Geometric realization and coincidence for reducible non-unimodular Pisot tiling spaces with an application to β-shifts

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Abstract

1 Introduction

We are interested here in the pure discrete spectrum property for Pisot substitutions. Traditionally, this would mean considering the pure discreteness of the unitary operator $f \mapsto f \circ \sigma$ on $L^2(X_\phi)$ where X_ϕ is the (discrete) substitutive system associated with a substitution ϕ of Pisot type. We find it more convenient to study the tiling flow $T^t : \mathcal{T}_\phi \to \mathcal{T}_\phi$, $t \in \mathbb{R}$, on the space of tilings associated with ϕ; (X_ϕ, σ) can be recovered by taking a cross-section. The advantage of $(\mathcal{T}_\phi, T^t)$ over (X_ϕ, σ) lies in the existence of the inflation-and-substitution homeomorphism $\Phi : \mathcal{T}_\phi \to \mathcal{T}_\phi$ that interacts with T^t via $\Phi \circ T^t = T^{\lambda t} \circ \Phi$, λ the dominant eigenvalue of ϕ. In fact, the inflation-and-substitution dynamics allows one to define the tiling space as a global attractor in a geometrical setting. "Geometric realization" of the tiling space onto a compact abelian group is then a simple matter and the preeminent question in the subject — whether or not the tiling flow has pure discrete spectrum — reduces to the question of a.e. one-to-oneness of geomteric realization.

In case the abelianization of ϕ is unimodular, has irreducible characteristic polynomial, and has dominant eigenvalue a Pisot-Vijayaraghavan number (the "irreducible, unimodular Pisot" case), the geometric theory of the tiling flow alluded to above is developed in [5]. Here we extend the theory to cover all primitive substitutions of Pisot type.

The earliest instance of geometric realization for Pisot substitutions occurs in [21]. There Rauzy constructed three topological disks in the plane that tile the plane periodically as well as aperiodically. The union of the three disks (the *Rauzy fractal*) is a fundamental domain for a two-torus on which an irrational translation is defined whose orbits, when coded by the three disks, yield the substitutive system associated with the Tribonacci substitution ($1 \mapsto 12$, $2 \mapsto 13$, $3 \mapsto 1$). Using a more arithmetical approach, Thurston ([29]) produced tilings as a geometrical picture of the expansion of numbers in a Pisot base. These tilings lead to "arithmetic codings" of hyperbolic toral automorphisms, a process studied by Vershik, Sidorov, Kenyon, Schmidt, and others ([24, 17, 23], the survey [25]). The substitution based geometric approach initiated by Rauzy has been developed by Arnoux and Ito, and Cantorini and Siegel ([3, 9]), and recast from the Iterated Function Systems point of view by Sirvent and Wang ([26]). Further advances were made independently in [16] and [5] where an optimal coincidence condition in the irreducible unimodular case was introduced. The optimality alludes here to equivalence with various *good properties* ranging from some very specific tiling and metric properties of the (generalized) Rauzy fractals to the general measure theoretical property of pure discrete spectrum (c.f. [28]) of the tiling flow; see [5] for a comprehensive discussion. The related number-theoretic investigations have been undertaken in a number of works by Akiyama, Frougny, Ito, Rao, Solomyak, Steiner, and Thuswaldner([1, 2, 13, 16, 30]). For a recent survey of the connections between tilings, Pisot arithmetics, and substitutions consult [7].

As mentioned above, the main issue in all of this is the question of pure discrete spectrum. It is proved in [10] (see also Cor. 5.7 in [5] for a strand based proof) that, for irreducible Pisot ϕ, the tiling flow T^t has pure discrete spectrum if and only if the substitutive shift σ does. In the reducible case, the relation[1] between the two spectra is not as simple: pure discrete spectrum for σ implies that for T^t but the opposite implication typically fails. The following conjecture has become known as the Pisot conjecture ([7]).

Conjecture 1.1 (Pisot Conjecture) *The tiling flow associated with an irreducible Pisot substitution has pure discrete spectrum.*

In Sections 2 and 3 below we construct the geometric realization of the tiling flow associated with a general susbtitution of Pisot type. In Section 4 we state the Geometric Coincidence Condition (GCC) and prove that it holds if and only if geometric realization is a.e. one-to-one. In Section 5 we identify the eigenvalues of the tiling flow and prove that pure discrete spectrum of the flow is equivalent to the GCC. We also provide an example to show that the validity of the Pisot Conjecture does not extend to arbitrary reducible Pisot substitutions, even assuming the expansion λ is a Pisot unit. In Section 6 we establish a powerful criterion (Theorem 6.1) that allows us to verify (in Section 7) the Pisot Conjecture for a particular class of substitutions that arise from β-shifts for certain Parry numbers. As a corollary, we increase the scope of β-shifts with Pisot β for which the natural extension is known to be naturally isomorphic to an automorphism of a compact group — c.f. [23], and see the pertaining discussion in Section 7 for more detail. Finally, in Section 8, we explain how injectivity of geometric realization (as established in Section 7) provides an explanation for a phenomenon observed by Ei and Ito ([11]), namely that the natural domain exchange on the Rauzy fractal corresponding to certain classes of reducible β-substitutions is induced by a toral translation; that is, it is the first return, under an appropriate toral translation, to the Rauzy fractal.

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2 Strand Space

We fix a substitution $\phi : \mathcal{A} \to \mathcal{A}^*$ on an alphabet $\mathcal{A}$ of n letters, which we may well take to be $\mathcal{A} = \{1, \ldots, n\}$, with values in the set $\mathcal{A}^*$ of finite nonempty words over $\mathcal{A}$. A substitution ϕ extends to words by concatenation and hence may be iterated. The *abelianization of* ϕ is given by an $n \times n$ matrix $A = (a_{ij})$ with a_{ij}

[1] To be elaborated elsewhere.

equal to the number of ocurrences of i in $\phi(j)$. By the Perron Frobenius Theorem, the spectral radius λ of A is its dominant eigenvalue; let ω be a corresponding non-negative eigenvector,

$$A\omega = \lambda\omega.$$

Throughout this paper ϕ is *primitive* ($A^m > 0$ for some $m \in \mathbb{N}$) and *translation aperiodic* (if $\phi^n(i) = uw^kv$ then $k < N$ for $N > 0$ independent of $u, v, w \in \mathcal{A}^*$, $i \in \mathcal{A}$, $k \in \mathbb{N}$) and satisfies the following definition.

Definition 2.1 *ϕ is Pisot iff λ is a Pisot number, i.e., $\lambda > 1$ and all conjugates of λ over $\mathbb{Q}$ are of modulus less than one.*

The Fibonacci substitution ($1 \mapsto 12, 2 \mapsto 1$) and the Morse substitution ($1 \mapsto 12, 2 \mapsto 21$) are both Pisot.

The monic minimal polynomial of A decomposes into irreducible (over $\mathbb{Q}$) monic factors

$$p_A(x) = p_1(x)p_2(x)^{m_2} \cdots p_k(x)^{m_k} \tag{2.1}$$

where $p_{\min} := p_1$ is the minimal monic polynomial of λ (with no exponent because λ is simple). Taking $q(x) := p_2(x)^{m_2} \cdots p_k(x)^{m_k}$ we have an A-invariant decomposition[2]

$$\mathbb{R}^n = V \oplus W \tag{2.2}$$

so that $p_1(x)$ and $q(x)$ are the characteristic polynomials of the restrictions $A|_V$ and $A|_W$, respectively. Here both V and W are rational in the sense that they are linear spans over $\mathbb{R}$ of their intersections with $\mathbb{Q}^n$. The dynamical meaning of the Pisot hypothesis is that $A|_V$ is hyperbolic and has the stable/unstable splitting

$$V = E^u \oplus E^s$$

with E^u of dimension 1; $E^u = \text{lin}(\omega)$. We shall denote by

$$\text{pr}_V : \mathbb{R}^n \to V, \quad \text{pr}_s : V \to E^s, \quad \text{pr}_u : V \to E^u$$

the projections along W, E^u and E^s, respectively. We have an A-invariant lattice

$$\Gamma := \text{pr}_V(\mathbb{Z}^n) \subset \mathbb{Q}^n \cap V.$$

From $A\Gamma \subset \Gamma$ and 0 being an attractor in E^s, $E^s \cap \Gamma = \{0\}$ making E^s *totally irrational* (i.e. $E^s \cap \mathbb{Q}^n = \{0\}$). Also, E^u is *non-resonant*[3] in the sense that E^u taken modulo Γ yields a dense subgroup of the torus V/Γ.[4] Thus $\text{pr}_u : V \to E^u$ is injective on the rational points of V and $\text{pr}_s(\Gamma)$ is dense in E^s.

[2]Concretely, upon choosing $s_1(x), s_2(x) \in \mathbb{Q}[x]$ so that $s_1(x)p_{\min}(x) + s_2(x)q(x) = 1$, one checks that the matrices $P_1 := s_1(A)p_{\min}(A)$ and $P_2 := s_2(A)q(A)$ yield the complementary projections onto V and W so that $V := \ker(P_1)$ and $W := \ker(P_2)$.

[3]To see that, note that the orthogonal complement of E^u is the stable space E^s for the transpose of A, and $E^s \cap \mathbb{Q}^d = \{0\}$.

[4]As a consequence, if ϕ is Pisot then it is automatically translationally aperiodic if $\dim(V) \geq 2$.

Denoting by e_i, $i = 1, \ldots, n$, the standard basis vectors in $\mathbb{R}^n$, set

$$v_i := \mathrm{pr}_V(e_i)$$

and let

$$\sigma_i := \{tv_i : \ 0 \le t \le 1\}$$

be the edge (an oriented segment) representing v_i. We would like to distinguish between σ_i and σ_j even if $\sigma_i = \sigma_j$ for $i \neq j$, thus we shall consider each σ_i as a *labeled edge* with the label — also referred to as *type* — being i. An oriented broken line γ in V obtained by stringing together tip-to-tail a sequence of translated copies of the basic edges, $(\sigma_{i_k} + x_k)$, $x_k \in V$, will be called a *strand*. Taken together with the sequence of labels (i_k), such γ is called a *labeled strand*.

We shall denote the *space of the bi-infinite strands in* V by

$$\mathcal{F} := \{\gamma : \ \gamma \text{ is a bi-infinite labeled strand in } V\}.$$

The substitution ϕ naturally induces a map

$$\Phi : \mathcal{F} \to \mathcal{F}.$$

Namely, given an edge I labeled i with its initial vertex denoted $x = \min I$, $\Phi(I)$ is the finite strand beginning at Ax and labeled by $\phi(i)$. Acting edge-by-edge as above yields Φ on arbitrary strands.

Thus defined Φ is the factor via pr_V of the map Φ on strands in $\mathbb{R}^n$ defined in [5]; and most of the pertaining dicussion in [5] can be repeated in the present context. In particular (c.f. Lemma 5.1 in [5]), taking $|\cdot|_s$ to be the stable adapted semi-norm for $A|_V$, there is $R_0 > 0$ such that the set $\mathcal{F}^{R_0}$ of strands that are contained in the diameter R_0 cylinder about E^u,

$$\mathcal{C}^{R_0} := \{x \in V : |x|_s \le R_0\},$$

is forward invariant under Φ and eventually absorbs iterates of every strand that lies within bounded distance from E^u (i.e. $\forall_{R>0} \forall_{\gamma \in \mathcal{F}^R} \exists_{n \in \mathbb{N}} \ \Phi^n(\gamma) \in \mathcal{F}^{R_0}$).

Definition 2.2 *The strand space of ϕ is the space of bi-infinite orbits of Φ that stay within a bounded distance from E^u,*

$$\mathcal{F}_\phi^{\leftarrow} := \{(\gamma_k)_{k=-\infty}^{\infty} : \ \gamma_{k+1} = \Phi(\gamma_k), \ \gamma_k \in \mathcal{F}^{R_0}, \ k \in \mathbb{Z}\}. \tag{2.3}$$

In other words, $\mathcal{F}_\phi^{\leftarrow}$ is the inverse limit of Φ restricted to $\mathcal{F}_\phi := \bigcap_{n \in \mathbb{N}} \Phi^n(\mathcal{F}^{R_0})$, which intersection served as the definition (in [5]) of the strand space of ϕ in the irreducible unimodular case (i.e., when $\det(A) = \pm 1$ and the characteristic polynomial of A is irreducible over $\mathbb{Q}$). We shall check now that $\mathcal{F}_\phi^{\leftarrow}$ is just a presentation of the *tiling space* $\mathcal{T}_\phi$ associated to ϕ,

$$\mathcal{T}_\phi := \bigcap_{n \in \mathbb{N}} \Phi^n(\mathcal{T}),$$

where $\mathcal{T}$ stands for the space of equivalence classes of bi-infinite strands in $\mathbb{R}^n$ with two strands being equivalent iff they differ by a translation along $W \oplus E^s$. Note that $\mathcal{T}$ can be thought of as the quotient of $\mathcal{F}$ by the translations along E^s and that Φ factors to a map on $\mathcal{T}$, which we have denoted with the same letter. Also, the topologies taken on $\mathcal{F}$ and $\mathcal{T}$ are those of uniform convergence on compact subsets of V and $\mathbb{R}^n/(W \oplus E^s)$, respectively. In particular, both $\mathcal{F}_\phi^{\leftarrow}$ and $\mathcal{T}_\phi$ are a priori compact[5].

