } t \in B_{L_0}. \tag{2.11}$$

Set $y_0 := y$ and $s_0 := 0$ and $\kappa_0 := 1$. We shall generate by induction a sequence $s_1, s_2, \ldots$ of points in B_{r_0} so that $y_1 := y_0 + \Lambda^{-1} s_1, y_2 := y_1 + \Lambda^{-2} s_2, \ldots$ satisfy

$$\mathsf{d}(\Phi^n x + t, \Phi^n y_n + t) < \delta \text{ for all } t \in \kappa_n \cdots \kappa_0 \Lambda^n B_{L_0}. \tag{2.12}$$

For $n = 0$, (2.12) coincides with (2.11). If y_{n-1} is already constructed for some $n \geq 1$ then application of Φ to the version of (2.12) with n replaced by $n-1$ yields (by the choice of δ)

$$\mathsf{d}(\Phi^n x + t, \Phi^n y_{n-1} + t) < \Delta_0 \text{ for all } t \in \kappa_{n-1} \cdots \kappa_0 \Lambda^n B_{L_0}.$$

Invoking Fact 2.1 for each $t \in \kappa_n \cdots \kappa_0 \Lambda^n B_{L_0} \subset B_{-R_1}(\kappa_{n-1} \cdots \kappa_0 \Lambda^n B_{L_0})$ (where we used (2.9)) produces $s(t) \in B_{\mu_0^{-1}(\delta)} \subset B_{l_0}$ with $\mathsf{d}(\Phi^n x + t, \Phi^n y_{n-1} + t + s(t)) < \delta_1$. From $(\mathrm{PS})_{r_0, l_0, \Delta_0}$ applied to $U = \kappa_n \cdots \kappa_0 \Lambda^n B_{L_0}$, there is $s_n \in B_{\mu_0^{-1}(\delta)} \subset B_{l_0} \subset B_{r_0}$ such that

$$\mathsf{d}(\Phi^n x + t, \Phi^n y_{n-1} + t + s_n) < \delta \text{ for all } t \in \kappa_n \cdots \kappa_0 \Lambda^n B_{L_0}.$$

That is $y_n := y_{n-1} + \Lambda^{-n} s_n$ satisfies (2.12) concluding the induction step.

In view of (2.12) and (2.5), the constructed sequence $(s_n) \subset B_{r_0}$ is such that

$$y_n = y + \Lambda^{-1} s_1 + \Lambda^{-2} s_2 + \ldots + \Lambda^{-n} s_n \tag{2.13}$$

satisfy

$$\mathsf{d}(\Phi^n x + t, \Phi^n y_n + t) < \delta \text{ for all } t \in \kappa_* \Lambda^n B_{L_0}. \tag{2.14}$$

Furthermore, for $m \leq n$, we have $y_n = y_m + \Lambda^{-(m+1)} s_{m+1} + \ldots + \Lambda^{-n} s_n$, so that, for all $t \in \kappa_* \Lambda^m B_{L_0}$, the stipulation (2.6) lets us estimate

$$\begin{aligned} \mathsf{d}(\Phi^m x + t, \Phi^m y_n + t) &\leq \mathsf{d}(\Phi^m x + t, \Phi^m y_m + t) \\ &\quad + \mathsf{d}(\Phi^m y_m + t, \Phi^m y_m + \Lambda^{-1} s_{m+1} + \ldots + \Lambda^{-(n-m)} s_n + t) \\ &\quad < \delta + \Delta_0/2 < \Delta_0. \end{aligned} \tag{2.15}$$

Thus, if we let

$$y_* := y + t_*, \quad t_* := \sum_{k=1}^{\infty} \Lambda^{-k} s_k \tag{2.16}$$

then $t_* \in B_{r_0}$ by (2.6); and taking the limit with $n \to \infty$ in (2.15) yields

$$\mathsf{d}(\Phi^m x + t, \Phi^m y_* + t) < \Delta_0 \quad \text{for all} \;\; t \in \kappa_* \Lambda^m B_{L_0}. \tag{2.17}$$

For $\tau \in B_{r_0}$, substituting $t = t' + \Lambda^m \tau$ yields

$$\mathsf{d}(\Phi^m (x + \tau) + t', \Phi^m (y_* + \tau) + t') < \Delta_0 \text{ for all } t' \in \Lambda^m B_{-r_0}(\kappa_* B_{L_0}). \tag{2.18}$$

This means that our goal relation (2.3) holds because $B_R \subset B_{-r_0}(\kappa_* B_{L_0})$ on the force of (2.7). □

Proof of Remark 2.4: That $y_* + \tau \in W^s_\epsilon(x + \tau, \Phi)$ for $\tau \in B_{r_0}$ follows immediately from (2.3) and Fact 2.2 upon taking $R > 0$ large enough. We turn then to the uniqueness of y_*. There is $\lambda_1 > 1$ that depends on Λ so that any orbit $(\Lambda^m s)_{m \in \mathbb{Z}}$ for $s \neq 0$ intersects the "shell"

$$\Omega := \{ s' : \; r_0/\lambda_1 \leq |s'| \leq r_0 \}.$$

Let $\epsilon > 0$ be so small that $2\epsilon \leq \mu_0(r_0/\lambda_1)$ and

$$\forall_{m \in \mathbb{Z}} \; \Lambda^m s \in \Omega \text{ and } 0 < |s| < \mu_0^{-1}(2\epsilon) \implies m \geq 0.$$

If $(y + B_{r_0}) \cap W_\epsilon(x, \Phi)$ contained, beside y_*, also $y_* + s$ for some $s \in B_{2r_0} \setminus \{0\}$, then

$$\mathsf{d}(\Phi^m y_*, \Phi^m y_* + \Lambda^m s) \leq \mathsf{d}(\Phi^m y_*, \Phi^m x) + \mathsf{d}(\Phi^m x, \Phi^m (y_* + s)) \leq 2\epsilon \quad (\forall m \geq 0). \tag{2.19}$$

Taking $m = 0$, we get $\mathsf{d}(y_*, y_* + s) \leq 2\epsilon$ so that $|s| \leq \mu_0^{-1}(2\epsilon)$. Also, taking the m with $\Lambda^m s \in \Omega$, the definition of $\mu_0(r_0/\lambda_1)$ yields a contradiction: $2\epsilon \geq \mathsf{d}(\Phi^m y_*, \Phi^m y_* + \Lambda^m s) \geq \mu_0(r_0/\lambda_1)$. □

Proof of Lemma 2.5: From (2.16), y_* is of the form $y + s$ where s is a sum of an absolutely convergent series and, given an arbitrary $\gamma > 0$, there is $N \in \mathbb{N}$ such that, for all sequences $(s_n) \subset B_{r_0}$ and Λ' in a small neighborhood of fixed expanding map Λ, we have a uniform bound on the *tail*: $\sum_{k=N}^{\infty} \|\Lambda'^{-k} s_k\| < \gamma/2$. Moreover, being specified via strict inequalities, the shifts $s_1, \ldots, s_N$ constructed for y, x, Φ continue to satisfy the requirements of the construction (chiefly (2.12)) for all small perturbations y', x', Φ'. Therefore, if Λ and Λ' are so close that also $\sum_{k=1}^{N} \|\Lambda'^{-k} - \Lambda^{-k}\| \|s_k\| < \gamma/2$, then $y'_* = y' + s'$ and $y_* = y + s$ with $|s - s'| < \gamma/2 + \gamma/2 = \gamma$.

Taking into account continuity of the $\mathbb{R}^d$-action, that $[x, y]_\Phi = y_* = y + s$ is continuous in y, x, Λ follows by the arbitrariness of γ. □

3 Markov Tilings: proof of Theorem 1.9

As already indicated in the introduction, with the local product structure at our disposal, the proof of the theorem hinges on invoking Bowen's construction of Markov partitions ([Bow70]). A tiling space that factors to X is then constructed as the *impression* of the Markov boxes on a single $\mathbb{R}^d$-orbit. We give an outline omitting the routine details.

Sketch of proof of Theorem 1.9: The local product structure allows us to cover X by a finite collection of open sets W_i, $i = 1, \dots, N$, where W_i is the homeomorphic image under the $[\cdot,\cdot]_\Phi$ map of the cartesian product $(x_i + B_{r_0}) \times W^s_\epsilon(x_i, \Phi)$ for some points $x_i \in X$. A set $Q \subset X$ is called a ***proper rectangle*** iff it is contained in some W_i and $Q = [Q^u, Q^s]_\Phi$ (which stands for $\{[x, y]_\Phi : \; x \in Q^u, y \in Q^s\}$) where $Q^u \subset x_i + B_{r_0}$ and $Q^s \subset W^s_\epsilon(x_i, \Phi)$ are closures of their interiors. The boundary of Q is contained in the union $\partial^u Q \cup \partial^s Q$ where $\partial^u Q := [Q^u, \partial Q^s]_\Phi$ and $\partial^s Q := [\partial Q^u, Q^s]_\Phi$ are referred to as the ***unstable boundary of*** Q and the ***stable boundary of*** Q, respectively. Bowen's construction reworks the covering $(W_i)_{i=1}^N$ by the rectangles into a covering $(Q_j)_{j=1}^M$ by proper rectangles with pairwise disjoint interiors that satisfies the ***Markov property***: Φ maps $\bigcup_j \partial Q^s_j$ into itself and Φ^{-1} maps $\bigcup_j \partial Q^u_j$ into itself. The diameters of Q_j can be made arbitrarily small; we ask that they do not exceed the expansivity constant Δ_0.

Because the orbit of $\bigcup_j \partial^u Q_j$ under (T^t) is easily expressed as a countable union of closed nowhere dense sets, there is a dense G_δ set G of points $x \in X$ such that $x + \mathbb{R}^d$ does not intersect $\bigcup_j \partial^u Q_j$ and G is invariant under Φ, $\Phi(G) = G$. For each $x \in G$, the immersion $\mathbb{R}^d \ni t \mapsto x - t$ pulls back the Markov partition of X to an essentially disjoint covering τ_x of $\mathbb{R}^d$ into sets, each translationally congruent to some Q^u_j. In particular, τ_x is a tiling of $\mathbb{R}^d$ with prototiles Q^u_j. For $x \notin G$, $x + \mathbb{R}^d$ may have points in more then one Q_j making the pull back ambiguous so that x is associated not to a single tiling but to a possibly infinite family of tilings, which we still denote τ_x.

Because there are only finitely many Q_j and they are all proper rectangles, the number of distinct R-patches centered at 0 appearing in the tilings in τ_x is finite for any $R > 0, x \in X$. As a consequence, we cannot have $R > 0$ and a sequence $(x_n) \subset G$ convergent to some $x \in X$ for which the R-patches centered at 0 are pairwise translationally incongruent. By compactness of X, this assures that, for any $R > 0$, there are only finitely many incongruent R-patches in $\bigcup_{x\in X} \tau_x$. In particular, the tiling τ_x has finite local complexity for any $x \in G$.

Upon fixing $x \in G$, we can form the *translation hull* of τ_x

$$Y := \mathrm{cl}\left\{\tau_x + t : \ t \in \mathbb{R}^d\right\} \tag{3.1}$$

where the closure is with respect to the standard topology on tilings. Because the association $G \ni x \mapsto \tau_x$ is continuous and $\tau_{x+t} = \tau_x + t$, the minimality of the action (T^t) assures that Y does not depend on $x \in G$.

Now, by sending $Y \ni \tau_x \mapsto x \in X$ we have a densely defined map on Y which is uniformly continuous because, if τ_x and $\tau_{x'}$ are close to each other, then there is a small translation s such that $x+t$ and $x'+s+t$ reside in the same Markov boxes for all t in a ball of large radius; and the expansivity axiom assures then that x and $x'+s$ are close to each other in X. Therefore, $G \ni \tau_x \mapsto x$ uniquely extends to a continuous map $\pi : Y \to X$. Of course, π factors the translation action on tilings to the action (T^t) on X: $\pi(\tau) + t = \pi(\tau + t)$. The translation on Y is minimal because (T^t) is minimal on X, and π is injective on $\pi^{-1}(G)$ by design.

$\Phi|_G$ induces a densely defined map on Y; namely, $\pi^{-1} \circ \Phi|_G \circ \pi$ on $\pi^{-1}(G)$. It again extends uniquely to all of Y by continuity. Moreover, the Markov property guarantees that this mapping, which we continue to denote by Φ, is associated with some inflation-substitution on the prototiles Ω^u_j, $j = 1, \ldots, M$. Thus Y is a minimal tiling space of finite local complexity generated by an inflation-substitution rule.

To finish the proof of the main assertion of the theorem, we have to argue that π is at most K to one for some $K > 0$. By the recognizability result in [Sol98], $\Phi : Y \to Y$ is a homeomorphism. Upon fixing $R_0 > 0$, we already know that there are only finitely many R_0-patches centered at the origin that can appear in $\tau \in \bigcup_{x\in X} \pi^{-1}(x)$. Let their number be K. Consider $x \in X$ and $R > 0$. Any R-patch of $\tau \in \pi^{-1}(x)$ centered at the origin is contained in the result of the n-fold inflation-substitution applied to the R_0-path in the $\Phi^{-n}(\tau)$ provided n is large enough. Thus there are at most K such R-patches among all $\tau \in \pi^{-1}(x)$. By arbitrariness of R, the cardinality of $\pi^{-1}(x)$ is at most K.

Finally, "the moreover part" hinges on the fact that, when X is locally a product of a Cantor set and $\mathbb{R}^d$, the initial covering $(W_i)_{i=1}^N$ in Bowen's construction can be taken such that each W_i is clopen when projected onto its local Cantor component. Hence, $\bigcup_j \partial^u W_j = \emptyset$ and thus also $\bigcup_j \partial^u Q_j = \emptyset$. This assures that the good set G is all of X, making π injective. □

For the sake of the proof of recognizability in Appendix 8, we record the following immediate corollary of the existence of Markov partitions in the above argument.

Corollary 3.1 *Under the assumptions of Theorem 1.9, the map $\Phi : X \to X$ in (EXP) is a continuous a.e. 1-1 factor of a two sided mixing topological Markov chain. In particular, the periodic points of Φ are dense in X.*

4 Reparametrizations of Actions and Averaging of Cocycles

This section collects some fundamentals about reparametrizations and the associated cocycles for uniquely ergodic $\mathbb{R}^d$-actions on a compact space X (cf. [Kat03]). (It does not use the expanding dynamics Φ.) Lemma 4.1 below will be instrumental in the proof of Theorem 1.1.

By a *(continuous) reparametrization* of $(T^t)_{t\in\mathbb{R}^d}$ we understand any other action $(S^t)_{t\in\mathbb{R}^d}$ for which there are continuous $\alpha, \beta : X \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$T^t x = S^{\alpha(x,t)} x \quad \text{and} \quad S^t x = T^{\beta(x,t)} x \qquad (\forall x \in X, t \in \mathbb{R}^d). \tag{4.1}$$

The relation between (T^t) and (S^t) is clearly symmetric. The *group property* of the actions, $T^{t_1+t_2} = T^{t_1} \circ T^{t_2}$ and $S^{t_1+t_2} = S^{t_1} \circ S^{t_2}$, implies that α has the *cocycle property*:

$$\alpha(x, t_1 + t_2) = \alpha(x, t_1) + \alpha(T^{t_1} x, t_2) \qquad (\forall t_1, t_2 \in \mathbb{R}^d, x \in X). \tag{4.2}$$

The analogous cocycle property holds for β. Note that α and β are related by $\alpha(x, \beta(x,t)) = t$, so β can be recovered from α.

Any function $\alpha : X \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying (4.2) is called a *cocycle over* (T^t). Given such a cocycle for which $t \mapsto \alpha(x,t)$ is invertible with the inverse defining a continuous $\beta : X \times \mathbb{R}^d \to \mathbb{R}^d$, the cocycle property of α assures that the formula $S^t x = T^{\beta(x,t)} x$ defines a reparametrization of (T^t). [6]

There are always plenty of reparametrizations for any given (T^t) of which the simplest are the *(linear) rescalings* given by $S^t x := T^{At} x$ where $A : \mathbb{R}^d \to \mathbb{R}^d$ is a linear isomorphism. A common scenario giving rise to reparametrizations entails being presented with another $\mathbb{R}^d$-action $(\tilde{T}^t)$ on some $\tilde{X}$ for which there is a homeomorphism $h_0 : X \to \tilde{X}$ mapping the orbits of (T^t) to the orbits of $(\tilde{T}^t)$. Then $S^t := h_0^{-1} \circ \tilde{T}^t \circ h_0$ defines a reparametrization of (T^t). Although, any reparametrization (S^t) of (T^t) is obtainable in this way (as one can simply take $h_0 = \mathrm{Id}$ and let $\tilde{T}^t = S^t$), this is no longer true when one is faced with some a priori restrictions on $(\tilde{T}^t)$. For instance, in the extreme case when one insists that $\tilde{X} = X$ and $(\tilde{T}^t) = (T^t)$, (S^t) has to be conjugated to (T^t). From this perspective, Theorem 1.1 concerns the class of reparametrizations arising when both (T^t) and $(\tilde{T}^t)$ are stipulated to be abstract self-similar tiling actions and it asserts that the reparametrization is conjugated to a linear rescaling of the original action (T^t). The following variant of the uniform ergodic theorem will be instrumental in identifying the right rescaling matrix A.

[6] To see the group property for S one needs the cocycle property for β: $\beta(x, t_1 + t_2) = \beta(x, t_1) + \beta(S^{t_1} x, t_2)$. Upon setting $s_1 := \beta(x, t_1)$ and $s_2 := \beta(S^{t_1} x, t_2) = \beta(T^{s_1} x, t_2)$, this rewrites as $\beta(x, \alpha(x, s_1) + \alpha(T^{s_1} x, s_2)) = s_1 + s_2$ and reduces to the tautology $\beta(x, \alpha(x, s_1 + s_2)) = s_1 + s_2$ via the cocycle property of α.

Lemma 4.1 (averaging) *Suppose that (T^t) is a uniquely ergodic $\mathbb{R}^d$-action on a compact space X and that $\alpha : X \times \mathbb{R}^d \to \mathbb{R}^d$ is a continuous cocycle, then there exists a linear map $A : \mathbb{R}^d \to \mathbb{R}^d$ such that*

$$\lim_{s\to+\infty} \frac{1}{s}\alpha(x, sv) = Av \qquad (\forall v \in \mathbb{R}^d,\ x \in X). \tag{4.3}$$

The limit is uniform in the sense that

$$\lim_{s\to+\infty} \sup_{|v|=1, x\in X} \left|\frac{1}{s}\alpha(x, sv) - Av\right| = 0. \tag{4.4}$$

Remark 4.2 *If additionally α arises from a reparametrization (S^t) which is also uniquely ergodic then A must be a linear isomorphism.*

Proof of Remark 4.2: Because (S^t) is also uniquely ergodic, we can apply the lemma to the cocycle β in (4.1) to get the corresponding average $B : \mathbb{R}^d \to \mathbb{R}^d$ linear. Then taking any $v \in \mathbb{R}^d$, passage to the limit with $s \to \infty$ in $\frac{1}{s}\alpha(x, \beta(x, sv)) = v$ yields $ABv = v$. Likewise, $BAv = v$. Hence, $B = A^{-1}$. □

Before proving the lemma observe that, if $v = u_1 + \cdots + u_k$, then the cocycle property yields

$$\alpha(x, v) = \alpha(x, u_1) + \alpha(x + u_1, u_2) + \cdots + \alpha(x + u_1 + \cdots + u_{k-1}, u_k). \tag{4.5}$$

In particular, taking $k \in [|v|, |v| + 1) \cap \mathbb{N}$ and $u_i := v/k$, we conclude that

$$|\alpha(x, v)| \leq C_0|v| + C_0 \qquad (\forall x \in X, v \in \mathbb{R}^d) \tag{4.6}$$

where $C_0 := \sup_{x\in X,|v|\leq 1} |\alpha(x, v)|$. [7]

The formula (4.5) also suggests recovering $\alpha(x, v)$ from its "derivative" $\alpha'(x, v) := \lim_{\Delta\to 0} \frac{1}{\Delta}\alpha(x, \Delta v)$. If the limit exists α is branded a *smooth cocycle* and (4.5) yields $\alpha(x, v) = \int_0^1 \alpha'(x + tv, v)\, dt$. However, we only need a cruder fact, valid for all continuous α.

Fact 4.3 *Given a cocycle α and $\Delta > 0$, there is a constant $C > 0$ such that, for any unit vector $v \in \mathbb{R}^d$ and any $x \in X$, we have*

$$\left|\alpha(x, sv) - \int_0^s \frac{1}{\Delta}\alpha(x + tv, \Delta v)\, dt\right| \leq C. \tag{4.7}$$

[7] In fact, any cocycle is homologous to one that is Lipschitz (in the sense that $|\alpha(x, v)| \leq C_0|v|$). See [Kat03].

Proof of Fact 4.3: Fix α and $\Delta > 0$. In view of (4.6), we may well restrict attention to s that are integer multiples of Δ, i.e., $s = m\Delta$ for some $m \in \mathbb{N}$.[8]

The integral can be approximated by its mnth Riemann sum with a step $h := \frac{m\Delta}{mn}$, as given by

$$\begin{aligned}&\frac{\alpha(x,\Delta v)}{\Delta}h + \frac{\alpha(x+hv,\Delta v)}{\Delta}h + \cdots + \frac{\alpha(x+(mn-1)hv,\Delta v)}{\Delta}h\\ =&\frac{1}{n}\left[\alpha(x,\Delta v) + \alpha(x+hv,\Delta v)\cdots + \alpha(x+(mn-1)hv,\Delta v)\right].\end{aligned}$$

This sum can be reworked by vigorous application of the cocycle property to become

$$\frac{1}{n}\left(n\alpha(x,sv) - \sum_{j=1}^{n}\alpha(x,jhv) + \sum_{j=1}^{n}\alpha(x+sv,jhv)\right) = \alpha(x,sv) + \text{ Error}$$

where $|\text{ Error }| \leq \sup_{y\in X,\ |u|\leq\Delta}|\alpha(y,u)| < +\infty$ since $jh \leq \Delta$. The inequality (4.7) obtains by letting $n \to \infty$. $\square$

Proof of Lemma 4.1: First of all, upon fixing $v \in \mathbb{R}^d$, for a.e. x, Birkhoff's Ergodic Theorem applied to the $\mathbb{R}$-sub-action along v assures existence of the limit

$$A(x,v) := \lim_{s\to+\infty}\frac{1}{s}\alpha(x,sv) = \lim_{s\to+\infty}\frac{1}{s}\int_0^s \frac{1}{\Delta}\alpha(x+tv,\Delta v)dt. \tag{4.8}$$

Of course, this does not give us the independence on x and the uniformity of the convergence.

Claim 4.4 $\forall_{\epsilon>0}\ \exists_{s_0>0}\ \ s, s' > s_0 \implies |\frac{1}{s'}\alpha(x',s'v) - \frac{1}{s}\alpha(x,sv)| < \epsilon$ *for all* $|v| \leq 1$ *and* $x, x' \in X$.

The claim readily implies that $\frac{1}{s}\alpha(x,sv)$ converges to a limit $A(v)$ independent of x in a way that (4.4) holds. That $v \mapsto A(v)$ is linear can be then seen by passing to the limit in

$$\frac{1}{s}\alpha(x,-sv) = -\frac{1}{s}\alpha(x-sv,sv)$$

to get $A(-v) = -A(v)$, and in

$$\frac{1}{s}\alpha(x,s(a_1v_1+a_2v_2)) = a_1\frac{1}{sa_1}\alpha(x,sa_1v_1) + a_2\frac{1}{sa_2}\alpha(x+sa_1v_1,sa_2v_2)$$

to get $A(a_1v_1+a_2v_2) = a_1A(v_1) + a_2A(v_2)$ for $a_i \geq 0$ and $v_i \in \mathbb{R}^d$, $i = 1, 2$.

[8] Indeed, $\sup_{y\in X,\ |v|=1}|\frac{1}{\Delta}\alpha(y,\Delta v)| < +\infty$ and $|\alpha(x,sv) - \alpha(x,\lfloor s/\Delta\rfloor\Delta v)| = |\alpha(x - \lfloor s/\Delta\rfloor\Delta v, (s-\lfloor s/\Delta\rfloor\Delta)v)| \leq \sup_{y\in X,\ |t|\leq\Delta}\alpha(x,t)$.

It remains to prove the claim. Suppose that the claim fails and we have $\epsilon > 0$, unit vectors $v_k \in \mathbb{R}^d$, $x_k, x'_k \in X$, and $s_k, s'_k \to \infty$ such that

$$\left| \frac{1}{s'_k}\alpha(x'_k, s'_k v_k) - \frac{1}{s_k}\alpha(x_k, s_k v_k) \right| \geq \epsilon \quad (\forall k \in \mathbb{N}).$$

By passing to a subsequence and using (4.6), we can replace v_k by their accumulation point v and still have

$$\left| \frac{1}{s'_k}\alpha(x'_k, s'_k v) - \frac{1}{s_k}\alpha(x_k, s_k v) \right| \geq \epsilon - \frac{(C_0 s'_k |v - v_k| + C_0)}{s'_k} - \frac{(C_0 s_k |v - v_k| + C_0)}{s_k} > \epsilon/2 \tag{4.9}$$

for sufficiently large k. Furthermore, by using Fact 4.3 with $\Delta = 1$, we get

$$\left| \frac{1}{s'_k}\int_0^{s'_k} \alpha(x'_k + tv, v)dt - \frac{1}{s_k}\int_0^{s_k} \alpha(x_k + tv, v)dt \right| > \epsilon/4 \tag{4.10}$$

for sufficiently large k.

Now, let J_k be the segment from 0 to $s_k v$, $J_k := [0, s_k]v$, and let W_k be the $(d-1)$-disk of radius[9] $r_k := s_k^{1/2}$ centered at the origin and perpendicular to v. Thus $J_k + W_k$ is a slender cylinder in $\mathbb{R}^d$. Denote by δ_k the normalized volume measure carried by the orbit piece $x_k + (J_k + W_k)$, that is, for any continuous *test* function u on X, we set

$$\int u \, d\delta_k := \frac{1}{\mathrm{Vol}(J_k + W_k)} \int_{J_k + W_k} u(x_k + \tau) \, |d\tau|_d \tag{4.11}$$

where $|d\tau|_d$ indicates the Lebesgue integration in $\mathbb{R}^d$. A set of parallel definitions produces δ'_k; and we may assume (upon passing to subsequences) that δ_k and δ'_k weak* converge to some Borel probability measures μ and μ', respectively. By construction, μ is invariant under (T^t). Indeed, upon fixing $t \in \mathbb{R}^d$, the ratio of the volume of the symmetric difference of $(J_k + W_k) + t$ and $J_k + W_k$ to the volume of $J_k + W_k$ is of order $q_k := \frac{|t|}{s_k}$ for $d = 1$ and otherwise of order

$$q_k := \frac{|t| r_k^{d-1} + |t| r_k^{d-2} s_k}{r_k^{d-1} s_k} = |t| \left(\frac{1}{s_k} + \frac{1}{r_k} \right) \qquad (d \geq 2).$$

For any continuous $u : X \to \mathbb{R}$, we have then

$$\left| \int u \, d\delta_k - \int u \circ T^t d\delta_k \right| \leq q_k \sup_x |u(x)| \to 0,$$

yielding $\int u \, d\mu = \int u \circ T^t \, d\mu$. Likewise, μ' is also (T^t)-invariant. Hence, $\mu = \mu'$ by the unique ergodicity.

[9] All we need of r_k is that $r_k \to \infty$ and $r_k / s_k \to 0$.