Proposition 2.3 *The natural projection $\mathcal{F}_\phi^{\leftarrow} \to \mathcal{T}_\phi$ given by $(\gamma_k) \mapsto \gamma_0$ mod E^s is a homeomorphism.*

Proof: First we show surjectivity. $\mathcal{T}_\phi$ always contains *simple inflation periodic tilings*, i.e., tilings of the form η (mod E^s) where the labeled strand η has 0 as a vertex and is fixed by Φ^m. Such a tiling is clearly the image of $(\eta_k) \in \mathcal{F}_\phi^{\leftarrow}$ where $\eta_k := \Phi^{k \bmod n}(\eta)$. Thus one concludes that all of $\mathcal{T}_\phi$ is in the image by virtue of the union of translation orbits of simple inflation periodic tilings being dense in $\mathcal{T}_\phi$. We used here Proposition 4.3 from [5].

Injectivity hinges on the fact that Φ induces a homeomorphism on $\mathcal{T}_\phi$, which is a consequence of Mosse's recogizability result [19]. Indeed, suppose that $(\gamma_k), (\gamma'_k) \in \mathcal{F}_\phi^{\leftarrow}$ are such that $\gamma_0 \equiv \gamma'_0 \mod E^s$. Then $\gamma_k \equiv \gamma'_k \mod E^s$ for all $k \in \mathbb{Z}$ by the bijectivity of Φ on $\mathcal{T}_\phi$. That is $\gamma'_k = \gamma_k + x_k$ where $x_k \in E^s \oplus W$; and $|x_k| \leq C$ for some $C > 0$ independent of k because $\gamma_k, \gamma'_k \in \mathcal{C}^{R_0}$. Hence, for $k \in \mathbb{Z}$ and $m \in \mathbb{N}$, we can write

$$\gamma'_k = \Phi^m(\gamma'_{k-m}) = \Phi^m(\gamma_{k-m} + x_{k-m}) = \Phi^m(\gamma_{k-m}) + A^m x_{k-m} = \gamma_k + A^m x_{k-m}. \quad (2.4)$$

Thus $x_k = A^m x_{k-m}$ allowing us to write $x_k = \lim_{m\to\infty} A^m x_{k-m} = 0$ where $|x_{k-m}| \leq C$ facilitated computation of the limit. This shows $(\gamma_k) = (\gamma'_k)$. □

From the proposition, $(\gamma_k) \mapsto \gamma_0$ is a homeomorphism between $\mathcal{F}_\phi^{\leftarrow}$ and $\mathcal{F}_\phi = \bigcap_{m\in\mathbb{N}} \Phi^m(\mathcal{F}^{R_0})$.Our preference for the inverse limit $\mathcal{F}_\phi^{\leftarrow}$ in the non-unimodular setting is somewhat idiosyncratic and has to do with the group serving as the geometric realization of $\mathcal{T}_\phi$ being an inverse limit itself. (Besides, in most arguments, individual strands γ_0 will be invariably accompanied by their Φ-orbits making the notation γ_k for $\Phi^k(\gamma)$ pleasantly compact.)

3 Natural Lattice and Geometric Realization

In this section we shall construct the appropriate compact abelian group to serve as the geometric realization of $\mathcal{F}_\phi^{\leftarrow}$. The obvious candidate is the inverse limit of

[5]The definition of the tiling space we have given is a bit non-standard; in particular, it allows for the possible existence of finitely many orbits under the tiling flow that are non-recurrent, orbits that aren't included in the ususal "hull" definition, c.f. [28].

the endomorphism induced by A on the torus V/Γ, but it is optimal to replace Γ with an intrinsic lattice Σ that reflects the recurrence of the translation flow on $\mathcal{F}_\phi^{\leftarrow}$. Thus constructed, the geometric realization will have the property that it is an a.e. one-to-one presentation of the tiling flow if and only if the Pisot Conjecture holds for ϕ (see Corollary 5.2). We start with Σ.

The *recurrence vectors* of the letter i are

$$\Theta(i) := \{v \in \Gamma : \ \exists_{(\gamma_k)\in\mathcal{F}_\phi^{\leftarrow}} \exists_{k\in\mathbb{Z}}\ \gamma_k \text{ contains edges } I, I' \text{ labeled } i \text{ and } I' = I + v\}. \tag{3.1}$$

Since, given $i, j \in \mathcal{A}$, $\phi^m(i)$ contains j for large enough m by primitivity of ϕ, $v \in \Theta(i)$ implies $A^m v \in \Theta(j)$ by considering repetitions of j in $(\Phi^m(\gamma_k))$. Hence, $\bigcup_{k\in\mathbb{Z}} A^k\Theta(i)$ is independent of i and so is the subgroup of V generated by it:

$$\Sigma_\infty := \Big\langle \bigcup_{k\in\mathbb{Z}} A^k\Theta(i) \Big\rangle. \tag{3.2}$$

Clearly , $A\Sigma_\infty = \Sigma_\infty$ and $\Sigma_\infty \subset \Gamma_\infty$ where $\Gamma_\infty := \bigcup_{n\geq 0} A^{-n}\Gamma$. This makes

$$\Sigma := \Sigma_\infty \cap \Gamma \tag{3.3}$$

an A-invariant sublattice of Γ (as the irreducibility of $A|_V$ over $\mathbb{Q}$ implies that ranks of Γ and Σ coincide) from which Σ_∞ can be recovered via

$$\Sigma_\infty = \bigcup_{k\geq 0} A^{-k}\Sigma. \tag{3.4}$$

If A is irreducible Pisot (i.e. $W = \{0\}$) then $\Sigma = \Gamma$ by the argument in [5] but that is not generally the case by the example at the end of this section.

For $i, j \in \mathcal{A}$, in view of $\Theta(i), \Theta(j) \subset \Sigma$, there is a well defined element $w_{ij} \in \Gamma/\Sigma$ with the following property: if $(\gamma_k) \in \mathcal{F}_\phi^{\leftarrow}$ and I, J are edges in γ_k of type i and j, respectively, then

$$\min J - \min I \pmod{\Sigma} = w_{ij}. \tag{3.5}$$

Because $(w_{ij})_{i,j\in\mathcal{A}}$ is a coboundary in the sense that $w_{ij} + w_{jk} + w_{ki} = 0$, there are $u_i \in \Gamma/\Sigma$, $i \in \mathcal{A}$, such that

$$w_{ij} = u_j - u_i. \tag{3.6}$$

Observe that, if $i \in \mathcal{A}$ and i' is the first letter of $\phi(i)$, then

$$\tau := Au_i - u_{i'} \tag{3.7}$$

is independent of i. (Indeed, $Aw_{ij} = w_{i'j'}$ obtains by applying of Φ to γ_k.) Now, u_i being only unique up to an additive constant allows normalization $\tau = 0$. This entails replacing u_i by $u_i - (A - I)^{-1}\tau$ where we rely on the Pisot hypothesis for the existence of $(A - I)^{-1}$.[6]

[6]We compute $A(u_i-(A-I)^{-1}\tau)-(u_{i'}-(A-I)^{-1}\tau) = Au_i-u_{i'}-(A-I)(A-I)^{-1}\tau = \tau-\tau = 0$.

Denote by $\mathbb{T}_A$ the inverse limit of the endomorphism $A : V/\Sigma \to V/\Sigma$ induced by A, i.e.,

$$\mathbb{T}_A := \{(p_k)_{k=-\infty}^{\infty} : \ p_{k+1} = Ap_k, \ p_k \in V/\Sigma, \ k \in \mathbb{Z}\}. \tag{3.8}$$

One readily verifies that the following is an unambiguous definition.

Definition 3.1 *The geometric realization of $\mathcal{F}_\phi^{\leftarrow}$ is the map $h_\phi : \mathcal{F}_\phi^{\leftarrow} \to \mathbb{T}_A$ sending $(\gamma_k)_{k=-\infty}^{\infty}$ to $(p_k)_{k=-\infty}^{\infty}$ given by*

$$p_k := \min I - u_i, \ \ k \in \mathbb{Z}, \tag{3.9}$$

where I is an edge of γ_k with a label i (the choices of i and I being immaterial).

Thus defined, h_ϕ factors the dynamics on $\mathcal{F}_\phi^{\leftarrow}$ to nice algebraic actions on $\mathbb{T}_A$. First, we have a commuting diagram

$$\begin{array}{ccc} \mathcal{F}_\phi^{\leftarrow} & \xrightarrow{\Phi} & \mathcal{F}_\phi^{\leftarrow} \\ {\scriptstyle h_\phi}\downarrow & & {\scriptstyle h_\phi}\downarrow \\ \mathbb{T}_A & \xrightarrow{A} & \mathbb{T}_A \end{array} \tag{3.10}$$

where the automorphism denoted by A on $\mathbb{T}_A$ is given by $(p_k)_{k=-\infty}^{\infty} \mapsto (Ap_k)_{k=-\infty}^{\infty} = (p_{k+1})_{k=-\infty}^{\infty}$. Second, the natural translation action on $\mathcal{F}_\phi^{\leftarrow}$ whereupon the strands are translated in the direction of E^u,

$$T^t : (\gamma_k)_{k=-\infty}^{\infty} \mapsto (\gamma_k + \lambda^k t\omega)_{k=-\infty}^{\infty}, \ \ t \in \mathbb{R}, \tag{3.11}$$

factors down to the translation $T_\omega^t : (p_k)_{k=-\infty}^{\infty} \mapsto (p_k + \lambda^k t\omega)_{k=-\infty}^{\infty}$ along the one parameter dense subgroup $\mathcal{E}^u := \{(\lambda^k t\omega)_{k=-\infty}^{\infty} : \ t \in \mathbb{R}\} \subset \mathbb{T}_A$, i.e.,

$$\begin{array}{ccc} \mathcal{F}_\phi^{\leftarrow} & \xrightarrow{T^t} & \mathcal{F}_\phi^{\leftarrow} \\ {\scriptstyle h_\phi}\downarrow & & {\scriptstyle h_\phi}\downarrow \\ \mathbb{T}_A & \xrightarrow{T_\omega^t} & \mathbb{T}_A \end{array} \tag{3.12}$$

The inflation-substitution homeomorphism $\Phi : \mathcal{F}_\phi^{\leftarrow} \to \mathcal{F}_\phi^{\leftarrow}$ has a natural Markov partition with the transition matrix given by A, which makes it almost homeomorphically conjugate to a mixing Markov chain. The measure of maximal entropy of Φ serves as the invariant measure of the the tiling flow $T^t : \mathcal{F}_\phi^{\leftarrow} \to \mathcal{F}_\phi^{\leftarrow}$, which is uniquely ergodic (see e.g. [28]). Also, T^t is minimal on the complement $\mathcal{F}_{\phi,\min}^{\leftarrow}$ of a finite number of wandering orbits (via Proposition 3.5 in [5].)

We finish this section with a simple example for which Γ is finer then Σ.

Example 3.5 *(where $\Gamma \neq \Sigma$):* Consider $\phi : 1 \mapsto 12323, \ 2 \mapsto 1232, \ 3 \mapsto 323$. We have

$$A := \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

with the characteristic polynomial $(x-1)(x^2-4x+1)$. Taking $b_1 := [0,1,1]^T, b_2 := [1,0,0]^T, a := [1,0,-2]^T$, we have $V := \text{lin}(b_1, b_2)$, $W = \text{lin}(a)$ with $A|_V$ represented by

$$B := \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}.$$

One computes $\text{pr}_V(e_1) = e_1$, $\text{pr}_V(e_2) = [-1/2, 1, 1]^T$, $\text{pr}_V(e_3) = [1/2, 0, 0]^T$. Thus $\Gamma = \text{pr}_V(\mathbb{Z}^3) = \langle [-1/2, 1, 1]^T, [1/2, 0, 0]^T \rangle$. At the same time, upon renaming $c = 23$, ϕ factors to $\psi : 1 \mapsto 1cc,\ 2 \mapsto 1ccc$. Therefore, consecutive repetitions of 1 are separated by a word that is a power c^m and so the vectors of $\Theta(1)$ have the form $e_1 + m(\text{pr}_V(e_2) + \text{pr}_V(e_3)) = [1, m, m]^T \in \mathbb{Z}^3$ where $m \in \mathbb{N}$. Since both A and A^{-1} map $\mathbb{Z}^3$ to itself, we have $\Sigma \subset \mathbb{Z}^3$. □

Let us add that the above example arises by taking the toral automorphism associated to B and constructiong the Markov partition by cutting $\mathbb{T}^2$ into three boxes along the stable manifold of $[0, 0]$ and the unstable manifolds of $[0, 0]$ and $[1/2, 0]$.

4 Geometric Coincidence Condition

In this section we study the fiber of h_ϕ and develop a suitable *Geometric Coincidence Condition* allowing for algorithmic verification whether or not the geometric realization h_ϕ is a measure theoretical isomorphism for any given ϕ. It is conjectured that h_ϕ is an isomorphism for all Pisot ϕ that are irreducible (i.e. $\deg(p_{\min}) = n$) or arise from β-expansions. At the same time, as soon as one abandons the irreducibility hypothesis, h_ϕ may fail to be a.e. $1-to-1$ as exemplified by the Morse substitution and Example 5.3 ahead, in which the dominant eigenvalue is a Pisot unit (the product of λ with its conjugates is ± 1).

We say that a (finite or infinite) labeled strand γ *lies over* $p \in V/\Sigma$ iff $\min I - u_i \in p + \Sigma$ where I is any edge of γ and i is its label (and u_i is as in (3.6)).

Two labeled strands γ, η are *coincident*, denoted $\gamma \sim \eta$, iff $\Phi^k(\gamma)$ and $\Phi^k(\eta)$ share a labeled edge for some $k \geq 0$.