To obtain a contradiction, we consider the averages of the function $x \mapsto \alpha(x, v)$.

$$\begin{aligned} I_k &:= \int \alpha(x,v)\,\delta_k(dx) \\ &= \frac{1}{\mathrm{Vol}(W_k+J_k)} \int_{W_k+J_k} \alpha(x_k+\tau,v)\,|d\tau|_d \\ &= \frac{1}{\mathrm{Vol}(W_k)} \int_{W_k} \frac{1}{s_k}\left[\int_0^{s_k} \alpha(x_k+\tau+tv,v)\,dt\right] |d\tau|_{d-1}. \end{aligned} \tag{4.12}$$

For $\tau \in W_k$, the cocycle property and (4.6) yield

$$|\alpha(x_k+tv+\tau,v)-\alpha(x_k+tv,v)| = |\alpha(x_k+tv,\tau)+\alpha(x_k+tv+\tau+v,-\tau)| \le 2(C_0 r_k+C_0)$$

so that we can drop τ in the last integral in (4.12) with an error

$$\left| I_k - \frac{1}{s_k}\int_0^{s_k} \alpha(x_k+tv,v)\,dt \right| \le \frac{2(C_0 r_k + C_0)}{s_k} \to 0.$$

As a consequence, (4.10) forces

$$|I_k - I'_k| > \epsilon/4 > 0$$

for sufficiently large k. At the same time, from $\lim \delta_k = \mu = \mu' = \lim \delta'_k$, we have

$$\lim_{k\to\infty} I_k = \int \alpha(y,v)\,\mu(dy) = \int \alpha(y,v)\,\mu'(dy) = \lim_{k\to\infty} I'_k,$$

which is a contradiction. The claim is shown. □

5 Ironing homeomorphisms to conjugacies: proof of Theorem 1.1

Let $T = (T^t : X \to X)_{t\in\mathbb{R}^d}$ be an abstract self-affine tiling action and $\Phi : X \to X$ be a homeomorphism as in (EXP), i.e., $\Phi\circ T^t = T^{\Lambda t}\circ\Phi$ with Λ expanding. We shall require that $\|\Lambda^{-1}\| < 1$, which can always be achieved by using the norm on $\mathbb{R}^d$ *adapted* to Λ ([KH95]) or, more crudely, by passing to an iterate Λ^N for sufficiently large $N > 0$. We assume that beside (X, T, Φ, Λ) we also have another quadruple $(\tilde{X}, \tilde{T}, \tilde{\Phi}, \tilde{\Lambda})$ with the same properties and that we are given a homeomorphism $h_0 : X \to \tilde{X}$ that maps the orbits of T to the orbits of $\tilde{T}$. (Such is the case for the h_0 in Theorem 1.1.) As discussed in the previous section, the formula

$$S^t := h_0^{-1}\circ\tilde{T}^t\circ h_0 \tag{5.1}$$

defines a reparametrization of (T^t) and there are continuous cocycles α over (T^t) and β over (S^t) such that (4.1) holds.

Let A_α be the linear isomorphism provided by Lemma 4.1. The technical role of the conformality assumption on Λ and $\tilde{\Lambda}$ in Theorem 1.1 is revealed by the following lemma.

Lemma 5.1 *For any linear isomorphism A_α, if each of Λ and $\tilde{\Lambda}$ is similar to a scalar multiple of an orthogonal transformation of $\mathbb{R}^d$, then there are $(m_k)_{k=1}^\infty, (n_k)_{k=1}^\infty \subset \mathbb{N}$ with $m_k, n_k \to \infty$ and a linear isomorphism A such that*

$$\|\tilde{\Lambda}^{-n_k}\Lambda^{m_k} - I\| \to 0 \tag{5.2}$$

$$\|\tilde{\Lambda}^{-n_k}A_\alpha\Lambda^{m_k} - A\| \to 0. \tag{5.3}$$

$$\sup_{k\in\mathbb{N}} \|\Lambda^{m_k}\|\|\tilde{\Lambda}^{-n_k}\| < +\infty \quad \textit{and} \quad \sup_{k\in\mathbb{N}} \|\Lambda^{-m_k}\|\|\tilde{\Lambda}^{n_k}\| < +\infty. \tag{5.4}$$

That also $\|\Lambda^{-m_k}\tilde{\Lambda}^{n_k} - I\| \to 0$ and $\|\Lambda^{-m_k}A\tilde{\Lambda}^{n_k} - A_\alpha\| \to 0$ is automatic from (5.2,5.3,5.4). The proofs of the lemma and the remark below are relegated to the end of the section.

Remark 5.2 *The reverse implication in Lemma 5.1 holds as well: (5.4) implies that both Λ and $\tilde{\Lambda}$ are similar to a scalar multiple of an orthogonal transformation.*

The following theorem is the main result of this section.

Theorem 5.3 (technical version of Theorem 1.1) *If (5.2,5.3,5.4) hold, then there is a homeomorphism $h : X \to \tilde{X}$ such that*

$$h \circ T^t = \tilde{T}^{At} \circ h \quad (\forall t \in \mathbb{R}^d). \tag{5.5}$$

In view of Lemma 5.1, Theorem 1.1 follows readily.

The idea of our argument for Theorem 5.3 harks back to the oldest linearization results of hyperbolic dynamics. The averaging (Section 4) tells us that h_0 is approximately linear on *large scale* and we attempt to bring this linearity to the *microscopic scale* by *renormalizing* h_0 with the aid the high iterates Φ^{m_k} and $\tilde{\Phi}^{-n_k}$, which transport us between the two scales in the domain and in the range, respectively. Precisely, we consider

$$h_k := \tilde{\Phi}^{-n_k} \circ h_0 \circ \Phi^{m_k} \quad (k \in \mathbb{N}). \tag{5.6}$$

With this picture in mind, the following lemma is to be expected; although, the equicontinuity in the transverse direction (5.8) is not a forgone conclusion.

Lemma 5.4 *The family of homeomorphisms $(h_k)_{k\in\mathbb{N}}$ is equicontinuous.*

Proof: First we claim that the h_k are equicontinuous along the leaves (i.e., $\forall_{\epsilon>0}\,\exists_{\delta>0}\,|t| < \delta \implies \forall_k\, \mathsf{d}(h_k(x+t), h_k(x)) < \epsilon$). To see this, we use $h_0 \circ S^t = \tilde{T}^t \circ h_0$ and the definition of α to write

$$\begin{aligned} h_k(x+t) &= \tilde{\Phi}^{-n_k} \circ h_0(\Phi^{m_k}x + \Lambda^{m_k}t) \\ &= \tilde{\Phi}^{-n_k}(h_0(\Phi^{m_k}x) + \alpha(\Phi^{m_k}x, \Lambda^{m_k}t)) \\ &= \tilde{\Phi}^{-n_k} \circ h_0 \circ \Phi^{m_k}(x) + \tilde{\Lambda}^{-n_k}\alpha(\Phi^{m_k}x, \Lambda^{m_k}t) \\ &= h_k(x) + \tilde{\Lambda}^{-n_k}\alpha(\Phi^{m_k}x, \Lambda^{m_k}t) \end{aligned} \tag{5.7}$$

and exploit (4.6) to estimate:

$$|\tilde{\Lambda}^{-n_k}\alpha(\Phi^{m_k}x,\Lambda^{m_k}t)| \leq C_0\|\tilde{\Lambda}^{-n_k}\|\|\Lambda^{m_k}\||t| + \|\tilde{\Lambda}^{-n_k}\|C_0 \leq C_0'|t| + \|\tilde{\Lambda}^{-n_k}\|C_0$$

where the constant $C_0' > 0$ is secured via (5.4). Since $\|\tilde{\Lambda}^{-n_k}\|C_0 \to 0$, our claim follows readily.

Courtesy of the local product structure (see Section 2), it remains to show that the h_k are equicontinuous on the transversals given by the stable sets of Φ. To be specific, fix $\Delta \in (0,\Delta_0)$. Recalling the basis of open sets supplied by (2.2), it suffices to show that, given $\epsilon \in (0,\Delta)$, there is and $l > 0$ and a ball D centered at 0, such that

$$x|^{(\Delta)}_{\Lambda^l D} \overset{s}{=} y|^{(\Delta)}_{\Lambda^l D} \implies \mathsf{d}(h_k(x), h_k(y)) < \epsilon \tag{5.8}$$

for all $k \in \mathbb{N}$ that are sufficiently large (independently of $x, y \in X$).

We begin with selection of suitable constants needed in the argument. First, we use that Φ^{-1} is continuous (by Theorem 8.1) to secure $C'' > 0$ so that, for any sets $U, V \subset \mathbb{R}^d$, we have

$$x|^{(\Delta)}_U = y|^{(\Delta)}_U \implies \tilde{\Phi}^{-1}x|^{(\Delta)}_V = \tilde{\Phi}^{-1}y|^{(\Delta)}_V \qquad \text{provided } V \subset \tilde{\Lambda}^{-1}(B_{-C''}(U)). \tag{5.9}$$

Then we take $C' > 0$ so that

$$B_{C'+C''}(\tilde{\Lambda}t) \subset \tilde{\Lambda}B_{C'}(t) \quad (\forall t \in \mathbb{R}^d). \tag{5.10}$$

as made possible by $\|\tilde{\Lambda}^{-1}\| < 1$. This choice of C' secures

$$\tilde{\Lambda}^{-1}B_{-C'-C''}(\tilde{\Lambda}^{-m+1}U) \supset B_{-C'}(\tilde{\Lambda}^{-m}U) \quad (\forall m \in \mathbb{N},\ U \subset \mathbb{R}^d). \tag{5.11}$$

(Indeed, if $t \in B_{-C'}(\tilde{\Lambda}^{-m}U)$ so that $B_{C'}(t) \subset \tilde{\Lambda}^{-m}U$, then $B_{C'+C''}(\tilde{\Lambda}t) \subset \tilde{\Lambda}B_{C'}(t) \subset \tilde{\Lambda}^{-m+1}U$, i.e., $\tilde{\Lambda}t \in B_{-C'-C''}(\tilde{\Lambda}^{-m+1}U)$; and (5.11) follows.)

All this is to facilitate the induction on $m > 0$ and to forge (5.9) into

$$x|^{(\Delta)}_U = y|^{(\Delta)}_U \implies \tilde{\Phi}^{-m}x|^{(\Delta)}_V = \tilde{\Phi}^{-m}y|^{(\Delta)}_V \qquad \text{provided } V \subset B_{-C'}(\tilde{\Lambda}^{-m}U). \tag{5.12}$$

To verify the induction step, suppose $\tilde{\Phi}^{-m+1}x|^{(\Delta)}_{V_{m-1}} = \tilde{\Phi}^{-m+1}y|^{(\Delta)}_{V_{m-1}}$ where $V_{m-1} := B_{-C'}(\tilde{\Lambda}^{-m+1}U)$. Then $\tilde{\Phi}^{-m}x|^{(\Delta)}_{V_m'} = \tilde{\Phi}^{-m}y|^{(\Delta)}_{V_m'}$ where $V_m' = \tilde{\Lambda}^{-1}B_{-C''-C'}(\tilde{\Lambda}^{-m+1}U) \supset B_{-C'}(\tilde{\Lambda}^{-m}U) = V_m$, as desired.

By the definition of β, $\tilde{T}^{\tilde{t}} \circ h_0(x) = h_0 \circ T^{\beta(x,\tilde{t})}(x)$ ($\forall x \in X$, $\tilde{t} \in \mathbb{R}^d$), so taking $C_1 > 1$ to be the constant in the version of (4.6) applied to β gives

$$h_0(x) + \tilde{t} = h_0(x + \beta(x,\tilde{t})), \qquad |\beta(x,\tilde{t})| \leq C_1|\tilde{t}| + C_1. \tag{5.13}$$

By taking $D = B_\rho$ where ρ is large enough, say $\rho > C'C_1 + 1$, we can assure that the set

$$W' := B_{-C'}(\tilde{\Lambda}^{-n_k}B_{-1}(C_1^{-1}\Lambda^{m_k}D)) \tag{5.14}$$

is non-empty (and thus contains 0) for all but finitely many k. Indeed, for large m_k, the expanding of Λ assures $B_{-C_1}(\Lambda^{m_k}D) \supset \Lambda^{m_k}B_{-1}(D)$ and so

$$\begin{aligned} B_{-C'}(\tilde{\Lambda}^{-n_k}B_{-1}(C_1^{-1}\Lambda^{m_k}D)) &= C_1^{-1}B_{-C'C_1}(\tilde{\Lambda}^{-n_k}B_{-C_1}(\Lambda^{m_k}D)) \\ &\supset C_1^{-1}B_{-C'C_1}(\tilde{\Lambda}^{-n_k}\circ\Lambda^{m_k}B_{-1}(D)) \to C_1^{-1}B_{-C'C_1}(B_{-1}(D)) \end{aligned}$$

where we used the hypothesis (5.2) about $\tilde{\Lambda}^{-n_k}\circ\Lambda^{m_k}$ converging to I.

Finally, the continuity of h_0 lets us fix $l_1 > 0$ such that

$$x'|^{(\Delta)}_{\Lambda^{l_1}D} = y'|^{(\Delta)}_{\Lambda^{l_1}D} \implies \mathsf{d}(h_0(x'), h_0(y')) < \epsilon \quad (\forall\, x', y' \in X).$$

Passing to the main argument for (5.8), suppose that

$$x|^{(\Delta)}_{\Lambda^{l}D} \stackrel{s}{=} y|^{(\Delta)}_{\Lambda^{l}D}$$

where $l > l_1$ is also large enough that $D + \Lambda^j D \subset \Lambda^{l+j}D$ for all $j \geq 0$. Then $\Phi^{m_k}x|^{(\Delta)}_{\Lambda^{m_k+l}D} \stackrel{s}{=} \Phi^{m_k}y|^{(\Delta)}_{\Lambda^{m_k+l}D}$ so that, as soon as $m_k \geq l$, we have

$$\Phi^{m_k}x + t|^{(\Delta)}_{\Lambda^{l}D} = \Phi^{m_k}y + t|^{(\Delta)}_{\Lambda^{l}D} \qquad \text{for all } t \in \Lambda^{m_k}D$$

(because $\Lambda^l D + \Lambda^{m_k}D \subset \Lambda^{m_k+l}D$). Thus, by the choice of l_1,

$$\mathsf{d}(h_0(\Phi^{m_k}x + t), h_0(\Phi^{m_k}y + t)) < \epsilon \qquad \text{for all } t \in \Lambda^{m_k}D.$$

From (5.13),

$$\mathsf{d}(h_0(\Phi^{m_k}x) + \tilde{t}, h_0(\Phi^{m_k}y) + \tilde{t}) < \epsilon \qquad \text{for all } \tilde{t} \in C_1^{-1}B_{-C_1}(\Lambda^{m_k}D)\ ,$$

which is to say that

$$h_0\circ\Phi^{m_k}(x)|^{(\epsilon)}_{W} = h_0\circ\Phi^{m_k}(y)|^{(\epsilon)}_{W} \quad \text{for } W := B_{-1}(C_1^{-1}\Lambda^{m_k}D).$$

Applying $\tilde{\Phi}^{-n_k}$ to both sides and using (5.12) and (5.14) results in

$$\tilde{\Phi}^{-n_k}\circ h_0\circ\Phi^{m_k}(x)|^{(\epsilon)}_{W'} = \tilde{\Phi}^{-n_k}\circ h_0\circ\Phi^{m_k}(y)|^{(\epsilon)}_{W'}.$$

Thanks to our stipulation that $0 \in W'$ for all but finitely many k, we get

$$\mathsf{d}(\tilde{\Phi}^{-n_k}\circ h_0\circ\Phi^{m_k}(x), \tilde{\Phi}^{-n_k}\circ h_0\circ\Phi^{m_k}(y)) \leq \epsilon,$$

which shows (5.8). □

Proof of Theorem 5.3: The lemma can be also applied with the roles of Φ and $\tilde{\Phi}$ interchanged to see that $h_k^{-1} = \Phi^{-m_k}\circ h_0^{-1}\circ\tilde{\Phi}^{n_k}$ form an equicontinuous family. By passing to a subsequence, we can secure that both h_k and h_k^{-1} converge uniformly. Thus they must converge to h and h^{-1}, respectively, where the limit h

is a homeomorphism. To finish, we have to show that $h \circ T^t \circ h^{-1} = \tilde{T}^{At}$. From (5.1) and (4.1), we have

$$h_0 \circ T^t \circ h_0^{-1}(x) = \tilde{T}^{\alpha(h_0^{-1}(x),t)} x \qquad (\forall x \in \tilde{X}, t \in \mathbb{R}^d) \tag{5.15}$$

and we expect that

$$h_k \circ T^t \circ h_k^{-1}(x) = \tilde{T}^{\alpha_k(h_k^{-1}(x),t)} x \qquad (\forall x \in \tilde{X}, t \in \mathbb{R}^d)$$

where α_k are cocycles that get "increasingly linear" with $k \to \infty$ so that $\alpha_k(h_k^{-1}(x), t) \to At$.

To see this, for any $t \in \mathbb{R}^d$ and $x \in \tilde{X}$, we first write (via (5.15))

$$\begin{aligned}
h_k \circ T^t \circ h_k^{-1}(x) = \;& \tilde{\Phi}^{-n_k} \circ h_0 \circ \Phi^{m_k} \circ T^t \circ \Phi^{-m_k} \circ h_0^{-1} \circ \tilde{\Phi}^{n_k}(x) \\
= \;& \tilde{\Phi}^{-n_k} \circ h_0 \circ T^{\Lambda^{m_k} t} \circ h_0^{-1} \circ \tilde{\Phi}^{n_k}(x) \\
= \;& \tilde{\Phi}^{-n_k} \circ \tilde{T}^{\alpha(h_0^{-1} \circ \tilde{\Phi}^{n_k}(x), \Lambda^{m_k} t)} \circ \tilde{\Phi}^{n_k}(x) \\
= \;& \tilde{T}^{\tilde{\Lambda}^{-n_k} \alpha(h_0^{-1} \circ \tilde{\Phi}^{n_k}(x), \Lambda^{m_k} t)}(x).
\end{aligned} \tag{5.16}$$

Hence, using $h_0^{-1} \circ \tilde{\Phi}^{n_k} \circ h_k = \Phi^{m_k}$, we identify the renormalized cocycle as given by

$$\alpha_k(z, t) = \tilde{\Lambda}^{-n_k} \alpha(\Phi^{m_k} z, \Lambda^{m_k} t) \qquad (\forall z \in X, t \in \mathbb{R}^d).$$

Therefore, it is left to take the limit

$$\begin{aligned}
\tilde{\Lambda}^{-n_k} \alpha(\Phi^{m_k} x, \Lambda^{m_k} t) = \;& \tilde{\Lambda}^{-n_k} |\Lambda^{m_k} t| \frac{1}{|\Lambda^{m_k} t|} \alpha(\Phi^{m_k} x, \Lambda^{m_k} t) \\
= \;& \tilde{\Lambda}^{-n_k} |\Lambda^{m_k} t| \left(A_\alpha \frac{\Lambda^{m_k} t}{|\Lambda^{m_k} t|} + \text{Error} \right) \\
= \;& \tilde{\Lambda}^{-n_k} A_\alpha \Lambda^{m_k} t + |\Lambda^{m_k} t| \tilde{\Lambda}^{-n_k} \text{Error} \\
\to \;& At + 0
\end{aligned} \tag{5.17}$$

where A_α is supplied by Lemma 4.1 so that $|\text{Error}| \to 0$ as $k \to \infty$, and we used the hypothesis (5.3) and (5.4) to take the limit of $\tilde{\Lambda}^{-n_k} A_\alpha \Lambda^{m_k} t$ and to get $\sup_k |\Lambda^{m_k} t| \|\tilde{\Lambda}^{-n_k}\| < \infty$ as needed to estimate

$$\left| |\Lambda^{m_k} t| \tilde{\Lambda}^{-n_k} \text{Error} \right| \leq |\Lambda^{m_k} t| \|\tilde{\Lambda}^{-n_k}\| |\text{Error}| \leq \text{Const} |\text{Error}|.$$

□

Remark 5.5 *Later we shall need a more exact error bound in (5.17): for any* $\epsilon > 0$, *there is* $K > 0$ *so that* $k > K$ *implies*

$$|\tilde{\Lambda}^{-n_k} \alpha(y, \Lambda^{m_k} t) - At| \leq \epsilon + \epsilon t \quad (\forall y \in X, \; t \in \mathbb{R}^d). \tag{5.18}$$

Proof of Remark 5.5: From Lemma 4.1, there is s_0 such that, as soon as $s := |\Lambda^{m_k} t| > s_0$, we have

$$\frac{1}{|\Lambda^{m_k} t|} |\alpha(y, \Lambda^{m_k} t) - A_\alpha \Lambda^{m_k} t| < \frac{\epsilon/2}{\sup_k \|\tilde{\Lambda}^{-n_k}\| \|\Lambda^{m_k}\|} \quad (\forall y \in X). \tag{5.19}$$

Via (5.4), this yields

$$\left|\tilde{\Lambda}^{-n_k} \alpha(y, \Lambda^{m_k} t) - \tilde{\Lambda}^{-n_k} A_\alpha \Lambda^{m_k} t\right| < \|\tilde{\Lambda}^{-n_k}\| \frac{\epsilon/2}{\|\tilde{\Lambda}^{-n_k}\| \|\Lambda^{m_k}\|} |\Lambda^{m_k} t| \leq \frac{\epsilon}{2} |t|, \tag{5.20}$$

which gives (5.18) conditioned on $|\Lambda^{m_k} t| > s_0$, as soon as k large enough so that $|\tilde{\Lambda}^{-n_k} A_\alpha \Lambda^{m_k} t - At| \leq \epsilon/2|t|$ (as assured by (5.3)).

When $|\Lambda^{m_k} t| \leq s_0$, we simply use (4.6) and the expanding of Λ and $\tilde{\Lambda}$ to obtain, again for sufficiently large k,

$$\left|\tilde{\Lambda}^{-n_k} \alpha(y, \Lambda^{m_k} t)\right| \leq \|\tilde{\Lambda}^{-n_k}\| (C_0 |\Lambda^{m_k} t| + C_0) \leq \|\tilde{\Lambda}^{-n_k}\| (C_0 s_0 + C_0) < \epsilon/2. \tag{5.21}$$

To arrive at (5.18), note that our hypothesis $|\Lambda^{m_k} t| \leq s_0$ forces $|At| < \epsilon/2$ for large k (since $|t| \to 0$ with $k \to \infty$). $\square$

Proof of Lemma 5.1: We have $\Lambda = \lambda C Q C^{-1}$ and $\tilde{\Lambda} = \tilde{\lambda} \tilde{C} \tilde{Q} \tilde{C}^{-1}$ where $\lambda, \tilde{\lambda} > 0$ and the transformations $C, \tilde{C}$ are nonsingular and $Q, \tilde{Q}$ are orthogonal. Start with picking $m_k, n_k \to \infty$ such that $\lim_{k\to\infty} \tilde{\lambda}^{-n_k} \lambda^{m_k} = 1$. Then (5.4) holds easily. Upon passing to a subsequence, we can assure that $m_k^{(new)} := m_{k+1} - m_k$ and $n_k^{(new)} := n_{k+1} - n_k$ converge to infinity and, via compactness of the orthogonal group,

$$\tilde{\Lambda}^{-n_k} \Lambda^{m_k} = \tilde{\lambda}^{-n_k} \lambda^{m_k} \tilde{C} \tilde{Q}^{-n_k} \tilde{C}^{-1} C Q^{m_k} C^{-1}$$

converge in norm as $k \to \infty$. Thus the right hand side in

$$\|\tilde{\Lambda}^{-(n_{k+1} - n_k)} \Lambda^{m_{k+1} - m_k} - I\| \leq \|\tilde{\Lambda}^{-n_k} \Lambda^{m_k}\| \|\tilde{\Lambda}^{-n_{k+1}} \Lambda^{m_{k+1}} - \tilde{\Lambda}^{-n_k} \Lambda^{m_k}\|$$

converges to 0. That is (5.2) holds for $m_k^{(new)}$ and $n_k^{(new)}$. Also, having held for m_k and n_k, (5.4) holds for $m_k^{(new)}$ and $n_k^{(new)}$ as well. Securing (5.3) is done by invoking compactness of the orthogonal group and passing to a subsequence if necessary. $\square$

Proof of Remark 5.2: Let λ_+ and λ_- be the largest and the smallest modulus of an eigenvalue of Λ, and $\tilde{\lambda}_+$ and $\tilde{\lambda}_-$ be the analogues for $\tilde{\Lambda}$. From (5.4), the sequences $\lambda_+^{m_k} \tilde{\lambda}_-^{-n_k}$ and $\lambda_-^{-m_k} \tilde{\lambda}_-^{n_k}$ are bounded, which is only possible when $\lambda_+ = \lambda_-$ and $\tilde{\lambda}_+ = \tilde{\lambda}_-$ because $\lambda_- \leq \lambda_+$ and $\tilde{\lambda}_- \leq \tilde{\lambda}_+$. This shows that $\Lambda = \lambda_+ U$ and $\tilde{\Lambda} = \tilde{\lambda}_+ \tilde{U}$ where all the eigenvalues of U and $\tilde{U}$ are of modulus one. Because $\|\Lambda^{m_k}\| \|\tilde{\Lambda}^{-n_k}\| = \lambda^{m_k} \tilde{\lambda}^{-n_k} \|U^{m_k}\| \|\tilde{U}^{-n_k}\|$ and $\|\Lambda^{-m_k}\| \|\tilde{\Lambda}^{n_k}\| = \lambda^{-m_k} \tilde{\lambda}^{n_k} \|U^{-m_k}\| \|\tilde{U}^{n_k}\|$ are bounded while $\|U^{\pm m_k}\|, \|\tilde{U}^{\pm n_k}\| \geq 1$, it must be that $\lambda^{-m_k} \tilde{\lambda}^{n_k}$, $\lambda^{m_k} \tilde{\lambda}^{-n_k}$, $\|U^{\pm m_k}\|$, $\|\tilde{U}^{\pm n_k}\|$ are also bounded (uniformly in k). By inspecting the Jordan form of U^{m_k}, one concludes that for $\|U^{\pm m_k}\|$ to be bounded, U has to be diagonalizable, and thus similar to an orthogonal transformation. The same goes for $\tilde{U}$. $\square$

6 Discreteness and Homotopy: proof of Theorems 1.2 and 1.4

The main result of this section is Lemma 6.1 below. It easily encompases Theorem 1.2. Togeather with Lemma 6.2 and the elements of proof of Theorem 5.3, it gives Theorem 1.4.

Lemma 6.1 (discreteness) *Suppose that T is a minimal action on a compact space X and $(LF)_{r_0}$ holds for some $r_0 > 0$. Let $\Psi, \Psi' \in \mathcal{H}(X, x_0)$ and $\Theta, \Theta' : \mathbb{R}^d \to \mathbb{R}^d$ be such that*

$$\Psi(x_0 + t) = x_0 + \Theta(t) \quad \text{and} \quad \Psi'(x_0 + t) = x_0 + \Theta'(t) \qquad (\forall t \in \mathbb{R}^d).$$

Suppose that Θ is linear and $\eta_0(t) := \Theta'(t) - \Theta t$ satisfies

$$|\eta_0(t)| \le L|t| + L \quad (\forall t \in \mathbb{R}^d). \tag{6.1}$$

There is $L_0 > 0$ (that depends on Ψ only) such that $L < L_0$ implies that η_0 is bounded. Moreover, $\Psi = \Psi'$ if Θ' is also linear.

To parley the boundedness of the η_0 (above) into homotopy of Ψ and Ψ', we record the following useful result.

Lemma 6.2 (homotopy) *Suppose that T is a minimal aperiodic action on a compact space X. Let $\Psi, \Psi' : X \to X$ be continuous and both fix x_0. If there is a bounded function $\eta_0 : \mathbb{R}^d \to \mathbb{R}^d$ such that*

$$\Psi(x_0 + t) = \Psi'(x_0 + t) + \eta_0(t) \quad (\forall t \in \mathbb{R}^d) \tag{6.2}$$

then there is a continuous $\eta : X \to \mathbb{R}^d$ such that

$$\Psi(x) = \Psi'(x) + \eta(x) \quad (\forall x \in X). \tag{6.3}$$

Moreover, Ψ and Ψ' are homotopic along the orbits of the $\mathbb{R}^d$-action.

Proof of Lemma 6.2: Fix $x \in X$. Take t_n so that $\lim_{n\to\infty} x_0 + t_n = x$. Let v be any accumulation point of $\eta_0(t_n)$. By continuity of Ψ and Ψ', we have $\Psi(x) = \Psi'(x) + v$. Note that if there were another v' so that $\Psi(x) = \Psi'(x) + v'$ then $\Psi(x) = \Psi(x) + v - v'$ violating aperiodicity of the action. Thus v is uniquely determined by x and we can set $\eta(x) := v$. To see that $x \mapsto \eta(x)$ is continuous it suffices to note that, if $x_n \to x$ and $\eta(x_n) \to v'$, then $\Psi(x_n) = \Psi'(x_n) + \eta(x_n)$ yields $\Psi(x) = \Psi'(x) + v'$, and so $v' = \eta(x)$ by the uniqueness. Finally, the formula $H(x, t) = \Psi'(x) + t \cdot \eta(x)$ gives the desired homotopy $H : X \times [0, 1] \to X$. □

To prepare for the proof of Lemma 6.1, recall that, for $r < R$, a set $D \subset \mathbb{R}^d$ is r-***separated*** iff $\inf\{|t-s| : t \ne s,\ t, s \in D\} > r$ and R-***dense*** iff $\inf\{|t-s| : s \in D\} < R$ for all $t \in \mathbb{R}^d$. Sets D with both attributes are called ***Delone sets***. The following lemma isolates the basic rigidity mechanism rooted in the simple observation that the identity $\mathrm{Id} : D \to D$ on a Delone set has no small Lipschitz perturbations (other than itself).