Definition 4.1 *The coincidence rank of ϕ, denoted by cr_ϕ, is the maximal number of strands in $\mathcal{F}$ that lie over the same point of V/Σ and no two of which are coincident with each other. We say that the Geometric Coincidence Condition (GCC) holds for ϕ iff $cr_\phi = 1$.*

The a priori finiteness and algorithmic computability of cr_ϕ will be made apparent by Remark 4.3 ahead.

Theorem 4.2 (Coincidence Theorem) *The geometric realization map h_ϕ is uniformly finite-to-one (i.e. $\exists_{C>0} \forall_{p \in \mathbb{T}_A} \# h_\phi^{-1}(p) \leq C$) and almost everywhere cr_ϕ-to-1. Precisely, there is a full Haar measure G_δ-subset $G_\phi^u \subset \mathbb{T}_A$ such that, for*

$p \in G^u_\phi$, *we have*

$$\# h_\phi^{-1}(p) = \min\{\# h_\phi^{-1}(q) : \ q \in \mathbb{T}_A\} = cr_\phi. \tag{4.1}$$

Moreover, the map $p \mapsto h_\phi^{-1}(p)$ *is continuous at* $p \in G^u_\phi$ *and, if* $h_\phi((\gamma_k)) = h_\phi((\eta_k)) \in G^u_\phi$ *for* $(\gamma_k) \neq (\eta_k)$*, then* γ_k *and* η_k *are noncoincident for every* $k \in \mathbb{Z}$*.*

Regarding the last assertion of the theorem, we point out that, given arbitrary $(\gamma_k), (\eta_k) \in \mathcal{F}_\phi^{\leftarrow}$, we have an obvious equivalence:

$$\exists_{k\in\mathbb{Z}} \ \gamma_k \sim \eta_k \iff \forall_{k\in\mathbb{Z}} \ \gamma_k \sim \eta_k. \tag{4.2}$$

Also, G^u_ϕ is a priori invariant under A and the flow T^t_ω.

Proof of Theorem 4.2: Let us start with some preliminary observations. Given $c \in \mathbb{N}$, consider points p with the fiber $h_\phi^{-1}(p)$ that can be covered by c balls of small diameter; precisely, for $N \in \mathbb{N}$, we set

$$U_{N,c} := \{p \in \mathbb{T}_A : \ \# \pi_{-N}(h_\phi^{-1}(p))|_{-N}^N \leq c\} \tag{4.3}$$

where $\pi_{-N}((\gamma_k)) := \gamma_{-N}$ and $\eta|_{-N}^N$ denotes the *central substrand of* η of the unstable length $2N$ (i.e. the smallest substrand of η containing ${\mathrm{pr}_u}^{-1}[-N\omega, N\omega] \cap \eta$). One readily checks that $U_{N,c}$ is A-invariant and that

$$\{p : \ \# h_\phi^{-1}(p) \leq c\} = \bigcap_{N\in\mathbb{N}} \mathrm{int}(U_{N,c}). \tag{4.4}$$

Thus $\{p : \ \# h_\phi^{-1}(p) \leq c\}$ is a full measure G_δ for $c := \min\{\# h_\phi^{-1}(q) : \ q \in \mathbb{T}_A\}$ by ergodicity of $A : \mathbb{T}_A \to \mathbb{T}_A$.

To show $c \leq cr_\phi$, it suffices to prove the last assertion of the theorem; namely, that if $h_\phi^{-1}(p) = \{\gamma^1, \ldots, \gamma^c\}$ is a fiber of minimal possible cardinality then γ^i_k and $\gamma^{i'}_k$ are noncoincident for $i \neq i'$ and $k \in \mathbb{Z}$.

Suppose that $\gamma^i_{k_0}$ and $\gamma^{i'}_{k_0}$ are coincident for some $i \neq i'$ and $k_0 \in \mathbb{Z}$. Then for large enough $r \in \mathbb{N}$, $\gamma^i_{k_0+r} = \Phi^r(\gamma^i_{k_0})$ and $\gamma^{i'}_{k_0+r} = \Phi^r(\gamma^{i'}_{k_0})$ contain a common finite labeled substrand of length increasing to ∞ as $r \to \infty$. Upon replacing p by its translate $T^t_\omega(p)$, if necessary, we may require that the common finite substrand intersects E^s. Now, given any $N \in \mathbb{N}$, by taking sufficiently large $r \in \mathbb{N}$, the $\xi^i \in \mathcal{F}_\phi$ defined by

$$\xi^i_k := \Phi^r(\gamma^i_{k_0+k}), \ k \in \mathbb{Z}, \tag{4.5}$$

have the property that $\#\{\xi^i_{-N}|_{-N}^N : \ i = 1, \ldots, c\} \leq c - 1$ because some two of the labeled strands have their central $|_{-N}^N$ substrands coalesced into one. Since, by construction, $\{\xi^i : \ i = 1, \ldots, c\} = h_\phi^{-1}(q)$ where $q = A^{r+k_0}p$, we see that $U_{N,c-1} \neq \emptyset$, and thus it is of full measure. In this way, $\{p : \ \# h_\phi^{-1}(p) \leq c - 1\} = \bigcap_{N\in\mathbb{N}} U_{N,c-1} \neq \emptyset$ contradicting the minimality of c.

Second, we see that cr_ϕ is a lower bound on the cardinality of the fiber $h_\phi^{-1}(p)$. Suppose that no two of the strands $\eta^1, \ldots, \eta^{cr_\phi} \in \mathcal{F}$ are coincident and they lie over the same point of V/Σ. Then, for any $m \in \mathbb{N}$, the same can be said about $\Phi^m(\eta^1), \ldots, \Phi^m(\eta^{cr_\phi}) \in \mathcal{F}$ as well as about $\gamma^1, \ldots, \gamma^{cr_\phi} \in \mathcal{F}$ obtained as limits $\gamma^i = \lim_{j\to\infty} \Phi^{m_j}\eta^i$ (provided the limits exist). Now, choose[7] the sequence $m_j \to \infty$ so that, for every $k \in \mathbb{Z}$, $\Phi^{m_j+k}(\eta^i)$ converges, and denote the limit by γ^i_k. By this construction, the $\gamma^i := (\gamma^i_k)_{k=-\infty}^{\infty}$ belong to $\mathcal{F}_\phi^{\leftarrow}$ and map under h_ϕ to the same point $p \in \mathbb{T}_A$. Also, the γ^i are distinct with some definite distance separating any two, as follows from the fact that, for $i \neq j$ and $k \in \mathbb{Z}$, γ^i_k and γ^j_k are noncoincident and thus do not share any labeled edges.

Since any $q \in \mathbb{T}_A$ is a limit $q = \lim_{j\to\infty} T_\omega^{t_j}(p)$ for some $t_j \to \infty$, we conclude that $h_\phi^{-1}(q)$ must contain the set of the cr_ϕ distinct elements of $\mathcal{F}_\phi^{\leftarrow}$ obtained as a (Hausdorff) limit point of the sequence of sets $h_\phi^{-1}(T_\omega^{t_j}(p)) = \{T^{t_j}\gamma^1, \ldots, T^{t_j}\gamma^{cr_\phi}\}$. This shows that $\min\{\#h_\phi^{-1}(q) : \; q \in \mathbb{T}_A\} \geq cr_\phi$.

As to the global bound on the cardinality of the fiber, consider $M \in \mathbb{N}$, and suppose that $h_\phi^{-1}(p) \geq M$ for some $p \in \mathbb{T}_A$. Because, $\pi_0 : \mathcal{F}_\phi^{\leftarrow} \to \mathcal{F}$ is a homeomorphism onto its image (by Proposition 2.3), we have $\#\pi_0(h_\phi^{-1}(p)) \geq M$ and thus also $\#\pi_0(h_\phi^{-1}(p))|_{-N}^{N} \geq M$ for some $N \in \mathbb{N}$. However, for large enough $m \in \mathbb{N}$, Φ^m maps $\pi_{-m}(h_\phi^{-1}(p))|_{-1}^{1}$ to a family of substrands which properly contain the substrands in $\pi_0(h_\phi^{-1}(p))|_{-N}^{N}$. Thus $M \leq \#\pi_0(h_\phi^{-1}(p))|_{-N}^{N}$ cannot exceed the maximal number of strands of the form $\eta|_{-1}^{1}$ contained in $\mathcal{C}^{R_0}$ and lying over the same point of V/Σ.

Finally, upper semicontinuity $p \mapsto h_\phi^{-1}(p)$ at any p is a general property of continuous mappings between compact spaces. We leave it to the reader to see that, if the lower semicontinuity failed at p then p could not have a minimal cardinality fiber. □

Before leaving, let us characterize cr_ϕ in terms of coincidence of individual edges and thus reconnect with the development in the irreducible unimodular case (Definition 7.1 in [5]). Thus, for $q \in V/\Sigma$, we introduce the set of *states over* q:

$$\mathbb{S}_q := \{I : \;\; I \text{ is an edge over } q \text{ and } (I \setminus \max I) \cap E^s \neq \emptyset\} \tag{4.6}$$

and its finite subset $\mathbb{S}_q^{R_0} := \{I \in \mathbb{S}_q : \; I \subset \mathcal{C}^{R_0}\}$. Of course every strand $\gamma \in \mathcal{F}$ determines a state, denoted by $\hat{\gamma}$.

Remark 4.3 *For any $q \in V/\Sigma$, cr_ϕ coincides with the maximal cardinality of a pairwise non-coincident family in $\mathbb{S}_q$. Moreover, such a family of cr_ϕ states can be found in $\mathbb{S}_q^{R_0}$.*

In particular, cr_ϕ can be algorithmically computed along the lines of Proposition 17.1 in [5].

[7]By using a diagonal argument and compactness of the space of strands contained in $\mathcal{C}^{R_0}$.

Proof of Remark 4.3: Fix $q \in V/\Sigma$. Let c be the maximum cardinality of a pairwise non-coincident family in $\mathbb{S}_q$.

Pick any $p = (p_k) \in G_\phi^u$ and let $h_\phi^{-1}(p) = \{\gamma^1, \dots, \gamma^{cr_\phi}\}$. The strands $\gamma_0^1, \dots, \gamma_0^{cr_\phi}$ are strictly inside $\mathcal{C}^{R_0}$ and are pairwise non-coincident. Since E^u winds densely in V/Σ, we can find $x \in V$ so that $p_0 + x = q$ and $\gamma_0^1 + x, \dots, \gamma_0^{cr_\phi} + x$ are still in $\mathcal{C}^{R_0}$. Thus $\{\widehat{\gamma_0^1 + x}, \dots, \widehat{\gamma_0^{cr_\phi} + x}\}$ is a non-coincident family in $\mathbb{S}_q^{R_0}$. In particular, $c \geq cr_\phi$.

For the opposite inequality, suppose that $\{I_1, \dots, I_c\}$ is a non-coincident family in $\mathbb{S}_q$. After possibly performing a small translation along E^u, we can assume that the I_i intersect E^s in an interior point. Thus, as $m \to \infty$, the $\Phi^m(I_k)$ grow indefinitely on both sides of E^s (i.e. $\mathrm{pr}_u(\Phi^m(I_k))$ converges to $\mathbb{R}$), and we can repeat the arguments of the third paragraph of the proof of the theorem to construct limiting bi-infinite strands $\gamma_k^i := \lim_{j\to\infty} \Phi^{m_j+k}(I_i)$, $i = 1, \dots, c$, so that $(\gamma_k^i)_{k\in\mathbb{Z}}$ are in the same fiber of h_ϕ and $\gamma_k^i \not\sim \gamma_k^j$ for $i \neq j$, which implies $c \leq cr_\phi$. $\square$

5 Discrete Spectrum

In this section, we identify the discrete spectrum of the tiling flow T^t and show that T^t has pure discrete spectrum iff $cr_\phi = 1$. We then use the result to exhibit a Pisot substitution with λ a unit for which T^t fails to have pure discrete spectrum.

Recall first some fundamentals regarding pure discrete spectrum of the algebraic flow $T_\omega^t : \mathbb{T}_A \to \mathbb{T}_A$, $T_\omega^t : (p_k) \mapsto (p_k + \lambda^k t\omega)$. We shall use the linear dual V^T of V realized as the subspace of $\mathbb{R}^n$ orthogonal to W, $V^T := W^\perp$, so that the ordinary dot product $\langle\cdot|\cdot\rangle$ provides the pairing $V^T \times V \to \mathbb{R}$. V^T is invariant under the action of the transpose A^T and so is the ***dual lattice*** of Σ defined by

$$\Sigma^* := \{u \in V^T : \ \langle u|v\rangle \in \mathbb{Z} \ \text{ for all } v \in \Sigma\}. \tag{5.1}$$

The subgroup of V^T given by

$$\Sigma^*_\infty := \bigcup_{l\geq 0} (A^T)^{-l}\Sigma^* \tag{5.2}$$

is the Pontryagin dual of $\mathbb{T}_A$; the characters on $\mathbb{T}_A$ are indexed by $u \in \Sigma^*_\infty$ and given on $p = (p_k) \in \mathbb{T}_A$ by

$$\chi_u(p) := \exp(\langle (A^l)^T u | p_{-l}\rangle) \tag{5.3}$$

where $\exp(t) := e^{2\pi i t}$ and $l \in \mathbb{N}$ is taken sufficiently large so that $(A^l)^T u \in \Sigma^*$ (which makes the scalar product well defined). Each χ_u, $u \in \Sigma^*_\infty$, is an eigenfunction for the flow T_ω^t with the eigenvalue $\langle u|\omega\rangle$; indeed,

$$\chi_u(T_\omega^t p) = \exp(\langle (A^l)^T u | p_{-l} + t\lambda^{-l}\omega\rangle) = \exp(t\langle u|\omega\rangle)\chi_u(p), \quad p \in \mathbb{T}_A,\ t \in \mathbb{R}.$$

Theorem 5.1 *The eigenvalues of the tiling flow T^t consist of numbers $\langle u|\omega\rangle$ where $u \in \Sigma^*_\infty$, and $\chi_u \circ h_\phi$ serves as an eigenfunction corresponding to the eigenvalue $\langle u|\omega\rangle$.*

By observing that functions $\chi_u \circ h_\phi$ are constant on the fibers of h_ϕ and thus cannot form a dense subset[8] of $L^2(\mathcal{F}^{\leftarrow}_\phi)$ unless h_ϕ is a measure theoretical isomorphism, Theorem 5.1 can be combined with Theorem 4.2 to yield the following counterpart of Corollary 9.4 in [5].