Lemma 6.3 (lipschitz to bounded) *Suppose $r, R, L > 0$ and $D, D' \subset \mathbb{R}^d$ are two sets that are r-separated and R-dense and both contain 0. Let $\Theta, \Theta' : \mathbb{R}^d \to \mathbb{R}^d$ be such that $\Theta(D) \subset B_{r/4}(D')$ and $\Theta'(D) \subset B_{r/4}(D')$ and Θ is linear while Θ' is of the form*

$$\Theta'(v) = \Theta v + \eta_0(v) \quad (\forall v \in \mathbb{R}^d) \tag{6.4}$$

with

$$|\eta_0(u) - \eta_0(v)| \leq L|u - v| + L \quad (\forall u, v \in \mathbb{R}^d). \tag{6.5}$$

If

$$L < \frac{r/2}{2R + 1}, \tag{6.6}$$

then η_0 is bounded and

$$\sup_{v \in \mathbb{R}^d} |\eta_0(v)| < r. \tag{6.7}$$

Proof of Lemma 6.3: Order the points of D and D' into sequences so that $D = \{v_0, v_1, v_2, \ldots\}$ and $D' = \{v'_0, v'_1, v'_2, \ldots\}$ with $v_0 = v'_0 = 0$. It suffices to show that

$$|v'_j - \Theta v_i| < r/4 \;\Leftrightarrow\; |v'_j - \Theta'(v_i)| < r/4 \quad (\forall i, j \geq 0). \tag{6.8}$$

Indeed, then the lemma follows since, given $v \in \mathbb{R}^d$, there is i with $|v_i - v| < R$ and we can estimate

$$\begin{aligned} |\eta_0(v)| &= |(\Theta'(v) - \Theta v) - (\Theta'(v_i) - \Theta v_i)| + |\Theta'(v_i) - \Theta v_i| \\ &\leq |\eta_0(v) - \eta_0(v_i)| + |\Theta'(v_i) - v'_j| + |v'_j - \Theta'(v_i)| \\ &\leq LR + L + \frac{r}{4} + \frac{r}{4} < r. \end{aligned} \tag{6.9}$$

To finish, we use induction on n to prove (6.8) for i restricted to the set

$$I_n := \{i : |v_i| \leq nR\} \quad (n \geq 0).$$

For $n = 0$, $i = 0$ is the only element of I_0, and (6.8) is clear. Suppose then that (6.8) holds for all $i \in I_n$, $j \geq 0$. Consider $i \in I_{n+1}$. Use the R-density to pick $k \in I_n$ so that $|v_k - v_i| < 2R$. From $\Theta(D) \subset B_{r/4}(D')$ and the induction hypothesis, there is $l \in \mathbb{N}$ such that $|\Theta v_k - v'_l| < r/4$ and $|\Theta'(v_k) - v'_l| < r/4$. We estimate

$$\begin{aligned} |\Theta'(v_i) - \Theta v_i| &\leq |\Theta'(v_k) - \Theta v_k| + |\Theta'(v_i) - \Theta v_i - \Theta'(v_k) + \Theta v_k| \\ &\leq r/4 + r/4 + |\eta_0(v_i) - \eta_0(v_k)| \\ &\leq r/2 + (L2R + L) \\ &< r. \end{aligned} \tag{6.10}$$

D' being r-separated, the last inequality guarantees that, if $|\Theta'(v_i) - v'_{j_1}| < r/4$ and $|\Theta v_i - v'_{j_2}| < r/4$ for some $j_1, j_2 \geq 0$, then $j_1 = j_2$. This shows that (6.8) holds for all $i \in I_{n+1}$ and finishes the induction step. □

Proof of Lemma 6.1: By the minimality of T, given an $\epsilon_0 > 0$, there is $R_0(\epsilon_0) > 0$ such that the set

$$\mathcal{R}_{\epsilon_0} := \{t \in \mathbb{R}^d : \ \mathsf{d}(x_0 + t, x_0) < \epsilon_0\} \tag{6.11}$$

is $R_0(\epsilon_0)$-dense in $\mathbb{R}^d$. By $(LF)_{r_0}$, for sufficiently small $\epsilon_0 > 0$, we have a dichotomy

$$t, s \in \mathcal{R}_{\epsilon_0} \implies (\, |t - s| < r_0/4 \text{ or } |t - s| > r_0\,)\,. \tag{6.12}$$

(Otherwise, there are $r_0/4 \leq |t_n - s_n| \leq r_0$ with $\mathsf{d}(x_0 + t_n, x_0 + s_n) \leq \mathsf{d}(x_0 + t_n, x_0) + \mathsf{d}(x_0, x_0 + s_n) \to 0$ forcing violation of $(LF)_{r_0}$ in the limit.) Given the two homeomorphisms $\Psi, \Psi' \in \mathcal{H}(X, x_0)$, we can pick $\epsilon_1 \in (0, \epsilon_0)$ so that

$$\mathsf{d}(x_0, x_0 + t) < \epsilon_1 \implies (\,\mathsf{d}(x_0, \Psi(x_0 + t)) < \epsilon_0 \text{ and } \mathsf{d}(x_0, \Psi'(x_0 + t)) < \epsilon_0\,)\,, \tag{6.13}$$

which secures

$$\Theta(\mathcal{R}_{\epsilon_1}) \subset \mathcal{R}_{\epsilon_0} \text{ and } \Theta'(\mathcal{R}_{\epsilon_1}) \subset \mathcal{R}_{\epsilon_0}. \tag{6.14}$$

Observe that the relation $\sim$ given by $t \sim s$ iff $|t - s| < r_0/4$ is an equivalence relation on both $\mathcal{R}_{\epsilon_1}$ and $\mathcal{R}_{\epsilon_0}$. Take $D \subset \mathcal{R}_{\epsilon_1}$ to contain a single point from each equivalence class of $\sim$, and take an analogous $D' \subset \mathcal{R}_{\epsilon_0}$. Thus $\mathcal{R}_{\epsilon_1} \subset B_{r_0/4}(D)$, $\mathcal{R}_{\epsilon_0} \subset B_{r_0/4}(D')$, D and D' are $R_0 + r_0/4$-dense and r_0-separated, and (6.14) implies

$$\Theta(D) \subset B_{r_0/4}(D') \text{ and } \Theta'(D) \subset B_{r_0/4}(D'). \tag{6.15}$$

We can now invoke Lemma 6.3 to conclude that, for sufficiently small $L_0 > 0$, $\eta_0(t)$ is a bounded function of $t \in \mathbb{R}^d$. This proves the main assertion of the lemma.

If additionally Θ' is linear, then $\Theta = \Theta'$ because $\sup_{t \in \mathbb{R}^d} |(\Theta - \Theta')t| = \sup_{t \in \mathbb{R}^d} |\eta_0(t)| < +\infty$. From $\Theta = \Theta'$, $\Psi = \Psi'$ follows by density of the orbit $\{x_0 + t : \ t \in \mathbb{R}^d\}$ in X. □

We are ready to prove the two theorems stated in the introduction.

Proof of Theorem 1.2: To show that $\mathbb{G}l(x_0)$ is discrete we have to show that if $A \in \mathbb{G}l(x_0)$ is close to the identity $I \in \mathbb{G}l(x_0)$ then $A = I$. However, once $\|A - I\| < L_0$ this is so by Lemma 6.1 (applied with $\Theta = I$ and $\Theta' = A$). □

Proof of Theorem 1.4: Taking h_k and h as in the proof of Theorem 5.3, we have $h^{-1} \circ h_k(x_0) = x_0$ so that $h^{-1} \circ h_k \in \mathcal{H}(X, x_0)$. By combining (5.7) and (5.5), we get

$$h^{-1} \circ h_k(x_0 + t) = x_0 + A^{-1}\tilde{\Lambda}^{-n_k}\alpha(x_0, \Lambda^{m_k} t) = x_0 + t + \eta_k(t) \quad (\forall t \in \mathbb{R}^d)$$

where

$$\eta_k(t) := A^{-1}\tilde{\Lambda}^{-n_k}\alpha(x_0, \Theta^{m_k} t) - t \quad (\forall t \in \mathbb{R}^d).$$

By Remark 5.5, given arbitrarily small $\epsilon > 0$, as soon as k is large enough, we have

$$|\eta_k(t)| \leq \epsilon |t| + \epsilon \quad (\forall t \in \mathbb{R}^d).$$

Taking $\epsilon > 0$ sufficiently small, Lemmas 6.1 and 6.2 combine to guarantee that η_k is bounded and $h^{-1} \circ h_k$ is homotopic to the identity for k large enough. Upon fixing one such k, we see that h_k is homotopic to h and so $h_0 = \tilde{\Phi}^{n_k} \circ h_k \circ \Phi^{-m_k}$ is homotopic to $h_{lin} := \tilde{\Phi}^{n_k} \circ h \circ \Phi^{-m_k}$ (which is clearly linear). □

Proof of Corollary 1.5: The homeomorphism $h^{-1} \circ \tilde{\Phi} \circ h$ places $A^{-1}\tilde{\Lambda}A$ in $\mathbb{G}l(x_0)$. Thus, for any $m, n \in \mathbb{N}$,

$$\Lambda^m (A^{-1}\tilde{\Lambda}A)^{-n} = \Lambda^m A^{-1} \tilde{\Lambda}^{-n} A \in \mathbb{G}l(x_0). \tag{6.16}$$

By Lemma 5.1, we can select m, n so that $\|\Lambda^m A^{-1}\tilde{\Lambda}^{-n}A - I\|$ is small enough that $\Lambda^m A^{-1}\tilde{\Lambda}^{-n}A = I$, as forced by the discreteness of $\mathbb{G}l(x_0)$. Hence $\Lambda^m = A^{-1}\tilde{\Lambda}^n A$, which yields $\Phi^m = h^{-1} \circ \tilde{\Phi}^n \circ h$. □

* * *

7 Appendix I: Verification of Axioms for Concrete Tiling Spaces

For completeness, we verify that the translation action on the tiling space of a concrete tiling (as defined in the introduction) is indeed an example of an abstract tiling action. Below, we use $x|_{B_R}$ to denote the central R-patch of the tiling x, i.e., the union of all the tiles of x contained in the ball B_R.

Proposition 7.1 *Suppose that $(T^t : X \to X)_{t \in \mathbb{R}^d}$ is the translation action on the tiling space of a tiling x_0 that is repetitive and has finite local complexity. Then the axioms (LF), (WE), (PS) are satisfied for some positive constants r_0, l_0, R_0, Δ_0 with $l_0 < r_0$.*

We add that verification of (EXP) when the tiling x_0 is generated by an inflation-substitution procedure is a matter of applying the procedure to an arbitrary tiling $x \in X$ and thus constructing the *inflation-substitution map* $\Phi : X \to X$. This can be found for instance in [Sol98] (where our Φ is denoted by ω, see Definition on page 268 therein).

Proof: (LF): Take $r_0 > 0$ small enough that any prototile contains a ball of radius r_0 in its interior. Let σ be the tile of x that contains $0 \in \mathbb{R}^d$. If $x = x + t$ then both σ and $\sigma - t$ are tiles of x so they either coincide or have disjoint interiors. The first possibility is precluded by boundedness of σ and the second cannot happen for $|t| < r_0$ by the choice of r_0.

(WE): Looking at the above argument, we see that there is large $R > 0$ and small $\epsilon \in (0, r_0/2)$ such that

$$(t \in B_\epsilon \text{ and } x|_{B_R} = (x+t)|_{B_R}) \implies t = 0 \quad (\forall x \in X).$$

Pick Δ_0 so that

$$\mathsf{d}(x,y) \le \Delta_0 \implies \exists!_{s \in B_{\epsilon/4}}\ (x+s)|_{B_{2R}} = y|_{B_{2R}} \quad (\forall x, y \in X).$$

Suppose that $\mathsf{d}(x+t, y+t) < \Delta_0$ for all $t \in \mathbb{R}^d$. For every $t \in \mathbb{R}^d$, let $s(t) \in B_{\epsilon/4}$ be such that

$$x + t + s(t)|_{B_{2R}} = y + t|_{B_{2R}} \quad (\forall t \in \mathbb{R}).$$

This gives

$$x + t + s(t) + t'|_{B_R} = y + t + t'|_{B_R} \quad (\forall t \in \mathbb{R}, |t'| < R)$$

and also

$$x + t + t' + s(t + t')|_{B_R} = y + t + t'|_{B_R} \quad (\forall t \in \mathbb{R}, |t'| < R).$$

Because $s(t+t') - s(t) \in B_\epsilon$, it must be that $s(t+t') = s(t)$ (for all $t \in \mathbb{R}$ and $|t'| < R$) so that $t \mapsto s(t)$ is constant and equal to some s. Hence $x + t + s|_{B_{2R}} = y + t|_{B_{2R}}$ for all $t \in \mathbb{R}^d$, which yields $x + s = y$ (assuming R was not too small).

(PS): Fix $\delta \in (0, \Delta_0)$. Take $\epsilon, R > 0$ as before. By diminishing $\epsilon > 0$ and increasing R if necessary we can secure that

$$(s \in B_\epsilon \text{ and } x + s|_{B_{2R}} = y|_{B_{2R}}) \implies \mathsf{d}(x,y) < \delta \quad (\forall x, y \in X).$$

Set $l_0 := \epsilon/4$. Take $\delta_1 \in (0, \Delta_0)$ so that

$$\mathsf{d}(x,y) < \delta_1 \implies (\exists_{|s| \le \epsilon/4}\ \ x + s|_{B_{2R}} = y|_{B_{2R}}) \quad (\forall x, y \in X).$$

Suppose that x, y satisfy the hypothesis in (PS). For every $t \in U$, from $\mathsf{d}(x + t + l(t), y + t) < \delta_1 < \Delta_0$, there is a unique $s(t) \in B_{\epsilon/4}$ such that

$$x + t + l(t) + s(t)|_{B_{2R}} = y + t|_{B_{2R}}.$$

Thus

$$x + t + l(t) + s(t) + t'|_{B_R} = y + t + t'|_{B_R} \qquad (\forall\ |t'| < R)$$

and, for such $t' \in B_R$ with $t + t' \in U$, we also have

$$x + t + t' + l(t+t') + s(t+t')|_{B_R} = y + t + t'|_{B_R}.$$

Because $s(t+t') + l(t+t') - s(t) - l(t) \in B_\epsilon$, it must be that $s(t+t') + l(t+t') = s(t) + l(t)$. This shows that $t \mapsto s(t) + l(t)$ is locally constant near any $t \in U$ so that, by the connectedness of U, $s(t) + l(t) = s$ for some s and all $t \in U$. Summarizing, for $t \in U$, we got $x + t + s|_{B_{2R}} = y + t|_{B_{2R}}$ with $s \in B_{\epsilon/2+\epsilon/2}$. That $\mathsf{d}(x + t + s, y + t) < \delta$ for $t \in U$ follows by the choice of ϵ and R. □

8 Appendix II: Recognizability

The map Φ in the axiom (EXP) satisfies $\Phi(x+t) = \Phi x + \Lambda t$ ($\forall x \in X$, $t \in \mathbb{R}^d$), so it maps each $\mathbb{R}^d$-orbit bijectively to an $\mathbb{R}^d$-orbit. Because under our standing hypothesis of minimality, X is certainly a closure of some orbit, Φ must be surjective. A deeper observation that Φ is in fact injective was pioneered for one-dimensional tilings by [Mos92] (see also [BK06] for another exposition) and generalized to all dimensions in [Sol98]. We follow the same basic idea to supply an analogous result in our more abstract setting.