Corollary 5.2 *The tiling flow T^t has pure discrete spectrum iff $cr_\phi = 1$.*

In the argument below, $\omega^* > 0$ is a Perron eigenvector of A^T satisfying $A^T\omega^* = \lambda\omega^*$ and the normalization $\langle\omega|\omega^*\rangle = 1$. Thus $\mathrm{pr}_u(v) = \langle v|\omega^*\rangle\omega$ for $v \in V$. Also, recall that $\mathcal{F}^{\leftarrow}_{\phi,\min}$ denotes the unique subset of $\mathcal{F}^{\leftarrow}_\phi$ minimal under the tiling flow T^t.

Proof of Theorem 5.1: That every α of the postulated form is an eigenvalue is clear from (3.12) so we concentrate on showing the converse.

First we shall use the duality between eigenvalues and the return times — going back at least to [15, 28, 20, 31] — to show that, for any eigenvalue α of T^t and any $t := \langle v|\omega^*\rangle$ with $v \in \Sigma$, we have

$$\lim_{n\to\infty} \exp(\lambda^n \alpha t) = 1. \tag{5.4}$$

Since, for a fixed α, the set of t for which (5.4) holds is a priori an additive group, it suffices to argue for $v \in \bigcup_{k\in\mathbb{Z}} A^k\Theta(i)$. For such v, we can find $\gamma = (\gamma_k) \in \mathcal{F}^{\leftarrow}_{\phi,\min}$ so that $\gamma_0 \sim_{t\omega} \gamma_0 + v$ for all sufficiently small t. Thus $\mathrm{dist}(\Phi^m(\gamma_0), \Phi^m(\gamma_0 + v)) \to 0$ as $m \to \infty$ (since the two strands coincide on a progressively longer central substrand). From $t = \langle v|\omega^*\rangle$, we see that $\gamma_0 + v$ and $\gamma_0 + t\omega$ differ by a translation along E^s and thus also $\mathrm{dist}(\Phi^m(\gamma_0), \Phi^m(\gamma_0 + t\omega)) \to 0$. It follows that

$$\lim_{m\to\infty} \mathrm{dist}(\Phi^m(\gamma), \Phi^m \circ T^t(\gamma)) = 0. \tag{5.5}$$

Moreover, since the containment $v \in Z_{\gamma_0}$ persists under a small perturbation of γ_0, we see that (5.5) holds on an open set of $\gamma \in \mathcal{F}^{\leftarrow}_{\phi,\min}$. This is to say that t is a *homoclinic return time* for a positive measure set of $\gamma \in \mathcal{F}^{\leftarrow}_{\phi,\min}$ and we can invoke Lemma 13.1 in [5] to conclude (5.4).

Now, characterizing α based on (5.4) is the object of the *Pisot theory*. Specifically, as in Lemma 13.3 in [5], one writes $\alpha = \langle u|\omega\rangle$ for some $u \in V^T$ and computes

$$\lambda^m t\alpha = \lambda^m\langle v|\omega^*\rangle\langle u|\omega\rangle = \langle A^m v|\omega^*\rangle\langle u|\omega\rangle = \langle A^m v|u\rangle - \langle A^m v - \langle A^m v|\omega^*\rangle\omega|u\rangle \tag{5.6}$$

to conclude that $\exp(\langle A^m v|u\rangle) \to 1$ by observing that $A^m v - \langle A^m v|\omega^*\rangle\omega = A^m \mathrm{pr}_s(v)$ decays exponentially. Now, Remark 1 in [18] applied to the action of A^T

[8] Use here that h_ϕ is locally injective on G^u_ϕ (from Theorem 4.2), cf. the proof of Corollary 9.4 in [5].

restricted to V^T, asserts that $u \in \bigcup_{m\geq 0}(A^T)^{-m}L_v^* + E_T^s$ where $E_T^s := \mathrm{lin}(\omega)^\perp$ is the stable space of $A^T|_{V^T}$ and L_v is the smallest sublattice of Σ that contains $A^m v$ for all $m \in \mathbb{N}$. By arbitrariness of $v \in \Sigma$, we must in fact have $u \in \bigcup_{m\geq 0}(A^T)^{-m}\Sigma^* + E_T^s$. Thus $\alpha = \langle u|\omega\rangle$ belongs to $\langle \Sigma_\infty^*|\omega\rangle$, which is what we set out to prove. □

Reacall that it is conjectured that T^t has pure discrete spectrum for any Pisot ϕ for which the abelianization A is irreducible. That the hypothesis of irreducibility is necessary is demonstrated by the well known example of the Morse substitution for which $\lambda = 2$ and h_ϕ can be easily seen to be a.e. two-to-one. To further clarify the role of irreducibility, we give below an example with λ that is a Pisot unit and h_ϕ is a.e. two-to-one.

Example 5.3: The idea is to take the tiling space assocaited to a Markov partition for a pseudo Anosov map that is a ramified covering of an Anosov automorphism on $\mathbb{T}^2$. Take then the toral automorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ induced by

$$B = \begin{pmatrix} 5 & 1 \\ 4 & 1 \end{pmatrix}.$$

Observe that $p = (0,0)$ and $q = (0, \frac{1}{2}) \pmod{\mathbb{Z}^2}$ are two fixed points. Cutting $\mathbb{T}^2$ along the stable and unstable manifolds of these fixed points, as indicated at the bottom of Figure 5.1, produces a Markov partition for which the associated substitution is easily seen to be

$$\psi : 1 \mapsto 132111, \qquad 2 \mapsto 132211, \qquad 3 \mapsto 32211. \tag{5.7}$$

The tilings of $\mathbb{R}$ constructed by associating to $p \in \mathbb{T}^2$ the embedding of $\mathbb{R}$ into a coset of $E^u + \mathbb{Z}^2$ via $t \mapsto t\omega + p$ and then decomposing $\mathbb{R}$ into intersections with individual Markov boxes are exactly those making up the tiling space $\mathcal{T}_\psi$ (with the possible exception of the countably many cosets containing whole boundary segments). This gives a measure theoretical isomorphism of $\mathcal{T}_\psi \simeq \mathcal{F}_\phi^{\leftarrow}$ to $\mathbb{T}^2$ that conjugates the tiling flow T_ψ^t to the Kronecker flow on $\mathbb{T}^2$.

Now, consider a genus two surface M presented as the pair of (near) rectangles depicted in the upper portion of Figure 5.1 and the ramified covering $\pi : M \to \mathbb{T}^2$ that identifies the corresponding points of the left and right rectangle. One checks (by applying the standard lifting theorem to π over the doubly puctured torus $\mathbb{T}^2 \setminus \{p, q\}$) that f lifts to $g : M \to M$, $f \circ \pi = \pi \circ g$. Thus obtained g is pseudo-Anosov with two four prong singularities of the stable/unstable foliations at p and q. The Markov partition of three boxes for f lifts to one of six boxes for g, and the associated substitution can be found to be:

$$\phi : 1 \mapsto 162111,\ 2 \mapsto 435211,\ 3 \mapsto 35211,\ 4 \mapsto 435444,\ 5 \mapsto 162544,\ 6 \mapsto 62544. \tag{5.8}$$

The (a.e. defined) holonomy flow on M along the leaves of the unstable foliation that factors via π to the Kronecker flow T_ω^t on $\mathbb{T}^2$ is measure theoretically conjugated to the tiling flow T_ϕ^t on $\mathcal{T}_\phi \simeq \mathcal{F}_\phi^{\leftarrow}$. (The isomorphism is constructed by assocaiting to $p \in M$ the tiling of the unstable manifold of p into intersection segments with the Markov partition.) We claim that this flow does not have pure discrete spectrum. One way to see this is to check that, modulo the above natural isomorphisms, π is the geometric realization map h_ϕ and so $cr_\phi = 2$. Another entails checking that the homoclinic return times of the holonomy flow on M are exacty those of T^t_ω, concluding that the discrete spectra of the two flows coincide (by using the ideas of the proof of Theorem 5.1), and observing that $L^2(M)$ cannot possibly be in the closed linear span of the eigenfunctions that must be the toral harmonics lifted from $\mathbb{T}^2$ to M via π. We leave the details as an exercise.

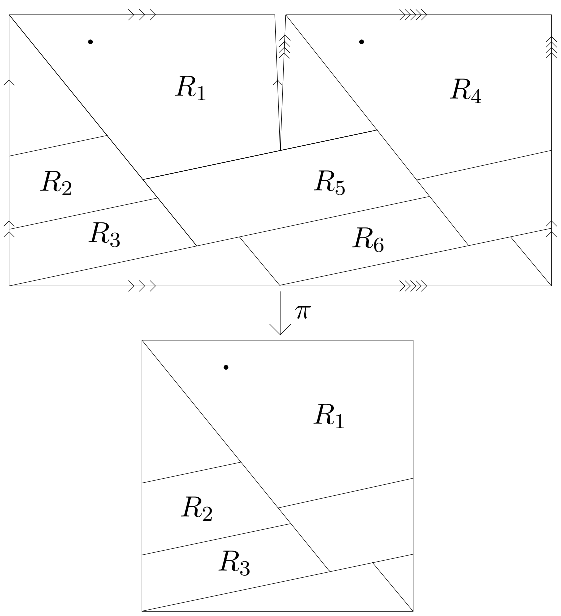

Figure 5.1: Two-to-one ramified covering of a toral automorphism.

6 Criteria for Coincidence Condition

In verifying the GCC one is greatly aided by the following result that again already appears in [5] in the irreducible unimodular context. Below, we use $\sim_{t\omega}$ for the equivalence relation of *coincidence along* $t\omega + E^s$: $\gamma \sim_{t\omega} \eta$ iff there is $k \geq 0$ such that $\Phi^k(\gamma)$ and $\Phi^k(\eta)$ share a labeled edge J and $J \setminus \max J$ intersects $\lambda^k t\omega + E^s$.

Let us precede the technical development by a rough outline. GCC holds in case $K \sim_{t\omega} L$ is typical for states K and L over an arbitrary point in the torus and small t. Theorem 6.1 bellow asserts that to guarantee that coincidence is typical it suffices to establish the existence of states, arbitrarily distant from each other, that are coincident along $t\omega + E^s$ for a dense G_δ set of $t \in [-\epsilon, \epsilon]$ for some fixed $\epsilon > 0$. The idea is that, if the latter condition holds, then repeated inflation will produce long strands that are generically coincident along $t\omega + E^s$ generically for

$t \in [-T, T]$ where T large. Carefully taking limits, this leads to $\gamma \in \mathcal{F}^{\leftarrow}_{\phi,\min}$ with $\gamma_0 \sim_{t\omega} \gamma_0 + v$ for generic $t \in \mathbb{R}$ and some nonzero $v \in \Sigma$. The irreducibility (over $\mathbb{Q}$) of the action of A on V forces the set of such v to be a finite index subgroup of Σ. Transitivity of the powers of Φ then forces this subgroup to equal Σ, from which $cr_\phi = 1$ follows immediately.

Theorem 6.1 *Suppose that $cr_\phi > 1$. For any $\epsilon > 0$ there is $D > 0$ such that if $K, L \in \mathbb{S}_p$, $p \in V/\Sigma$, and $K \sim_{t\omega} L$ for a dense G_δ set of $t \in [-\epsilon, \epsilon]$, then* $\operatorname{dist}(K, L) < D$.

The proof of the theorem will require some buildup including the following generalization of Lemma 10.1 in [5].

Lemma 6.2 *There is a full measure dense G_δ set $G^s_\phi \subset V/\Sigma$ that is invariant under translations along E^s and under the toral endomorphism induced by A and such that, for $p \in G^s_\phi$, we have*

(i) $\mathbb{S}_p$ consist of exactly cr_ϕ equivalence classes of $\sim_0$;

(ii) if $I \sim_0 J$ for $I, J \in \mathbb{S}_p$, then $I + z \sim_0 J + z$ for all sufficiently small $z \in V$;

(iii) there is an $R_1 > 0$ such that, for any $y \in E^s$, each equivalence class in $\mathbb{S}_p$ has a representative contained in the cylinder $y + \mathcal{C}^{R_1}$.

Proof of Lemma 6.2: We repeat the proof of Lemma 10.1 in [5], with obvious modifications, for the convenience of the reader.

Note first that there is $R_1 > 0$ so that, for any $p \in V/\Sigma$ and $R > R_1$, $\mathbb{S}^R_p$ has at least cr_ϕ equivalence classes of $\sim_0$. Indeed, if we fix any $q \in G^u_\phi$, write $p = q_0 + x$ for some x in a bounded fundamental domain of V/Σ, and let $h^{-1}_\phi(q) = \{\gamma^1, \ldots, \gamma^c\}$, $c = cr_\phi$, then $\{\widehat{\gamma^1_0 + x}, \ldots, \widehat{\gamma^c_0 + x}\}$ is a non-coincident family in $\mathbb{S}^{R_0+|x|_s}_p$.

To construct G^s_ϕ, for $R > R_1$, we define

$$D^n_R := \{p \in V/\Sigma : \ \#\widehat{\Phi^n(\mathbb{S}^R_p)} \le cr_\phi\}, \quad D_R := \bigcup_{n>0} D^n_R, \quad D := \bigcap_{R>0} \operatorname{int}(D_R). \quad (6.1)$$

Thus, from the definition of $\sim_0$, $p \in D_R$ iff $\mathbb{S}^R_p$ has at most (and thus, for $R > R_1$, exactly) cr_ϕ equivalence classes of $\sim_0$; and $p \in D$ iff $\mathbb{S}^R_p$ has exactly cr_ϕ such classes "stably" under small perturbation of p for any $R > R_1$. Note that D is E^s-invariant.

From now on we consider $R > R_1$. We claim that D_R is dense. Indeed, otherwise there would be $p \in V/\Sigma$, $\epsilon > 0$ and (since $\mathbb{S}^R_p$ is finite) a single $I \in \mathbb{S}^R_p$ such that $I \not\sim_{t\omega} (\gamma^i_0 + x)^\wedge$ for all $i = 1, \ldots, cr_\phi$ and all t with $|t| < \epsilon$ (where $\gamma^i_0 + x$ is as before). By applying Φ^m to I and the $\gamma^i_0 + x$ for n large enough (so that, say, $\lambda^m \epsilon > 100$), we would then get $cr_\phi + 1$ strands intersecting E^s along pairwise noncoincident states — in contradiction with Remark 4.3.

Moreover, we claim $D_R \subset \overline{\mathrm{int}(D_R)}$. Indeed, $p \in D_R$ means exactly that there is $m \in \mathbb{N}$ such that $\#\widehat{\Phi^m(\mathbb{S}_p^R)} = cr_\phi$. But then $\#\widehat{\Phi^m(\mathbb{S}_{\tilde{p}}^R)} = cr_\phi$ for all $\tilde{p} := p - t\omega$ where $0 \le t < \epsilon$ and $\epsilon > 0$ is sufficiently small. Coupled with E^s-invariance of $\sim_0$, this yields $\#\widehat{\Phi^m(\mathbb{S}_{\tilde{p}}^R)} = cr_\phi$ for all $\tilde{p}$ in a neighborhood of $p - \frac{\epsilon}{2}\omega$ thus placing $p - \frac{\epsilon}{2}\omega$ in $\mathrm{int}(D_R)$.

So far we know that $\mathrm{int}(D_R)$ is a dense open set. At the same time, for $R > R_0$, $\Phi(\widehat{\mathbb{S}_{A^{-1}p}^R}) \subset \mathbb{S}_p^R$ yields $A^{-1}(D_R) \subset D_R$, so $\mathrm{int}(D_R)$ is in fact of full measure by ergodicity of the toral endmorphism A. Thus D is a full measure dense G_δ invariant under actions of E^s and A, and so is

$$G_\phi^s := D \setminus (E^s + \Sigma) .$$

(i) follows immediately from $G_\phi^s \subset D$ and the construction of D.

(ii) alone can be easily seen to hold for all $p \notin E^s + \Sigma$.

As for (iii), we deal first with the special case of the cylinder centered at $y = 0$. From our initial discussion, we know that $\mathbb{S}_p^{R_1}$ contains representatives of cr_ϕ equivalence classes for every $p \in V/\Sigma$. For $p \in D$, there are no more classes in $\mathbb{S}_p$ and thus all are represented in $\mathbb{S}_p^{R_1}$.

To get (iii) in full generality, we translate along E^s: for $y \in E^s$, all the states of $\mathbb{S}_p$ in $y + \mathcal{C}^{R_1}$ constitute $\mathbb{S}_{p-y}^{R_1} + y$ and $p - y \in D$ whenever $p \in D$. □

Corollary 6.3 *The equivalence classes of $\sim_0$ on $\mathbb{S}_p$ depend continuously on p at $p \in G_\phi^s$ in the sense that, if $z_n \to 0$ and $I, J \in \mathbb{S}_p$ for $p \in G_\phi^s$, then we have $I \sim_0 J$ iff $I + z_n \sim_0 J + z_n$ for sufficiently large n.*

Proof: The implication $\Rightarrow$ is the object of (ii). We show $\Leftarrow$ now. First we fix representatives, $K_i := \widehat{\gamma_0^i + x}$, $i = 1, \ldots, cr_\phi$, of the equivalence classes of $\sim_0$ on $\mathbb{S}_p$ as supplied by the first paragraph of the proof of the lemma. Should $I \not\sim_0 J$ then $I \sim_0 K_i$ and $J \sim_0 K_j$ for some $i \neq j$. By (ii), for sufficiently large n, we have $I + z_n \sim_0 K_i + z_n$ and $J + z_n \sim_0 K_j + z_n$, which yields $K_i + z_n \sim_0 K_j + z_n$ by transitivity. In particular, $\gamma_0^i \sim \gamma_0^j$, contrary to Theorem 4.2. □

For $\gamma \in \mathcal{F}$, we define

$$\mathcal{Z}_\gamma := \{v \in \Sigma : \ \gamma \sim_0 \gamma + v\}. \tag{6.2}$$

Proposition 6.4 *The map $\gamma \mapsto \mathcal{Z}_\gamma$ is continuous (with the compact open topology in the range) at γ that lie over points in the generic set G_ϕ^s. Moreover, if $\mathcal{Z}_{\gamma_0} = \Sigma$ for a single $\gamma = (\gamma_k)_{k\in\mathbb{Z}} \in \mathcal{F}_\phi^{\leftarrow}$ then $cr_\phi = 1$.*

Proof: The continuity follows from Corollary 6.3 and the definition of $\mathcal{Z}_\gamma$.

As for $cr_\phi = 1$, from Remark 4.3, it suffices to show that, given two states J, K lying over the same point of V/Σ as γ_0, we must have $J \sim K$. Since γ_0 has edges of all types, there are $u, w \in \Gamma$ such that J is an edge of $\gamma_0 + u$ and K is an edge of

$\gamma_0 + w$. Since necessarily $u, w \in \Sigma = \mathcal{Z}_{\gamma_0}$, we have $\gamma_0 + u \sim_0 \gamma_0$ and $\gamma_0 + w \sim_0 \gamma_0$. By transitivity, $\gamma_0 + u \sim_0 \gamma_0 + w$, which is to say that $J \sim_0 K$. □

We also define, having fixed an arbitrary $\gamma = (\gamma_k)_{k\in\mathbb{Z}}$ in the minimal set $\mathcal{F}^{\leftarrow}_{\phi,\min}$ of the tiling flow,

$$\mathcal{Z}_\infty := \{v \in \Sigma : \ \gamma_0 \sim_{t\omega} \gamma_0 + v \ \text{ for generic } t \in \mathbb{R}\}. \tag{6.3}$$

Here, *generic* refers to a dense full measure G_δ subset.

Fact 6.5 *$\mathcal{Z}_\infty$ is either $\{0\}$ or a finite index subgroup of Σ independent of the choice of $\gamma = (\gamma_k)_{k\in\mathbb{Z}} \in \mathcal{F}^{\leftarrow}_{\phi,\min}$.*

Proof: Let us first show that $\mathcal{Z}_\infty$ is independent of the choice of γ. Fix then $\gamma, \eta \in \mathcal{F}^{\leftarrow}_{\phi,\min}$ and suppose that $\gamma_0 \sim_{t\omega} \gamma_0 + v$ for generic $t \in \mathbb{R}$. Let $l > 0$ be arbitrary. By minimality of the tiling flow, the $2l$ long central substrand of η_0 can be approximated by $\gamma_0+t\omega$: there is $t \in \mathbb{R}$ and $y \in E^s$ such that $(\gamma_0+t\omega)|_{-l}^{l} = \eta_0|_{-l}^{l}+y$ and $(\gamma_0 + v + t\omega)|_{-l}^{l} = (\eta_0 + v)|_{-l}^{l} + y$. From $\gamma_0 \sim_{t\omega} \gamma_0 + v$ for generic $t \in \mathbb{R}$, we get then that $\eta_0 \sim_{t\omega} \eta_0 + v$ for generic $t \in [-l, l]$. Hence, $v \in \{v \in \Sigma : \ \eta_0 \sim_{t\omega} \eta_0 + v \ \text{ for generic } t \in \mathbb{R}\}$ by arbitraryness of l (and stability of genericity under countable intersections).

As for $\mathcal{Z}_\infty$ being a subgroup, if $v, w \in \mathcal{Z}_\infty$ then $\gamma_0 \sim_{t\omega} \gamma_0 + v$ for generic $t \in \mathbb{R}$ and $\gamma_0 - v \sim_{t\omega} \gamma_0 - v + w$ for generic $t \in \mathbb{R}$, where we used the definition of $\mathcal{Z}_\infty$ with γ replaced by $T^{-\mathrm{pr}_u(v)}(\gamma)$ for the second one (as facilitated by Fact 6.5 and E^s invariance of $\sim_{t\omega}$). By transitivity of $\sim_{t\omega}$, we get $\gamma_0 \sim_{t\omega} \gamma_0 - v + w$ for generic $t \in \mathbb{R}$. That is, $w - v \in \mathcal{Z}_\infty$.

Finally, once $\mathcal{Z}_\infty \neq 0$, it is of finite index because irreducibility (over $\mathbb{Q}$) of the action of A on V forces the invariant subspace $\mathrm{lin}_{\mathbb{Q}}(\mathcal{Z}_\infty)$ to coincide with V. □

Lemma 6.6 *If $cr_\phi > 1$ then $\mathcal{Z}_\infty = \{0\}$.*

Proof: First we need to see that, for $\eta \in \mathcal{F}^{\leftarrow}_{\phi,\min}$, $\mathcal{Z}_{\eta_0}$ is a union of cosets of $\mathcal{Z}_\infty$. Suppose that $v \in \mathcal{Z}_{\eta_0}$ so that $\eta_0 \sim_{t\omega} \eta_0+v$ for all non-negative t near zero and that $w \in \mathcal{Z}_\infty$ by virtue of $\eta_0 + v \sim_{t\omega} \eta_0 + v + w$ for generic $t \in \mathbb{R}$. It follows that, for some $t_n \to 0$, $\eta_0 - t_n\omega \sim_0 \eta_0 + v - t_n\omega \sim_0 \eta_0 + v + w - t_n\omega$ so that $v + w \in \mathcal{Z}_{\eta_0 - t_n\omega}$. Hence $v + w \in \mathcal{Z}_{\eta_0}$ via Corollary 6.3. This shows $v + \mathcal{Z}_\infty \subset \mathcal{Z}_{\eta_0}$ for $v \in \mathcal{Z}_{\eta_0}$.

Suppose that $\mathcal{Z}_\infty$ is nontrivial and thus of finite index in Σ by Fact 6.5. Consider $g : \eta \mapsto \mathcal{Z}_{\eta_0}/\mathcal{Z}_\infty$ as a function on $\mathcal{F}^{\leftarrow}_{\phi,\min}$ taking values in subsets of $\Sigma/\mathcal{Z}_\infty$. From the definition of $\mathcal{Z}_{\eta_0}$, we have $g \circ \Phi = A \circ g$. Thus, $\Sigma/\mathcal{Z}_\infty$ being finite, we have $g \circ \Phi^{n_0} = g$ for some $n_0 \in \mathbb{N}$. Since Φ^{n_0} is transitive, g is constant on its continuity set: there is $Z \subset \Sigma$ such that $\mathcal{Z}_{\gamma_0} = Z$ for the set D consisting of $\gamma \in \mathcal{F}^{\leftarrow}_{\phi,\min}$ with γ_0 lying over $p \in G^s_\phi$ (see Corollary 6.3).

We claim that $Z = \mathcal{Z}_\infty$. Indeed, G^s_ϕ being E^s invariant, any coset of E^u must intersect G^s_ϕ along a generic subset. Therefore, having fixed any $\gamma \in \mathcal{F}^{\leftarrow}_{\phi,\min}$, we have then that $T^t(\gamma) \in D$ for generic $t \in \mathbb{R}$. Consequently, for any $v \in Z$,

$\gamma_0 + v - t\omega \sim_0 \gamma - t\omega$ for generic $t \in \mathbb{R}$. That is $v \in \mathcal{Z}_\infty$ and so $Z \subset \mathcal{Z}_\infty$, making $Z = \mathcal{Z}_\infty$ (since Z is consists of cosets of $\mathcal{Z}_\infty$).

Finally, having fixed any $i \in \mathcal{A}$, one easily sees that $\Theta(i) \subset \bigcup_{\gamma \in D} \mathcal{Z}_{\gamma_0} = \bigcup_{\gamma \in D} Z = Z$. It follows that $\Sigma \subset \langle Z \rangle = \langle \mathcal{Z}_\infty \rangle = \mathcal{Z}_\infty$. Proposition 6.4 secures $cr_\phi = 1$. □

We are ready to prove the theorem now.