Theorem 8.1 (recognizability) *For a minimal aperiodic abstract self-affine tiling action, the continuous mapping $\Phi : X \to X$ satisfying (EXP) is a homeomorphism.*

The *strong repetitivity* property of self-affine tilings asserts that, given a disk $D = B_\rho$, there is $k_D > 0$ so that any $\Lambda^{n+k_D} D$ admissible patch contains any other admissible $\Lambda^n D$ patch. Our variant of it is recovered below by a juxtaposition of the minimality of the action and the local product structure developed in Section 2 (which did not require that Φ be a homeomorphism). For convenience, we assume that $\|\Lambda^{-1}\| < 1$ (as secured by passing to an iterate of Φ or by using the adapted norm).

Fact 8.2 (strong repetitivity) *Fix $\Delta \in (0, \Delta_0/2)$. Given a sufficiently large disk $D = B_\rho$, ($\rho > 0$), there is $k_D > 0$ with a property that,*

$$\forall_{x,y \in X}\ \forall_{k \in \mathbb{N}}\ \exists_{s \in \Lambda^{k+k_D} D}\ \ x + s|^{(\Delta)}_{\Lambda^k D} = y|^{(\Delta)}_{\Lambda^k D}.$$

Proof: Select $R > 0$, say $R := 1$. Let $\epsilon_1 > 0$ be as in Lemma 2.3 applied with Δ_0 replaced by Δ. Require D to be large enough so that

$$\Lambda^{k_0} D + B_{r_0} \subset \Lambda^{k_0+1} D \quad \text{ and } \quad x + s|^{(\Delta)}_D = y|^{(\Delta)}_D \implies \mathsf{d}(x+s, y) < \epsilon_1. \tag{8.1}$$

Take $\delta > 0$ so that $\mathsf{d}(x+s,y) < \delta \implies x + s|^{(\Delta)}_D = y|^{(\Delta)}_D$. The minimality secures $k_0 > 0$ such that $x + \Lambda^{k_0} D$ is δ-dense in X for any $x \in X$. Thus, upon fixing some arbitrary $x, y \in X$, there is $s \in \Lambda^{k_0} D$ with $x + s|^{(\Delta)}_D = y|^{(\Delta)}_D$.

From Lemma 2.3, $[y, x+s]_\Lambda = x + s'$ where $s' - s \in B_{r_0}$ so $s' = s + s' - s \in \Lambda^{k_0+1} D$. Crucially, (2.3) gives $x + s'|^{(\Delta)}_{B_R} \stackrel{s}{=} y|^{(\Delta)}_{B_R}$, i.e.,

$$\Phi^m x + \Lambda^m s'|^{(\Delta)}_{\Lambda^m B_R} = \Phi^m y|^{(\Delta)}_{\Lambda^m B_R} \quad (\forall\ m \geq 0).$$

Since we can find $k_1 > 0$ with $D \subset \Lambda^{k_1} B_R$, setting $m := k + k_1$ above yields

$$\Phi^m x + \Lambda^{k+k_1} s'|_{\Lambda^k D} = \Phi^m y|_{\Lambda^k D} \quad (\forall\ k \in \mathbb{N}).$$

This proves that $k_D := k_1 + k_0 + 1$ is as desired because (by surjectivity of Φ) $\Phi^m x$ and $\Phi^m y$ could be any two points in X and $\Lambda^{k+k_1} s' \in \Lambda^{k+k_1+k_0+1} D$. $\square$

At the heart of all proofs of recognizability lies an a priori bound (in terms of $|t|$) on the proximity of a (non-trivial) return of $x + t$ back to a vicinity of x.

Lemma 8.3 (no small approximate periods) *Given $r > 0$, there is $\Delta \in (0, \Delta_0)$, a large enough disk $D = B_\rho \subset \mathbb{R}^d$ (for some $\rho > 0$) and $m_D > 0$ with a property that, for $x \in X$, $m, k \in \mathbb{N}$, we have*

$$\left(x + \Lambda^k t|^{(\Delta/2)}_{\Lambda^{k+m}D} = x|^{(\Delta/2)}_{\Lambda^{k+m}D},\ t \in D,\ m \geq m_D\right) \implies \Lambda^k t \in B_r.$$

Proof of Lemma 8.3: Pick $\Delta < \mu_0^{-1}(r)$ where the function μ_0 is as in Remark 1.6. Let $m_D > 0$ be large enough so that $D \subset \frac{1}{2}\Lambda^{m_D - k_{D/2}} D$, where $k_{D/2}$ is supplied by Fact 8.2. Suppose that $t \in D$, $k \geq 0$, $m > m_D$, and

$$x + \Lambda^k t|^{(\Delta/2)}_{\Lambda^{k+m}D} = x|^{(\Delta/2)}_{\Lambda^{k+m}D}. \tag{8.2}$$

Then

$$x + s + \Lambda^k t|^{(\Delta/2)}_{\frac{1}{2}\Lambda^{k+m}D} = x + s|^{(\Delta/2)}_{\frac{1}{2}\Lambda^{k+m}D} \quad \text{for all } s \in \frac{1}{2}\Lambda^{k+m}D$$

and, given an arbitrary $y \in X$, strong repetitivity yields $s \in \frac{1}{2}\Lambda^{k+m}D$ with

$$x + s|^{(\Delta/2)}_{\frac{1}{2}\Lambda^{k+m-k_{D/2}}D} = y|^{(\Delta/2)}_{\frac{1}{2}\Lambda^{k+m-k_{D/2}}D}. \tag{8.3}$$

Since, by the hypothesis on m_D, $\Lambda^k t$ belongs to $\frac{1}{2}\Lambda^{k+m-k_{D/2}}D$ (and so does 0), (8.3) implies that

$$\mathsf{d}(x + s, y) < \Delta/2 \quad \text{and} \quad \mathsf{d}(x + s + \Lambda^k t, y + \Lambda^k t) < \Delta/2,$$

which combine to $\mathsf{d}(y, y + \Lambda^k t) < \Delta$. Because $y \in Y$ was arbitrary, we also have $\mathsf{d}(y + s, y + \Lambda^n t + s) < \Delta$ for all $s \in \mathbb{R}^d$ and so the expansiveness axiom (WE) forces that $y + \Lambda^k t = y + s$ for some $s \in B_{r_0}$. By Remark 1.6, $s \in B_r$. We are done because $\Lambda^k t = s$ by the aperiodicity. $\square$

Conclusion of Proof of Theorem 8.1: Fix $r > 0$ so that

$$\tau \in \Lambda^{-1} B_r \implies \mathsf{d}(x + \tau, x) < \Delta_0/2 \quad (\forall x \in X).$$

Let $\Delta \in (0, \Delta_0)$, D, and m_D be as in Lemma 8.3. Fix $R > 0$ so that $\Lambda^{m_D} D \subset B_R$ and implement Lemma 2.3 with Δ_0 replaced by $\Delta/4$ to get $\epsilon_1 > 0$ so that (2.3) holds.

Suppose that Φ is not 1-1 and we have $z_0 \in X$ with two distinct preimages $z_{-1}, z'_{-1} \in \Phi^{-1}(z_0)$. Because the orbits $z_{-1} + \mathbb{R}^d$ and $z'_{-1} + \mathbb{R}^d$ must be distinct, (WE) says that we can translate the points so that

$$\mathsf{d}(z_{-1}, z'_{-1}) > \Delta_0. \tag{8.4}$$

For every $n \in \mathbb{N}$, pick $z_{-n} \in \Phi^{-n+1}(z_{-1})$ and $z'_{-n} \in \Phi^{-n+1}(z'_{-1})$ and consider the two Φ-orbit pieces made of $z_{-n+i} = \Phi^i z_{-n}$ and $z'_{-n+i} = \Phi^i z'_{-n}$, $i = 0, \ldots, n-1$. By compactness of $X \times X$, there are n and $1 \leq k < l \leq n$ such that

$$\mathsf{d}(z_{-k}, z_{-l}) < \epsilon_1 \quad \text{and} \quad \mathsf{d}(z'_{-k}, z'_{-l}) < \epsilon_1. \tag{8.5}$$

Hence the local product structure can be used to define

$$y_{-l} := [z_{-l}, z_{-k}]_\Lambda = z_{-k} + t \quad \text{and} \quad y'_{-l} := [z'_{-l}, z'_{-k}]_\Lambda = z'_{-k} + t'$$

where $t, t' \in B_{r_0}$. Also, (2.3) gives

$$y_{-l}|^{(\Delta/4)}_{\Lambda^{m_D} D} \overset{s}{=} z_{-l}|^{(\Delta/4)}_{\Lambda^{m_D} D} \quad \text{and} \quad y'_{-l}|^{(\Delta/4)}_{\Lambda^{m_D} D} \overset{s}{=} z'_{-l}|^{(\Delta/4)}_{\Lambda^{m_D} D}. \tag{8.6}$$

Focus on $y_{-1} := \Phi^{l-1} y_{-l}$ and $y'_{-1} := \Phi^{l-1} y'_{-l}$. Since $y_{-1} = z_{-k+l-1} + \Lambda^{l-1} t$ and $y'_{-1} = z'_{-k+l-1} + \Lambda^{l-1} t'$ where $z'_{-k+l-1} = z_{-k+l-1}$ (due to $l > k$) we see that y'_{-1} and y_{-1} are on the same $\mathbb{R}^d$-orbit:

$$y'_{-1} = y_{-1} + \tau, \quad \tau := \Lambda^{l-1}(t - t'). \tag{8.7}$$

At the same time, applying Φ^{l-1} to (8.6) yields

$$y_{-1}|^{(\Delta/4)}_{\Lambda^{m_D+l} D} \overset{s}{=} z_{-1}|^{(\Delta/4)}_{\Lambda^{m_D+l-1} D} \ \text{and} \ y'_{-1}|^{(\Delta/4)}_{\Lambda^{m_D+l-1} D} \overset{s}{=} z'_{-1}|^{(\Delta/4)}_{\Lambda^{m_D+l-1} D} \tag{8.8}$$

so certainly

$$\mathsf{d}(y_{-1}, y'_{-1}) \geq \Delta_0 - \mathsf{d}(y_{-1}, z_{-1}) - \mathsf{d}(y'_{-1}, z'_{-1}) \geq \Delta_0 - 2\Delta/4 \geq \Delta_0/2.$$

On the other hand, because $\Phi z_{-1} = z_0 = \Phi z'_{-1}$, applying one more Φ to (8.8) gives $\Phi y_{-1}|^{(\Delta/2)}_{\Lambda^{m_D+l} D} \overset{s}{=} \Phi y'_{-1}|^{(\Delta/2)}_{\Lambda^{m_D+l} D}$ where $\Phi y'_{-1} = \Phi y_{-1} + \Lambda\tau$ so Lemma 8.3 secures $\Lambda\tau \in B_r$. By the choice of r, this contradicts $\mathsf{d}(y_{-1}, y'_{-1}) \geq \Delta_0/2$. □

9 Appendix III: Unique Ergodicity

Unique ergodicity of the translation actions associated to minimal self-affine tilings is traditionally shown by using the substitution rule for patch counting. We give a different argument applicable to abstract tiling spaces.

Theorem 9.1 *A minimal self-affine abstract tiling action is uniquely ergodic.*

Proof: Consider two probability measures μ and ν invariant and ergodic under the $\mathbb{R}^d$-action. Our goal is to see that $\mu = \nu$, for which it suffices to show that the two measures are not mutually singular. This, in turn, follows once we show that there is $a > 0$ such that, for any continuous non-negative $\phi : X \to \mathbb{R}$, we have $\int \phi \, d\mu > a \int \phi \, d\nu$. By minimality, there is $R > r_0$ such that, for any $x, y \in X$, $x + B_R$ passes through the product structure neighborhood of y, i.e., it enters $B_{\epsilon_1}(y)$ where ϵ_1 is as in Lemma 2.3. To be specific, there is $\epsilon_2 > 0$ that depends on R (and not on x, y) such that $\{s \in B_R : \ x + s \in B_{\epsilon_1}(y)\}$ contains some ϵ_2-ball $B_{\epsilon_2}(s_0)$.

Now, fix $x, y \in X$ that are generic for μ and ν, respectively. Given $N \in \mathbb{N}$, by our initial remarks applied to $\Phi^{-N}x$ and $\Phi^{-N}y$, there is $s_0 \in B_R$ such that, to any

$$z := \Phi^{-N}x + s \text{ with } s \in B_{\epsilon_2}(s_0)$$

we can associate (by using the local product structure)

$$w := [z, \Phi^{-N}y]_\Phi = \Phi^{-N}y + s - s_0.$$

(Precisely, select s_0 so that $[\Phi^{-N}x + s_0, \Phi^{-N}y]_\Phi = \Phi^{-N}y$ and invoke Remark 2.4 to get $[\Phi^{-N}x + s, \Phi^{-N}y]_\Phi = \Phi^{-N}y + s - s_0$ for $s \in B_{\epsilon_2}(s_0)$.)