Proof of Theorem 6.1: Again, we repeat the proof of Theorem 16.3 in [5] with obvious modifications. Suppose $cr_\phi > 1$ yet the assertion of the theorem fails. We claim that there are then $\epsilon > 0$, $p \in V/\Sigma$ and an infinite unbounded family of states in $\mathbb{S}_p$, $J_1, J_2, \ldots$, such that $J_i \sim_{t\omega} J_j$ for all $i, j \in \mathbb{N}$ and generic $t \in [-\epsilon, \epsilon]$. Indeed, by our hypothesis, there exist $\epsilon > 0$ and $K_m, L_m \in \mathbb{S}_{p_m}$, $m \in \mathbb{N}$, such that $\mathrm{dist}(K_m, L_m) > m$ and $K_m \sim_{t\omega} L_m$ for generic $t \in [-2\epsilon, 2\epsilon]$ and with all K_m of the same type. By compactness, one can arrange that p_m converge to some $p \in V/\Sigma$. Taking $v_m \in V$ so that $p_m + v_m = p$ and $v_m \to 0$, one readily sees that $J_1 := K_m + v_m$, $J_2 := L_m + v_m$, $J_3 := L_{m+1} + v_{m+1}$, $J_4 := L_{m+2} + v_{m+2}$, ... are as desired provided m is large enough.

In view of Lemma 6.6, it suffices to show that $\mathcal{Z}_\infty \neq \{0\}$. To do that, for every $k \in \mathbb{N}$, pick from among the partial strands $\Phi^k(J_1)|_{-\lambda^k\epsilon}^{\lambda^k\epsilon}, \Phi^k(J_2)|_{-\lambda^k\epsilon}^{\lambda^k\epsilon}, \ldots$ two, call them α_k and β_k, that are disjoint and determine the same word $a_k := [\alpha_k] = [\beta_k]$, and intersect E^s at points x_k and y_k that are further than $100R_0$ apart. This assures $\alpha_k \subset x_k + \mathcal{C}^{2R_0}$ and $\beta_k \subset y_k + \mathcal{C}^{2R_0}$. What is more, by replacing α_k and β_k with $\Phi^l(\alpha_k)|_{-\lambda^k\epsilon}^{\lambda^k\epsilon}$ and $\Phi^l(\beta_k)|_{-\lambda^k\epsilon}^{\lambda^k\epsilon}$ for some large $l > 0$, we may require as well that $\mathrm{dist}(x_k, y_k) < 200\lambda R_0$. Finally, let us translate α_k and β_k by a common vector in E^s so that $\alpha_k, \beta_k \subset \mathcal{C}^{200\lambda R_0 + 4R_0}$.

By passing to a subsequence if necessary, we have $a_k \to a$, $\alpha_k \to \alpha$, $\beta_k \to \beta$ for some bi-infinite word a and bi-infinite strands α, β. By construction, $\alpha (\mathrm{mod}\ E^s), \beta (\mathrm{mod}\ E^s) \in \mathcal{T}_\phi$ and $\alpha \sim_{t\omega} \beta$ for generic $t \in \mathbb{R}$.

From $\alpha (\mathrm{mod}\ E^s) \in \mathcal{T}_\phi$, there is $x \in E^s$ so that $\gamma_0 := \alpha + x$ is a strand of some $(\gamma_k)_{k \in \mathbb{Z}} \in \mathcal{F}_{\phi,\min}^{\leftarrow}$. Also, $\beta + x = \gamma_0 + v$ for some $v \in V \setminus \{0\}$. Note that $v \in \Sigma$ because α_k and β_k lie over the same point of V/Σ and thus the same is true for α and β. From $\alpha \sim_{t\omega} \beta$, $\gamma_0 \sim_{t\omega} \gamma_0 + v$ for generic $t \in \mathbb{R}$ thus placing $v \neq 0$ in $\mathcal{Z}_\infty$. □

7 Coincidence Condition for a class of β-substitutions

Recall that $\beta > 0$ is a *Parry number* iff 1 is preperiodic under the action of the β-transformation $f_\beta : [0, 1] \to [0, 1]$ sending $x \mapsto \beta x - \lfloor \beta x \rfloor$. (This is to say that $\{{f_\beta}^n(1) : n \in \mathbb{N}\}$ is a finite set.) Any Pisot β is a Parry number [8, 22]. The sweeping conjecture asserts that, for Pisot β, f_β is *algebraic* in the sense that the natural extension of f_β is naturally almost homeomorphically conjugate to a compact abelian group automorphism. In our context, the natural extension of f_β can be realized as the tiling space of an approriate substitution. Thus the conjecture concerns injectivity of the geometric realization and can be attacked by verifying the GCC for a suitable class of substitutions.

We shall prove that f_β is algebraic for a broad subclass of *simple Parry numbers*, i.e., $\beta > 0$ such that $f_\beta^n(1) = 0$ for some $n \in \mathbb{N}$. The relevant substitution ϕ is given in the form

$$\phi = \phi_\beta : \begin{cases} 1 & \mapsto 21^{a_1} \\ 2 & \mapsto 31^{a_2} \\ \vdots & \\ n-1 & \mapsto n1^{a_{n-1}} \\ n & \mapsto 1^{a_n} \end{cases} \tag{7.1}$$

The numbers a_i are determined by the action of f_β. Setting $P_i := [0, f_\beta^{i-1}(1)]$, $i = 1, \ldots, n$, we see that f_β maps P_i a_i times across $P_1 = [0, 1]$ and once across P_{i+1} for $i = 1, \ldots, n-1$, and f_β maps P_n exactly a_n times across P_1. The intervals P_i are then proportional to the tiles of the tiling space $\mathcal{T}_{\phi_\beta}$. Indeed, there is a metric isomorphism of the inflationary dynamics of the tiling space, $(\mathcal{T}_{\phi_\beta}, \Phi_\beta)$, with the natural extension $(\lim_\leftarrow f_\beta, \hat{f}_\beta)$ given by $\gamma \overset{p}{\mapsto} (\ldots t_{-1}, t_0, t_1, \ldots)$ where $\mathrm{pr}_u(\Phi_\beta^n(\gamma)) = -t_n\omega$ and ω is normalized so that $\mathrm{pr}_u(e_1) = \omega$.

To connect with arithmetical properties of Pisot numbers, recall that each non-negative real number x has a *gready expansion* in base β, $x = \sum_{n=-N}^{\infty} x_n\beta^{-n}$ with $x_n \in \{0, \ldots, \lfloor\beta\rfloor\}$[9]. Each such greedy expansion determines a sequence $(\ldots, 0, 0, x_{-N}, \ldots, x_{-1}, x_0, x_1, x_2, \ldots) \in \{0, \ldots, \lfloor\beta\rfloor\}^{\mathbb{Z}}$; let Σ_β be the closure of the set of all such sequences in $\{0, \ldots, \lfloor\beta\rfloor\}^{\mathbb{Z}}$. The subshift (Σ_β, σ), the β-*shift*, is sofic for Pisot β and is of finite type in case β is a simple Parry number. The (a.e. defined) map

$$(\ldots, x_{-1}, x_0, x_1, \ldots) \overset{r}{\mapsto} \left(\ldots, \sum_{n=1}^{\infty} x_{n-1}\beta^{-n}, \sum_{n=1}^{\infty} x_n\beta^{-n}, \sum_{n=1}^{\infty} x_{n+1}\beta^{-n}, \ldots\right)$$

provides a metric isomprphism of (Σ_β, σ) with $(\lim_\leftarrow f_\beta, \hat{f}_\beta)$. The composition $g := h_{\phi_\beta} \circ p^{-1} \circ r$ is then a bounded-to-one semi-conjugacy between the β-shift and the algebraic system $(\mathbb{T}_A, A)$ that is continuous and satisfies $g(\underline{x}+\underline{x}') = g(\underline{x})+g(\underline{x}')$ for $\underline{x} = (\ldots, 0, 0, x_{-N}, \ldots, x_{-1}, x_0, x_1, \ldots)$ and $\underline{x}' = (\ldots, 0, 0, x'_{-K}, \ldots, x'_{-1}, x'_0, x'_1, \ldots)$ that come from β-expansions of real numbers x and x' (here $\underline{x} + \underline{x}'$ means $\underline{x + x'}$). Such g is called an *arithmetical coding* of $(\mathbb{T}_A, A)$ by Sidorov who proves that this coding is a.e. one-to-one in case β is a *weakly finitary* Pisot unit ([24]). The latter terminology is due to Hollander ([14]) and is defined as follows. Let $\mathrm{Fin}(\beta) := \{x \geq 0 : \ x \text{ has a finite greedy expansion in base } \beta\}$. Then β is weakly finitary

[9] For each $M \geq N$, $\left|x - \sum_{n=-N}^{M} x_n\beta^{-n}\right| < \beta^{-M}$.

provided, for each $\delta > 0$ and $x \in \mathbb{Z}[1/\beta]^+$, there is $y \in \mathrm{Fin}(\beta) \cap (0, \delta)$ so that $x + y \in \mathrm{Fin}(\beta)$. Akiyama ([1]) has proved that if ϕ_β is irreducible (that is, the algebraic degree of β is the n in (7.3)), then GCC for ϕ_β is equivalent with β being weakly finitary. Theorem 7.1 below assures that certain Pisot numbers are weakly finitary and that a wide class of arithmetical codings are a.e. one-to-one.

Let us briefly mention another interpretation of the result below associated with the (generalized) β-integeres, $\Sigma_\beta^- := \{(\ldots, x_{-1}, x_0.0, 0, \ldots) \in \Sigma_\beta\}$. There is an "adic" action that takes each $\underline{x} \in \Sigma_\beta^-$ to its immediate succesor (this is defined by extending the succesor map on the (finite) β-integers, which are ordered as real numbers). This adic action is measurably conjugate to the substitutive system (X_{ϕ_β}, σ). Theorem 7.1 below assures that, for certain β with the algebraic degree $\deg_{\mathbb{Q}}(\beta) = n$ (and thus with irreducible ϕ_β), the adic action and the substitutive system have pure discrete spectrum. (If $\deg_{\mathbb{Q}}(\beta) \neq n$, we can only conclude that these $\mathbb{Z}$-actions are induced as return maps under a flow with pure discrete spectrum.)

For β Pisot and a simple Parry number and ϕ as in (7.1), it is automatic that:

(i) $a_1 > 0$ and $a_n > 0$;

(ii) the largest modulus root β of $t^n - a_1 t^{n-1} - \cdots - a_{n-1}t - a_n$ is a Pisot number.

We will also require the following hypothesis:

(iii) the algebraic degree $d := \deg \beta$ satisfies $d > n/p$ where $p > 1$ is the smallest prime divisor of n.

Theorem 7.1 *Under the above hypotheses (i), (ii) and (iii) the substitution ϕ satisfies GCC, i.e., $cr_\phi = 1$.*

Observe that this result completely resolves the irreducible case when $n = d$, which was previously tackled only under extra hypotheses: $a_1 \geq a_2 \geq \cdots \geq a_{d-1} \geq a_d = 1$ in [13] and $a_1 > \sum_{i=2}^d a_i$ in [14]. That last hypothesis was weakened in [2] to $a_1 > \sum_{i=2}^n |a_i|$, which covers some Pisot β that are not simple Parry numbers and lie outside the scope of our result.

The following argument proceeds in the spirit of our initial improvement of these results in [6].

We observe that taking $b_1 := 1$, $b_2 := \phi(1), \ldots, b_n := \phi^{n-1}(1)$ we have $\phi(b_n) = \phi(n b_1^{a_{n-1}} \cdots b_{n-1}^{a_1}) = b_1^{a_n} b_2^{a_{n-1}} \cdots b_n^{a_1}$. Thus we are lead to abandon ϕ in favor of a more managable substitution

$$\psi : \begin{cases} 1 & \mapsto 2 \\ 2 & \mapsto 3 \\ \vdots & \\ n-1 & \mapsto n \\ n & \mapsto 1^{a_n} 2^{a_{n-1}} \cdots n^{a_1} \end{cases} \tag{7.2}$$

Fact 7.2 *The tiling flows and inflation substitution actions on $\mathcal{T}_\phi$ and $\mathcal{T}_\psi$ are homeomorphically conjugated.*

Proof: For $k = 1, \dots, n$, set $\sigma(k) := \phi^{k-1}(1)$ and

$$\rho(k) := \begin{cases} n & \text{for } k = 1 \\ (k-1)^{a_n} k^{a_{n-1}} \dots (n-1)^{a_k} & \text{for } 1 < k < n \\ (n-1)^{a_k} & \text{for } k = n. \end{cases} \quad (7.3)$$

Then $\sigma \circ \rho = \phi^{n-1}$ and $\rho \circ \sigma = \psi^{n-1}$. This is to say that ϕ^{n-1} and ψ^{n-1} are shift equivalent in the category of substitutions and one can conclude that the $(n-1)$st powers of inflation-substitution dynamics on tiling spaces are conjugated via Williams' theory of generalized solenoids [32]. An explicit development in the language of tilings can be found in [4]: Lemma 3.1 there yields $G_\rho : \mathcal{T}_{\phi^{n-1}} \mapsto \mathcal{T}_{\psi^{n-1}}$ and $G_\sigma : \mathcal{T}_{\psi^{n-1}} \mapsto \mathcal{T}_{\phi^{n-1}}$ that intertwine the inflation-substitution dynamics Ψ^{n-1} and Φ^{n-1}; moreover, naturality yields $G_\rho \circ G_\sigma = G_{\rho\sigma} = \Psi^{n-1}$ so G_ρ and G_σ are homeomorphisms. That G_ρ and G_σ intertwine the flows is stated in the beginning of the third paragraph of the proof of Lemma 3.1 in [4]. □

Note that the matrix A of ψ is a companion matrix. In particular,

$$|v_i|_u = |\mathrm{pr}_u(v_i)| = \beta^{i-1}|v_1|_u, \; i = 1, \dots, n, \quad (7.4)$$

so that $|v_1|_u < |v_2|_u < \dots < |v_n|_u$.

For $x \in \mathcal{A}$, we can *grow* σ_x into an infinite strand starting with σ_x:

$$\gamma^x := \bigcup_{k \geq 0} \Psi^{kn}(\sigma_x). \quad (7.5)$$

Lemma 7.3 *For each $y > n/p$, there is $x \leq n/p$ so that $\gamma^x \sim_{t\omega} \gamma^y$ for generic $t \in [0, \infty)$.*

Proof: In view of $a_1 > 0$, $\psi^{n-1}(l) = \dots n$ for all $l \in \mathcal{A}$. Corollary 3.5 of [5] applies then to yield: there are $k, l \in \mathcal{A}$, $k \neq l$, and $\epsilon > 0$ so that $\sigma_k \sim_{t\omega} \sigma_l$ for generic $t \in [0, \epsilon)$. It follows that $\gamma^k \sim_{t\omega} \gamma^l$ for generic $t \in [0, \infty)$. As $\Psi(\gamma^i) = \gamma^{i+1}$ with $i \in \mathcal{A}$ taken mod n, we also have that $\gamma^{k+i} \sim_{t\omega} \gamma^{l+i}$ for generic $t \in [0, \infty)$ and all i. Combined with transitivity of $\sim_{t\omega}$, this assures that $K := \{k : \gamma^{y+k} \sim_{t\omega} \gamma^y \text{ for generic } t \in [0, \infty)\}$ is a nontrivial additive (cyclic) subgroup of $\mathcal{A}$ (mod n). The order of K being at least p, K has to have an element k in the (cyclic) segment $\{-y+1, \dots, -y+n/p\}$ so that $x := y + k \in \{1, \dots, n/p\}$. □

Lemma 7.4 *There exists a finite strand γ with the following properties:*

(a) all segments of γ have type in $\{1, \dots, n/p\}$,

(b) $\min \gamma = 0$ and $|\max \gamma|_u \geq |v_{n/p+1}|_u$,

(c) $\gamma \sim_{t\omega} \sigma_{n/p+1}$ *for generic* $t \in [0, |v_{n/p+1}|_u]$.

Proof: Let $x_1 \in \{1, \ldots, n/p\}$ be such that $\gamma^{x_1} \sim_{t\omega} \gamma^{n/p+1}$ for generic $t \in [0, \infty)$, and let $I_1^1, I_2^1, \ldots$ be the consecutive segments of γ^{x_1}, i.e., $I_1^1 = \sigma_{x_1}$ and $\max I_i^1 = \min I_{i+1}^1$ for $i = 1, \ldots$. Let

$$m(1) := \inf\{i : \ I_i^1 \text{ is of type } y \text{ for some } y > n/p\}.$$

If $|\min I_{m(1)}^1| \geq |v_{n/p+1}|_u$, then we are done by taking

$$\gamma := I_1^1 \cup \cdots \cup I_{m(1)-1}^1. \tag{7.6}$$

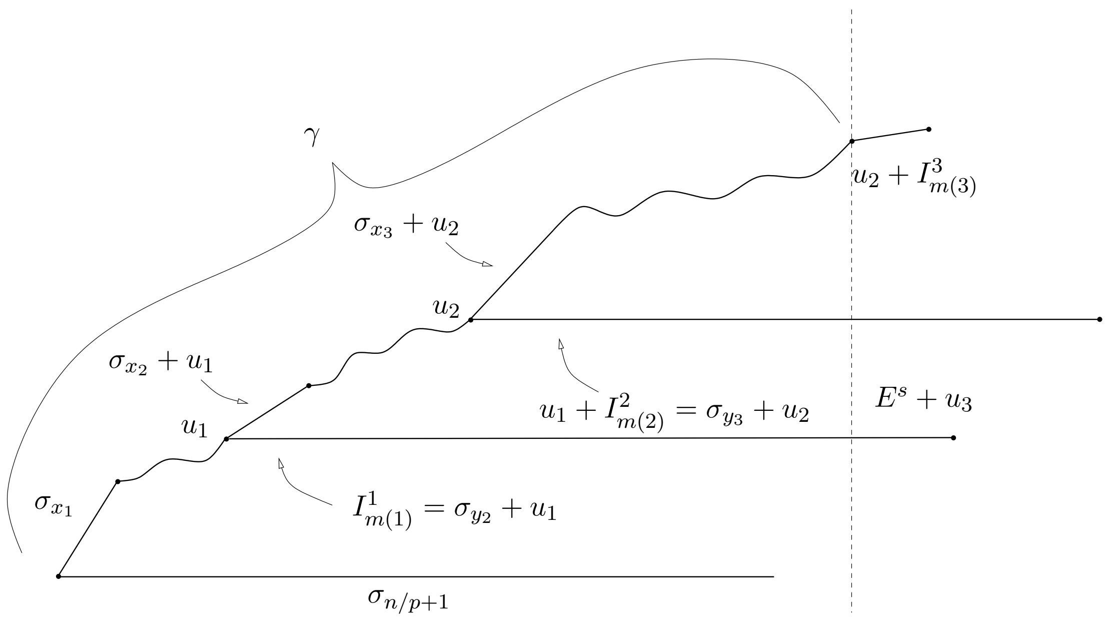

Figure 7.1: Construction of γ in the proof of Lemma 7.4.

If $|\min I_{m(1)}^1| < |v_{n/p+1}|_u$, let y_2 be the type of $I_{m(1)}^1$ and let $x_2 \in \{1, \ldots, n/p\}$ be so that $\gamma^{x_2} \sim_{t\omega} \gamma^{y_2}$ for generic $t \in [0, \infty)$. Let $I_1^2, I_2^2, \ldots$ be the consecutive segments of γ^{x_2}. Let

$$m(2) := \inf\{i : \ I_i^2 \text{ is of type } y \text{ for some } y > n/p\}.$$

If $|\min I_{m(2)}^2|_u + |\min I_{m(1)}^1|_u \geq |v_{n/p+1}|_u$, set

$$\gamma := I_1^1 \cup \cdots \cup I_{m(1)-1}^1 \cup (I_1^2 + u_1) \cup \cdots \cup (I_{m(2)-1}^2 + u_1) \tag{7.7}$$

where $u_1 := \max I_{m(1)-1}^1$.

That (a) and (b) hold for this γ is clear and so is (c) for $t \in [0, |u_1|_u]$. To account for (c) for $t \in [|u_1|_u, |v_{n/p+1}|_u]$, it suffices to observe that $I_{m(1)}^1 \sim_{t\omega} (\gamma^{x_2} + u_1)$ for generic $t \in [|u_1|_u, |u_1|_u + |v_{x_2}|_u]$ and $I^1_{m(1)} \sim_{t\omega} \sigma_{n/p+1}$ for generic $t \in [|u_1|_u, |v_{n/p+1}|_u]$. Beside the choice of x_2 and x_1, we used here $|\max I^1_{m(1)}|_u \geq |v_{n/p+1}|_u$, which is due to the type of $I^1_{m(1)}$ being in $\{n/p+1, \ldots, n\}$ and (7.4).

If $|\min I^2_{m(2)}|_u + |\min I^1_{m(1)}|_u < |v_{n/p+1}|_u$, let y_3 be the type of $I^2_{m(2)}$ and let $x_3 \in \{1, \ldots, n/p\}$ be so that $\gamma^{x_3} \sim_{t\omega} \gamma^{y_3}$ for generic $t \in [0, \infty)$. Let $I^3_1, I^3_2, \ldots$ be the consecutive segments of γ^{x_3}. Let

$$m(3) := \inf\{i : \ I^3_i \text{ is of type } y \text{ for some } y > n/p\}.$$

The strand γ is defined by stringing together portions of γ^{x_1}, $u_1 + \gamma^{x_2}$, and $u_2 + \gamma^{x_3}$ where $u_2 := \max I^2_{m(2)-1} + u_1$ following the pattern set by (7.7) and illustrated by Figure 7.1.

This process will terminate in a finite number of steps producing a finite strand γ with the desired properties. □

Let the consecutive segments of γ from the previous lemma be $J_1, J_2, \ldots$. Set $w_0 := v_{n/p+1}$ and define reccursively, for $k = 1, 2, \ldots$,

$$i_k := \inf\{i : \ |\max(J_i + w_{k-1} - v_{n/p+1})|_u \geq |v_{n/p+1})|_u\} \tag{7.8}$$

$$w_k := \max J_{i_k} + w_{k-1} - v_{n/p+1}. \tag{7.9}$$

The Figure 7.2 depicts the process generating the w_k as endpoints of the appropriate translated copies of γ. The role of the hypothesis (iii) is to assure that the resulting cluster of strands is infinite (and thus unbounded):

Claim 7.5 *We have $w_k \neq w_l$ for $k \neq l$, $k, l \in \mathbb{N}$ and also $\max |w_{k+1}|_u > |v_{n/p+1}|_u$ for $k = 0, 1, \ldots$.*

Proof: Note that w_k is of the form $\sum_{i=1}^{n/p} m_i v_i - (k-1) v_{n/p+1}$ for some non-negative integers m_i. If $w_k = w_l$ then $|w_k|_u - |w_l|_u = 0$ which (via (7.4)) has the form $\sum_{i=1}^{n/p} c_i \beta^{i-1} |v_1|_u - (k-l)\beta^{n/p}|v_1|_u = 0$ for some integers c_i. Thus $k \neq l$ would contradict (iii).

Likewise, $|w_{k+1}|_u - |v_{n/p+1}|_u = 0$ would contradict (iii) by yielding a relation of the form $\sum_{i=1}^{n/p} c_i \beta^{i-1} |v_1|_u - (k+1)\beta^{n/p}|v_1|_u$ for some integers c_i. □

We need the following direct consequence of the form of ψ.

Fact 7.6 *There is $\delta_1 > 0$ so that if $I \cup J$ is a 2-segment strand with $\max I = \min J$ and J is of type $j \in \{1, \ldots, n/p\}$ then $I \cup J \sim_{t\omega} \sigma_{n/p+1} + (\max J - v_{n/p+1})$ for all $t \in [|\min J|_u - \delta_1, |\max J|_u]$.*

Proof: The coincidence for $t \in [|\min J|_u, |\max J|_u]$ is effected by applying Ψ^{n-j} because $\psi^{n-j}(n/p+1) = \ldots n$ while $\psi^{n-j}(j) = n$ for $j \in \{1, \ldots, n/p\}$. Thanks to $\psi^n(i) = \ldots n$ for all i, subsequent application of Ψ^n extends the range of coincidence to $t \in [|\min J|_u - \delta_1, |\min J|_u]$ for some $\delta_1 > 0$. □

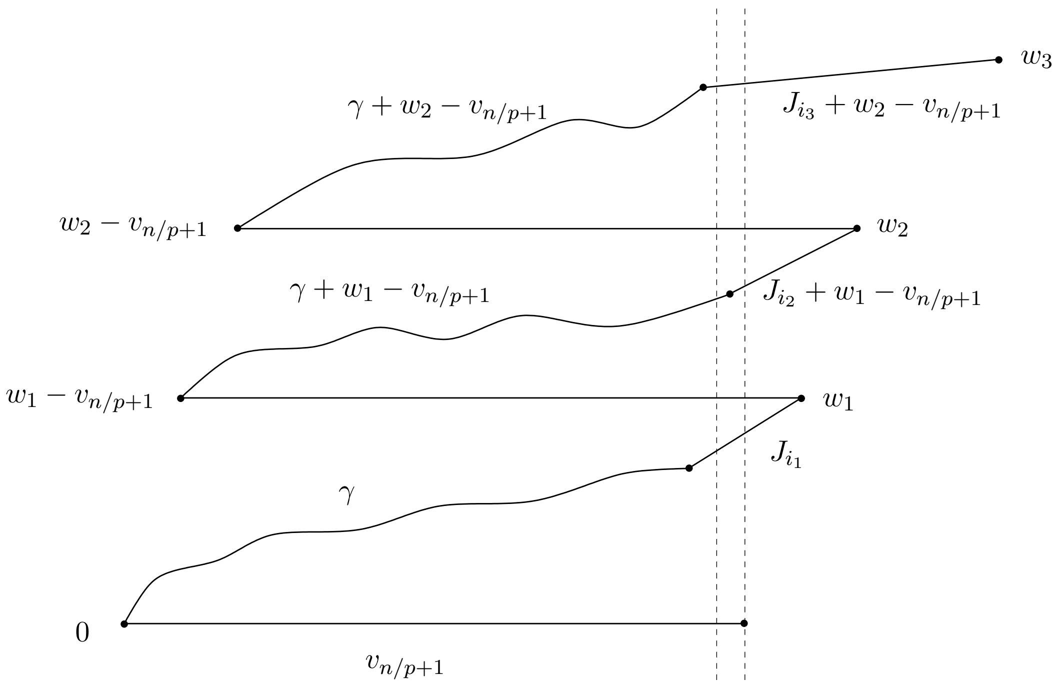

Figure 7.2: Stacking of strands $\gamma + w_k$.

Now let

$$\delta := \min\{\delta_1, |v_1|_u, (\beta^{n/p} - \beta^{n/p-1})|v_1|_u\}. \quad (7.10)$$

The following asserts that the strands in Figure 7.2 meeting at vertices $w_1, w_2, \ldots$ are coincident on the δ-strip between the dashed lines.

Claim 7.7 *For $k \geq 1$, we have*

$$\gamma + w_{k-1} - v_{n/p+1} \sim_{t\omega} \sigma_{n/p+1} + w_k - v_{n/p+1} \quad (7.11)$$

for all generic $t \in [|v_{n/p+1}|_u - \delta, |v_{n/p+1}|_u]$.