Upon applying Φ^n, to a point $\Phi^n z = x + \Lambda^n s$ with $s \in B_{\epsilon_2}(s_0)$ we associated $\Phi^n w = y + \Lambda^n(s - s_0)$ where $\tau := s - s_0 \in B_{r_0}$. Note that the distance $\mathrm{dist}(\Phi^n(z), \Phi^n(w)) \to 0$ exponentially and uniformly (in z) because w, z are in the same local stable set. This allows us to estimate

$$\int_{B_R} \phi(x + \Lambda^n s)\, |ds|_d \geq \int_{B_{\epsilon_2}(s_0)} \phi(x + \Lambda^n s)\, |ds|_d \geq \int_{B_{\epsilon_2}} \phi(y + \Lambda^n \tau)\, |d\tau|_d - \mathrm{error}_n$$

where $\mathrm{error}_n \to 0$ as $n \to \infty$ by uniform continuity of ϕ (and $|ds|_d$ is the Lebesgue volume element in $\mathbb{R}^d$). Therefore, via Birkhoff's Ergodic Theorem,

$$\begin{aligned}\int \phi\, d\mu &= \lim_{n\to\infty} \frac{1}{|B_R|} \int_{B_R} \phi(x + \Lambda^n \tau)\, |d\tau|_d \\ &\geq \frac{|B_{\epsilon_2}|}{|B_R|} \lim_{n\to\infty} \frac{1}{|B_{\epsilon_2}|} \int_{B_{\epsilon_2}(s_0)} \phi(y + \Lambda^n \tau)\, |d\tau|_d = \frac{|B_{\epsilon_2}|}{|B_R|} \int \phi\, d\nu.\end{aligned}$$

□

10 Appendix IV: MCG for Penrose Tiling

In order to give an example, we compute the general linear group $\mathbb{G}l(x_0)$ for the five-fold symmetric "sun" Penrose tiling[10] depicted in Figure 1.1. In passing, we record some general observations and corollaries of our main results. We shall heavily rely on detailed understanding ([Rob96]) of the almost everywhere 1-1 map γ sending the Penrose tiling space to a four-torus and factoring the tiling action to a Kronecker action. (γ is the maximal equicontinuous factor map.) The gist of our considerations is that $\mathbb{G}l(x_0)$ is narrowed down by the inhomogeneity of X (manifested by nontriviality of some fibers of γ) and the arithmetics of the Kronecker action. To illustrate the ideas, we include in our discussion the 1-dimensional Fibonacci tiling obtained from the fixed word of the substitution $\phi : 1 \mapsto 12, 2 \mapsto 1$, for which the analogous factor map γ is much easier to visualize (see e.g. *geometric realization* in [BK06]).

[10] The arguments for the "star" Penrose tiling are the same.

Theorem 10.1 *Let* $\lambda = (\sqrt{5}+1)/2$.
(i) The general linear group of the Fibonacci tiling is generated by the dilation $[\lambda]$ *and the reflection* $[-1]$.
(ii) The general linear group of the "sun" Penrose tiling is generated by the dilation by λ*, the rotation by* $2\pi/10$*, and the symmetry about the vertical axis, as given by*

$$\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}, \quad R_{2\pi/10} = \begin{bmatrix}\cos(2\pi/10) & \sin(2\pi/10)\\ -\sin(2\pi/10) & \cos(2\pi/10)\end{bmatrix}, \quad \textit{and} \quad F = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}.$$

Recall that while the Penrose tiling as a pattern is only 5-fold rotationally invariant it has been observed that it possesses 10-fold rotational symmetry in the *statistical sense* [Rad95]. We emphasize the fact that this 10 fold symmetry is in fact topological (i.e., effected by homeomorphisms). Similar is the case for the reflection symmetry of the Fibonacci tiling, which is not out-right symmetric under the flip $x \mapsto -x$ (Figure 10). It is of interest that the corresponding symmetries are enjoyed by the *dynamical spectrum* of x_0, by which we understand the spectral decomposition, $U^t = \int_{\mathbb{R}^d} e^{i\langle\omega|t\rangle} dE_\omega$, of the associated unitary action $U^t : u \mapsto u \circ T^t$, $U^t : L^2(X,\mu) \to L^2(X,\mu)$, μ being the unique invariant measure for (T^t). Indeed, we record the following general fact.

Fact 10.2 *If* $A \in \mathbb{G}l(x_0)$*, then the dynamical spectrum is invariant under the inverse transpose* $(A^T)^{-1}$*; namely, for any Borel subset* $\Omega \subset \mathbb{R}^d$*, the projections* $\int_\Omega dE_\omega$ *and* $\int_{(A^T)^{-1}\Omega} dE_\omega$ *are naturally unitarily equivalent.*

Proof: Suppose that $A \in \mathbb{G}l(x_0)$ and $h : X \to X$ is such that $h \circ T^t = T^{At} \circ h$. Then the push-forward $h_*(\mu)$ is also (T^t)-invariant (since $T^t_*(h_*(\mu)) = (T^t \circ h)_*(\mu) = (h \circ T^{A^{-1}t})_*(\mu) = h_*(T^{A^{-1}t}_*(\mu)) = h_*(\mu)$). By uniqueness of μ, $h_*(\mu) = \mu$ and the map $H : u \mapsto u \circ h$ constitutes a unitary isomorphism $L^2(X,\mu) \to L^2(X,\mu)$ that intertwines the actions (U^{At}) and (U^t), meaning that $HU^{At} = U^t H$ ($\forall t \in \mathbb{R}^d$). Therefore, if $\{E_\omega\}$ is the spectral resolution for (U^t), i.e., $U^t = \int e^{i\langle\omega|t\rangle} dE_\omega$, then $U^t = H^{-1}U^{At}H = H^{-1}\left(\int e^{i\langle\omega|At\rangle} dE_\omega\right) H = \int e^{i\langle A^T\omega|t\rangle} d(HE_\omega H^{-1}) = \int e^{i\langle\omega'|t\rangle} d(HE_{(A^T)^{-1}\omega'} H^{-1})$. By the uniqueness of the spectral resolution, for any Borel $\Omega \subset \mathbb{R}^d$, we have $\int_\Omega dE_\omega = \int_\Omega d(HE_{(A^T)^{-1}\omega'} H^{-1})$, which gives $\int_\Omega dE_\omega = H\left(\int_{(A^T)^{-1}\Omega} dE_\omega\right) H^{-1}$. □

Note that the physically relevant *diffraction spectrum* inherits symmetry properties from the dynamical spectrum because the intensity of the diffraction image can be recovered as $I(\omega)d\omega = d\langle E_\omega f|f\rangle$ for a suitable $f \in L^2(X,\mu)$ ([Dwo93], see also [Hof95]).

The rest of this section is devoted to the proof of the theorem. We start with the Fibonacci tiling, the construction of which as a *canonical cut-project tiling* associated to a line in $\mathbb{R}^2$ we briefly recount below. Consider the matrix

$$B_2 := \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$$

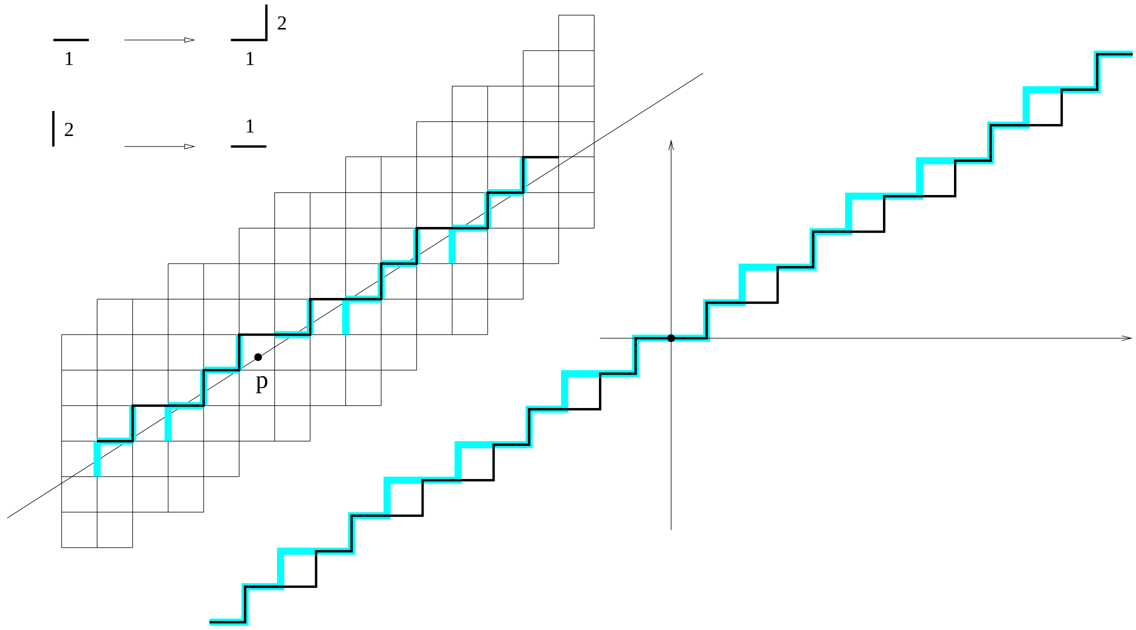

Figure 10.1: Up left, the Fibonacci inflation-substitution operation Ψ following the rule $1 \mapsto 12$, $2 \mapsto 1$. Center, an example of a (fragment of) the stepped line L_p assembled from the duals of the grid edges intersected by the line $p + E^u$. Lower right, the "upper" Fibonacci tiling x_0^+ (in solid black) is the stepped line associated to E^u. It is fixed by Ψ^2. Topological symmetry under the 180 deg flip notwithstanding, the flipped x_0^+ (in thick gray) disagrees with x_0^+.

acting on $\mathbb{R}^2$ and let E^u and E^s be its eigenspaces corresponding to the eigenvalues λ and $-1/\lambda$, respectively. Thus $E^u = \text{lin}(\omega)$ where $\omega = (\lambda, 1)$. Denote by $(e_i)_{i=1,2}$ the standard unit vectors in $\mathbb{R}^2$ and by σ_i the unit segment joining 0 and $0 + e_i$. To a ***grid edge***, i.e., a segment of the form $\sigma = \sigma_i + v$ where $v \in \mathbb{Z}^2$ and $i = 1, 2$, we associate its ***dual*** $\sigma^* := e_i + \sigma_j + v$ where $j \neq i$. Fix for a moment a point $p \in \mathbb{R}^2$ such that $p + E^u$ misses $\mathbb{Z}^2$. One can see that the duals of the grid edges intersected by $p + E^u$ form an infinite step line L_p (see Figure 10). If we consider $p + E^u$ as a copy of $\mathbb{R}$ via the parametrization $t \mapsto p + t\omega$, the Fibonacci tiling x_p associated to p is obtained by projecting L_p onto $p + E^u$ along E^s; the projection denoted by π. For p for which $p + E^u$ intersects $\mathbb{Z}^2$, by taking the limits of x_q with $q \to p$ one can obtain two different tilings x_q, denoted x_p^+ and x_p^-, depending on whether q converges to p from above or below. The *Fibonacci tiling* (in the theorem) is one of the tilings x_0^+ or x_0^-. We shall focus on x_0^+; the situation for x_0^- is completely analogous. All the tilings x_p and $x_p^\pm$ arising from the construction form the tiling space X of x_0^+. The map $\gamma : X \to \mathbb{R}^2/\mathbb{Z}^2$ factoring the tiling flow to a Kronecker flow simply maps x_p and $x_p^\pm$ to p mod $\mathbb{Z}^2$.

The self-similarity is that induced by the symbolic substitution $1 \mapsto 12$ and $2 \mapsto 1$: it transforms a tiling by inflating each tile $\pi(e_i + v)$ by a factor λ and replacing it according to the rule:

$$\lambda\pi(e_1+v)\mapsto\pi(\sigma_1+B_2v)\cup\pi(\sigma_2+e_1+B_2v),$$

$$\lambda\pi(e_2+v)\mapsto\pi(\sigma_1+B_2v).$$

Denote the induced map on X by Ψ. (It factors via γ to the automorphism of the torus $\mathbb{R}^2/\mathbb{Z}^2$ induced by B_2.) Observe that $\Psi(x_0^+)=x_0^-$ and $\Psi(x_0^-)=x_0^+$ so one needs to take the second iterate $\Phi:=\Psi^2$ to obtain a self-homeomorphism of X pointed at x_0^+. The corresponding element of $\mathbb{G}l(x_0^+)$ is $[\lambda^2]$. More elements will be flushed out by the following lemma.

Lemma 10.3 *Suppose that $A\in\mathbb{G}l_2(\mathbb{Z})$ preserves E^u and acts on it by multiplication by $a\in\mathbb{R}$. Then $[a]\in\mathbb{G}l(x_0^+)$.*

Sketch of Proof: The object is to see that the densely defined map of X given by $x_0^++t\mapsto x_0^++at$, $t\in\mathbb{R}$, is uniformly continuous so that it extends to a continuous map $h:X\to X$. Indeed, by recognizability, h is then a homeomorphism placing $[a]$ in $\mathbb{G}l(x_0^+)$.

To check the uniform continuity, it suffices to see that, if the tilings $x_0^++t=x_{t\omega}^+$ and $x_0^++s=x_{s\omega}^+$ are no further than small $\epsilon>0$ apart, then $x_0^++at=x_{at\omega}^+$ and $x_0^++as=x_{as\omega}^+$ are also close. The ϵ-proximity of the tilings means that, upon adjusting by a translation of order ϵ, which we may well do at the outset, the two tilings exactly coincide when restricted to $[-1/\epsilon,1/\epsilon]$. Interpreted in $\mathbb{R}^2$, this implies that the two segments $[t-1/\epsilon,t+1/\epsilon]\omega$ and $[s-1/\epsilon,s+1/\epsilon]\omega$ can be translated by vectors $u,w\in\mathbb{Z}^2$, respectively, to intersect exactly the same edges of the integral grid. For this they have to be not only very close to each other (as expressed by the proximity of $\gamma(x_0^++t)$ and $\gamma(x_0^++s)$, which follows from continuity of γ) but also have to pass on the same side (upper/lower) of any lattice points they get close to. The matrix A induces a linear transformation that preserves the lattice points and maps any pair of parallel segments in the direction of E^u that both pass just above (below) a lattice point v to a pair of such segments that are both just above (below) Av. This means that the segments $A([t-1/\epsilon,t+1/\epsilon]\omega+u)=[at-a/\epsilon,at+a/\epsilon]\omega+Au$ and $A([s-1/\epsilon,s+1/\epsilon]\omega+w)=[as-a/\epsilon,as+a/\epsilon]\omega+Aw$ are not only very close to each other but also pass on the same side (upper/lower) of any lattice points they (narrowly) miss. This is to say that the tilings $x_0^++at=x_{at\omega}^+$ and $x_0^++as=x_{as\omega}^+$ are separated by a distance of order ϵ/a. □

Considering $-I,B_2\in\mathbb{G}l_2(\mathbb{Z})$, the lemma places $[-1]$ and $[\lambda]$ in $\mathbb{G}l(x_0^+)$. Note that the homeomorphism $h:X\to X$ underlying $[\lambda]\in\mathbb{G}l(x_0^+)$ fixes x_0^+, unlike Ψ. In fact, $h\circ\Psi^{-1}:X\to X$ is an isometric involution on X that interchanges the two tilings $x_0^\pm$.

The following general proposition a priori limits the possibilities for the elements of $\mathbb{G}l(x_0^+)$ in the self-similar case.

Proposition 10.4 (commutation) *Suppose that T is a minimal self-similar abstract tiling action, the self-similarity is effected by $\Phi : X \to X$ with $\Phi \circ T^t = T^{\Lambda t} \circ \Phi$ ($\forall t \in \mathbb{R}^d$), and x_0 is fixed by Φ. If $A \in \mathbb{G}l(x_0)$, then there is $m \in \mathbb{N}$ such that Λ^m commutes with A.*

Proof: From the conformality of Λ, the sequences $(\Lambda^{-n} A \Lambda^n)_{n\in\mathbb{N}}$ and $(\Lambda^{-n} A^{-1} \Lambda^n)_{n\in\mathbb{N}}$ are bounded (e.g., $\|\Lambda^{-n} A \Lambda^n\| \le \|\Lambda^{-n}\|\|A\|\|\Lambda^n\| = \|A\|$) and so precompact in $\mathbb{G}l_d(\mathbb{R})$. Hence, there are $n_k \to \infty$ such that $(\Lambda^{-n_k} A \Lambda^{n_k})$ is Cauchy, and so eventually constant by the discreteness of $\mathbb{G}l(x_0)$. In particular, there is k with $\Lambda^{-n_k} A \Lambda^{n_k} = \Lambda^{-n_{k+1}} A \Lambda^{n_{k+1}}$. Then any $m := n_{k+1} - n_k > 0$ is as desired. □

Corollary 10.5 (canonical vertical) *In the context of the proposition, the linear homeomorphism $h : X \to X$ realizing A preserves the local stable sets of Φ, i.e.,*

$$\forall_{\epsilon>0}\ \exists_{\delta>0}\ \forall_{x\in X}\ h(W^s_\delta(x, \Phi)) \subset W^s_\epsilon(h(x), \Phi).$$

Proof: By the proposition, h and Φ^m commute for some $m \in \mathbb{N}$. This implies the preservation of the stable sets via a routine argument. □

In concrete setting, the corollary says that h maps any two tilings that exactly coincide on a large patch about the origin to a pair of tilings that have the same property on a perhaps smaller patch. Those preferring the Delone set description will easily deduce that the Delone set D obtained by intersecting the orbit of x_0 with a local stable set of x_0 actually maps into itself under $\Lambda^n \circ A$ for some large $n > 0$ (although more often than not not under A).

Finally, the arithmetic constraints on $\mathbb{G}l(x_0)$ are brought out by the following very general fact about minimal actions.

Proposition 10.6 (algebraization) *If T is a minimal action on X, $\gamma : X \to X_{alg}$ is a factor map onto its maximal equicontinuous factor action T_{alg}, and T_{alg} is aperiodic, then $\mathbb{G}l(T, x_0) \subset \mathbb{G}l(T_{alg}, \gamma(x_0))$ for any $x_0 \in X$.*

The proof of the proposition rests on the fact that the fibers of γ are the equivalence classes of the relation given by $p \sim q$ iff $u(p) = u(q)$ for every continuous eigenfunction u of T. One checks that, if $A \in \mathbb{G}l(T, x_0)$ by virtue of a homeomorphism $h : X \to X$ with $h \circ T^t = T^{At} \circ T$ and $u : X \to \mathbb{C}$, then both $u \circ h$ and u are eigenfunctions if only one of them is, and so $p \sim q$ iff $h(p) \sim h(q)$. Therefore, h induces a map $h_{alg} : X_{alg} \to X_{alg}$ which places A in $\mathbb{G}l(T_{alg}, \gamma(x_0))$. We leave details as an exercise because a more elementary converse of the last lemma is all we need here.

Proposition 10.7 (algebraization lite) *If $[a] \in \mathbb{G}l(x_0^+)$ then there is $A \in \mathbb{G}l_2(\mathbb{Z})$ that preserves E^u and acts on it by multiplication by $a \in \mathbb{R}$.*

Proof: Fix $[a] \in \mathbb{G}l(x_0^+)$ and let $h : X \to X$ be the underlying linear homeomorphism. Note that to every $p \in \mathbb{T}^2$ we can associate $\gamma^{-1}(p)$, which is either a singleton or a pair of tilings $x_p^\pm$ that are bi-asymptotic: $\lim_{t\to\pm\infty} \mathsf{d}(x_p^+ + t, x_p^- + t) = 0$. The relation of bi-asymptoticity is preserved by any linear homeomorphism. Thus, for $p \in \mathbb{T}^2$, there is a unique $q \in \mathbb{T}^2$ such that $h(\gamma^{-1}(p)) = \gamma^{-1}(q)$. Define $h_{alg} : \mathbb{T}^2 \to \mathbb{T}^2$ by $h_{alg}(p) := q$. h_{alg} easily inherits continuity from h. By repeating the arguments for h^{-1}, we see that h_{alg} is a homeomorphism. By construction, h_{alg} is the multiplication by a on the 1-parameter subgroup $E^u \pmod{\mathbb{Z}^2} \subset \mathbb{T}^2$. It follows that h_{alg} is an algebraic automorphism of $\mathbb{T}^2$, i.e., it is induced by some $A \in \mathbb{G}l_2(\mathbb{Z})$. □

Conclusion of Proof of (i) in Theorem 10.1: We already know from Lemma 10.3 that $[-1], [\lambda] \in \mathbb{G}l(x_0^+)$ but we have to see that these two elements generate $\mathbb{G}l(x_0^+)$.

To set the stage, E^u is totally irrational in the sense that $(E^u)^\perp \cap \mathbb{Z}^d = \{0\}$. As a consequence, any matrix A with rational entries preserving E^u is uniquely determined by its restriction $A|_{E^u}$ to E^u. Moreover, E^u being one-dimensional, any two such matrices commute. Thus the algebra of rational matrices preserving E^u equals $\mathcal{K} := \{A \in \mathbb{Q}^{2\times 2} : \ B_2A = AB_2\}$ and is a field. Since every $A \in \mathcal{K}$ is a zero of its quadratic characteristic polynomial, the field is a degree two extension of the rationals. Therefore, I and B_2 form a basis of $\mathcal{K}$ as a module over $\mathbb{Q}$. The isomorphism $\mathcal{K} \ni A \mapsto A|_{E^u} \in \mathbb{G}l_1(\mathbb{R}) \simeq \mathbb{R}$, maps $I \mapsto 1$ and $B_2 \mapsto \lambda$ so $\mathcal{K}$ is isomorphic with $\mathbb{Q}(\lambda) = \mathbb{Q}(\sqrt{5})$. The matrices in $\mathcal{K}$ that preserve $\mathbb{Z}^2$ (or any lattice for that matter) are exactly the algebraic integers $\mathcal{O}(\mathcal{K})$ of $\mathcal{K}$ and are integrally spanned by I and B_2 because $\mathcal{O}(\mathbb{Q}(\sqrt{5})) = \mathbb{Z}[\lambda]$.

Now, consider $[a] \in \mathbb{G}l(x_0^+)$. By Proposition 10.7, $[a]$ comes from a restriction $A|_{E^u}$ where $A \in \mathcal{O}(\mathcal{K})$ is multiplicatively invertible, i.e., it is a unit of $\mathcal{O}(\mathcal{K})$. This completes the proof because the group of units in $\mathbb{Z}[\lambda]$ is generated by -1 and λ. □

The Penrose tiling can be attacked along the same lines as it is also a canonical cut-project tiling: it can be constructed by taking duals of three dimensional grid faces intersected by a two dimensional eigenspace $E^u \subset \mathbb{R}^5$ of the matrix

$$B_5 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

The situation is complicated by the fact the right torus on which B_5 induces an automorphism is not $\mathbb{T}^5 := \mathbb{R}^5/\mathbb{Z}^5$ but the *anti-diagonal* subtorus $\mathbb{T}_0^4 := \{(u_0, \ldots, u_4) : u_0 + \ldots + u_4 = 0 \mod 1\}$. We note that B_5 induces an automorphism that is conjugate to the second Cartesian power $B_2 \times B_2$ of the Fibonacci automorphism[11].

[11]This conjugacy does not respect the 5-fold symmetry expressed by the commutation of B_5 with the cyclic permutation of the basis vectors in $\mathbb{R}^5$.

In particular, beside the uninteresting eigenvector $(1,1,1,1,1)$, B_5 has two two-dimensional eigenspaces E^u and E^s on which it acts by λI and $-\lambda^{-1}I$. Also, keeping track of the intersections of E^u with the grid cubes in $\mathbb{R}^5$ can be conveniently done directly in the plane $E^u \simeq \mathbb{R}^2$ via the *penta-grid formalism* of de Bruijn [Bru81] (see also [Sen95]). This allows one to explicitly construct the factor map $\gamma : X \to \mathbb{T}^4_0$ and ascribe its non-trivial fibers to singular pentagrid configurations. We assume that the reader is familiar with Robinson's description [Rob96] of the two possibilities: *Type A* pentagrid with an infinite string of 3-fold crossings along a single line and *Type B* pentagrid with a 5-fold crossing (and five lines of 3-fold crossings).[12]

For us it is important that, the fiber of 0 (which is the only fixed point of the automorphism of $\mathbb{T}^4_0$) contains two tilings: the *infinite star* (denoted x_0^+) and the *infinite sun* (denoted x_0^-). These two are interchanged by the inflation-substitution map $\Psi : X \to X$ which factors to the toral automorphism. Let us focus on computing $\mathbb{G}l(x_0^+)$.

Lemma 10.8 *The dilation λI, the rotation $R_{2\pi/10}$, and the reflection F all belong to $\mathbb{G}l(x_0^+)$.*

Sketch of Proof: Beside B_5 we shall use the matrices

$$R = \begin{bmatrix} 0&0&0&0&1\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 1&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&1&0&0&0 \end{bmatrix}.$$

Both leave invariant the splitting $\mathbb{R}^5 = E^u \oplus E^s \oplus \mathrm{lin}(1,1,1,1,1)$. R rotates E^u by $2\pi/5$ and E^s by $4\pi/5$. In both E^u and E^s, S is a symmetry about the projection of $\mathrm{lin}((1,0,0,0,0))$. (Note the commutation relations $RB_5 = B_5R$ and $SRS = R^{-1}$.) In what follows, A stands for either λI, the rotation $R_{2\pi/10}$, or F. Importantly, each of these three is generated by restricting to E^u a linear map inducing an automorphism of $\mathbb{T}^4_0$; namely, B_5, $-R^3$, and S, respectively. [13]

The strategy of the proof is analogous to that of Lemma 10.3. It suffices to see that if the tilings $x_0^+ + t$ and $x_0^+ + s$ agree on a large central patch for some $s, t \in \mathbb{R}^d$ then this is also so for $x_0^+ + At$ and $x_0^+ + As$. Again, it is clear that $\gamma(x_0^+ + t)$ and $\gamma(x_0^+ + s)$ are close in $\mathbb{T}^4_0$ and thus so are their images under the toral automorphism, $\gamma(x_0^+ + At)$ and $\gamma(x_0^+ + As)$. From this, on a large central patch, the pentagrids associated to $\gamma(x_0^+ + At)$ and $\gamma(x_0^+ + As)$ nearly agree and generate the same tiling except for the possibility that, at places where $\gamma(x_0^+ + At) + E^u$ and $\gamma(x_0^+ + As) + E^u$ pass very near a lattice point, the two (nearly singular) pentagrids resolve the singularity in different ways. However, the way the singularity is resolved ([Rob96])

[12] These two scenarios are related to *Conway's worms* and *cartwheels* ([GS87]).

[13] To check, $-R$ induces on E^u rotation by $2\pi/5 + \pi$ so $-R^3$ rotates by $3 \cdot 2\pi 7/10 \equiv 2\pi/10$.

is determined by containment of the vector describing (the shift of) the grids in certain halfplanes or 10-fold symmetric sectors inside E^s. The crux of the argument is in that those half-planes and sectors are manifestly invariant under the restriction of either R^3 or B_5 or S to E^s. Thus, since the grids underlying $x_0^+ + t$ and $x_0^+ + s$ resolved in the same way over a large central patch, this is also so for $x_0^+ + At$ and $x_0^+ + As$, so the two tilings agree on a large central patch. □

Lemma 10.9 *Any $A \in \mathbb{G}l(x_0^+)$, preserves the 10-fold star of lines obtained by rotating the vertical by the multiples of $2\pi/5$.*

Sketch of Proof: In the tiling space X there are lines K and pairs of tilings z_- and z_+ that are *bi-asymptotic away from* K in the sense that $\mathsf{d}(z_- + t, z_+ + t) \to 0$ as long as $|t| \to \infty$ so that the distance from t to K increases to infinity. Such pairs (z_-, z_+) arise from different resolutions of Type A singular pentagrids and the line K is exactly the line of 3-*fold crossings*. Clearly, under a linear homeomorphisms h corresponding to $A \in \mathbb{G}l(x_0^+)$, the relation of bi-asymptoticity between z_- and z_+ away from K translates to bi-asymptoticity between $h(z_-)$ and $h(z_+)$ away from AK. The five lines K through the origin form the 10-fold star that is preserved by A. □

Conclusion of Proof of (ii) in Theorem 10.1: Lemma 10.8 placed λI, $R_{2\pi/10}$, and F in $\mathbb{G}l(x_0^+)$. From Lemma 10.9, any $A \in \mathbb{G}l(x_0^+)$, possibly postcomposed with F and some power of $R_{2\pi/10}$, can be put in the form μI where $\mu > 0$. To finish, we have to show that μ is an integral power of the golden number λ. This follows from the arithmetic properties of the Kronecker $\mathbb{R}^2$-action associated to E^u. As we mentioned, this action is conjugate to the Cartesian square of two Fibonacci Kronecker actions. By repeating the arguments in the conclusion of the proof of part (i) of the theorem, μ must be a positive unit in the ring of algebraic integers $\mathbb{Z}[\lambda]$ and such units form an infinite cyclic group generated by λ. □

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