Proof: First we see that that the $w_k - v_{n/p+1}$ are to the left of the δ-strip in Figure 7.2. Indeed, by the minimality of i_k and (7.4) we get

$$\begin{aligned} |w_k - v_{n/p+1}|_u &< |\max J_{i_k} - \min J_{i_k}|_u \leq \beta^{n/p-1}|v_1|_u & (7.12)\\ &= (\beta^{n/p-1} - \beta^{n/p})|v_1|_u + |v_{n/p+1}|_u < |v_{n/p+1}|_u - \delta. & (7.13)\end{aligned}$$

Either J_{i_k} has no precedessor on γ, in which case $|\min J_{i_k} + w_{k-1} - v_{n/p+1}|_u = |w_{k-1} - v_{n/p+1}|_u < |v_{n/p+1}|_u - \delta$ by (7.12) so that

$$J_{i_k} + w_{k-1} - v_{n/p+1} \sim_{t\omega} \sigma_{n/p+1} + w_k - v_{n/p+1} \quad (7.14)$$

for all generic $t \in [|v_{n/p+1}|_u - \delta, |v_{n/p+1}|]$; or J_{i_k} has precedesor J_{i_k-1} on γ, in which case Fact 7.6 yields

$$(J_{i_k-1} \cup J_{i_k}) + w_{k-1} - v_{n/p+1} \sim_{t\omega} \sigma_{n/p+1} + w_{k-1} - v_{n/p+1} + \max J_{i_k} - v_{n/p+1} \quad (7.15)$$

for generic $t \in [|\min(J_{i_k} + w_{k-1} - v_{n/p+1})|_u - \delta, |\max(J_{i_k} + w_{k-1} - v_{n/p+1})|_u]$.

Now, (7.14) and (7.15), with an eye on $w_{k-1} - v_{n/p+1} + \max J_{i_k} = w_k$, secure (7.11). □

Corollary 7.8 *$\gamma + w_k - v_{n/p+1} \sim_{t\omega} \sigma_{n/p+1}$ for generic $t \in [|v_{n/p+1}|_u - \delta, |v_{n/p+1}|_u]$ and all $k = 0, 1, \ldots$.*

Proof: From (c) of Lemma 7.4, the strands meeting at $w_k - v_{n/p+1}$ are coincident on the δ-strip; namely, $\gamma + w_k - v_{n/p+1} \sim_{t\omega} \sigma_{n/p+1} + w_k - v_{n/p+1}$ for $k = 0, 1, 2, \ldots$ and for generic $t \in [|v_{n/p+1}|_u - \delta, |v_{n/p+1}|_u]$. From Claim 7.7, the strands meeting at vertices $w_1, w_2, \ldots$ are coincident on the δ-strip. Hence, the transitivity of $\sim_{t\omega}$, forces all the strands in Figure 7.2 to be coincident on the δ-strip. □

Conclusion of proof of Theorem 7.1: Taking $t_1 := |v_{n/p+1}|_u - \delta$, let L be the state $L := \sigma_{n/p+1} - t_1\omega$ and K_k be the state at which the finite strand $\gamma + w_k - v_{n/p+1} - t_1\omega$ intersects E^s. From Corollary 7.8, $K_k \sim_{t\omega} L$ for generic $t \in [0, \delta]$. From Claim 7.5, the vectors $w_k \in \Gamma$ must form an unbounded set and thus so do the distances $\text{dist}(L, K_k)$. Thus $cr_\phi = 1$ follows from Theorem 6.1. □

8 The substitutive system as an induced system of a group translation

The tiling flow T^t has a natural cross-section

$$X_\phi := \{(\gamma_k) \in \mathcal{F}_\phi^{\leftarrow} : \quad \gamma_0 \text{ has a vertex on } E^s\}. \tag{8.1}$$

The first return map $T_\phi : X_\phi \to X_\phi$ under the reversed flow (i.e. T^{-t}) constitutes the much studied ***substitutive system*** associated to ϕ. X_ϕ decomposes into

$$R_i^s := \{(\gamma_k) \in \mathcal{F}_\phi^{\leftarrow} : \quad \gamma_0 \text{ has an edge } I \text{ labeled } i \text{ with } \min I \in E^s\}, \; i = 1, \ldots, n, \tag{8.2}$$

which sets are the stable boundaries of the natural Markov boxes for Φ:

$$R_i := \{(\gamma_k) \in \mathcal{F}_\phi^{\leftarrow} : \quad \gamma_0 \text{ has an edge labeled } i \text{ intersecting } E^s\}, \; i = 1, \ldots, n. \tag{8.3}$$

A *geometric model* of X_ϕ, called the ***Rauzy fractal***, can be obtained by mapping X_ϕ via $(\gamma_k) \mapsto (\min \hat{\gamma}_k)$ (c.f. (4.6)) that sends (γ_k) to the sequence of vertices on E^s which can be thought of as an element of $E^s \times C$ where C is the totally disconnected subgroup of $\mathbb{T}_A$ that serves as its fiber over V/Σ, i.e., $C := \{(q_k) \in \mathbb{T}_A : \; q_0 = 0\}$. This procedure is particularly appealing when the stretching factor λ is a unit so that $A|_V$ is unimodular, C is just a single point, and the Rauzy fractal of ϕ becomes a subset of E^s; concretely,

$$\Omega^s := \bigcup_{i=1}^{n} \Omega_i^s \tag{8.4}$$

where the sets

$$\Omega_i^s := \{x \in E^s : \ x = \min \hat{\gamma_0} \ \text{ where } (\gamma_k) \in \mathcal{F}_\phi^{\leftarrow} \text{ and } \hat{\gamma_0} \text{ is labeled } i\} \tag{8.5}$$

are called the ***Rauzy pieces*** of ϕ.[10] In what follows, we restrict attention to the case in which λ is a Pisot unit.

If the union $\bigcup_{i=1}^n \Omega_i^s$ happens to be disjoint up to measure zero, the above construction factors the return map T_ϕ to the ***domain exchange*** $E_\phi : \Omega^s \to \Omega^s$ a.e. defined by (c.f. [3])

$$E_\phi(x) = x + w_i \ \text{ for } x \in \Omega_i^s, \ w_i := \mathrm{pr}_s(v_i). \tag{8.6}$$

When A is irreducible over $\mathbb{Q}$ and $cr_\phi = 1$, the Rauzy fractal Ω^s is a fundamental domain for the *anti-diagonal torus* $E^s/\Lambda \simeq \mathbb{T}^{d-1}$, $\Lambda := \langle w_i - w_j : \ i, j = 1, \ldots, d\rangle$, and the domain exchange is just the toral translation $x \mapsto x + w_1 \equiv x + w_i \mod \Lambda$.

In the Pisot unit case, regardless of whether or not GCC holds for ϕ, there is a dual tiling space $\mathcal{T}_\phi^*$ consisting of certain tilings of E^s by the Rauzy pieces. For irreducible ϕ, the definition of $\mathcal{T}_\phi^*$ is in [5] and it can be modified in a straightforward way for reducible ϕ. If ϕ satisfies GCC, then $\mathcal{T}_\phi^*$ is simply the space of tilings induced on the stable foliation $(x + E^s) + \Sigma$, $x \in E^u$, by intersecting with the Markov rectangles $(h_\phi)_0(R_i) := \{p + \Sigma : \ (\gamma_k) \in R_i, \ p_0 \text{ a vertex of } \gamma_0\}$ in the torus V/Σ.

The elements of $\mathcal{T}_\phi^*$ are *aperiodic* tilings of E^s by the Rauzy pieces. When GCC holds, the Rauzy pieces can also be assembled into periodic tilings of $\mathbb{R}^{d-1}$. For irreducible ϕ, $\Omega^s = \bigcup_{i=1}^d \Omega_i^s$ is simply a fundamental domain for the lattice Λ. For reducible ϕ, Ei and Ito ([11]) have observed that the Rauzy fractal Ω^s may fail to be a fundamental domain for any lattice in E^s (even though the geometric realization is a.e. one-to-one). Nevertheless, in all examples considered, they find some lattice in E^s and a translation on the associated torus, so that E_ϕ coincides with the first return map to Ω^s under the translation. That means that the Rauzy pieces tile $\mathbb{R}^{d-1}$ periodically because a fundamental domain of that lattice is obtained by taking the union of the translations of the Rauzy pieces prior to their return to Ω^s. The following simple general proposition guarantees that this is always the case.

Proposition 8.1 *Suppose that $cr_\phi = 1$ so that the geometric realization is a.e. one-to-one. Then the domain exhange E_ϕ is isometrically conjugated with the first return map to some domain induced by a minimal translation on the $d-1$-dimensional torus.*

Proof: For $R, \epsilon > 0$, we shall denote by $C_{\epsilon,R}$ the solid cylinder $B^s(R) \times B^u(\epsilon)$ obtained as the product of balls of radius R and ϵ in E^s and E^u, respectively, both centered at 0.

[10] The Ω_i^s are closures of their interiors and have boundary of zero measure, see [27].

Pick $R > 0$ large enough so that Ω^s is contained in the ball $B^s(R) \subset E^s$ and pick $\epsilon > 0$ small enough so that the natural projection $\pi : V \to V/\Sigma$ restricted to $C_{2\epsilon,R}$ is an embedding. Take $E \subset V$ to be a $d-1$-dimensional linear space that is totally rational (making $E/(E \cap \Sigma)$ a $d-1$-dimensional torus) and that approximates E^s well enough so that E passes through the sides of $C_{\epsilon,R}$ while avoiding its top and the bottom, i.e., $E \cap C_{\epsilon,R} \subset \partial B^s(R) \times B^u(\epsilon)$.

Let $\Lambda \subset E$ denote the image of the Rauzy fractal Ω^s under the projection $\mathrm{pr}_E : V \to E$ along E^u and denote by $h : \pi(\Omega^s) \to \pi(\Lambda^s)$ the bijection between the two sets obtained by conjugating pr_E via $\pi|_{C_{\epsilon,R}}$.

For the Kronecker flow in the direction ω, the domain exchange E_ϕ conjugated by π constitutes the first return to $\pi(\Omega^s)$. Because $\pi|_{C_{2\epsilon,R}}$ is an embedding, this return is conjugated via h to the first flow return to $\pi(\Lambda^s)$. (Indeed, the flow line segment J joining $p \in \pi(\Lambda^s)$ to its first return $q \in \pi(\Lambda^s)$ can encounter $\pi(\Omega^s)$ only inside $\pi(C_{\epsilon,R})$ and thus at no other points beside $h^{-1}(p)$ and $h^{-1}(q)$.)

Clearly, the first flow return to $\pi(\Lambda^s)$ is the first return to $\pi(\Lambda^s)$ under the minimal translation $\tau : E/(E \cap \Sigma) \to E/(E \cap \Sigma)$ induced by flow return to the global cross section $E/(E \cap \Sigma) \subset V/\Sigma$. □

The simplicity of the above argument exacts a toll: its practical implementation may lead to an excessively large torus $E/(E \cap \Sigma)$ of which the projected Rauzy fractal makes up only a small portion and takes many iterates to return to itself under the toral translation τ. Under an additional condition (condition (8.2) below) we will make an explicit construction (in the proof of Proposition 8.5) in a slightly different spirit, that limits the size of the $d-1$-dimensional torus and the return time to the Rauzy fractal. We restrict attention to substitutions of the form (7.1) that satisfy the hypotheses of Theorem 7.1 along with condition (8.2) below on the coefficients of the minimal polynomial of the expansion factor $\lambda = \beta$. These include the substitutions considered in [11].

Condition 8.2 *Let $t^d - b_1 t^{d-1} - \cdots - b_d$ be the minimal polynomial of β. Then either $b_i \geq 0$ for $i = 1, \ldots, d$ or $\sum_{i=1}^{l} b_i > 0$ for $l = 1, \ldots, d$.*

Fact 8.3 *Under the above condition, for every $m \in \mathbb{Z}^+$, there are $M_m \in \mathbb{Z}^+$ and $c_{m,i,j} \in \mathbb{Z}$, $i = 1, \ldots, d$, $j = 1, \ldots, M_m$, so that, for all j,*

$$\beta^{m-1} = \sum_{j=1}^{M_m} \eta_{m,j}, \qquad \eta_{m,j} := \sum_{i=0}^{d-1} c_{m,i,j} \beta^i > 0, \qquad \sum_{i=0}^{d-1} c_{m,i,j} = 1. \tag{8.7}$$

The proof is a straightforward induction on m. Of course, (8.7) is trivially satisfied for $m \in \{1, \ldots, d\}$ with $M_m = 1$, $c_{m,m-1,1} = 1$, and $c_{m,i,1} = 0$ for $i \neq m-1$.

Question 8.4 *Are there $c_{m,i,j} \in \mathbb{Z}$, $M_m \in \mathbb{Z}^+$ such that (8.7) holds for all $m \in \mathbb{Z}^+$ for every Pisot unit β?*

Proposition 8.5 *Suppose that the substitution ϕ is of the form (7.1) with β a unit and satisfies (i), (ii), (iii) of Theorem 7.1 together with conditions (8.2). Then there is a lattice $\Lambda \subset \mathrm{pr}_s(\Gamma)$ and $w \in E^s$, so that the domain exchange E_ϕ on the Rauzy fractal Ω^s coincides with the first return to Ω^s under the transitive toral translation $x \mapsto x + w \mod \Lambda$ on $E^s/\Lambda \simeq \mathbb{T}^{d-1}$.*

Proof: Let ψ be the substitution in (7.3). For a while all notation will be in the context of ψ, including the Rauzy pieces, the Markov boxes

$$\Omega_i := \bigcup_{0 \le t \le \beta_i} \Omega_i^s - t\omega, \qquad \beta_i := |v_i|_u = \beta^{i-1}|v_1|_u = \beta_1\beta^{i-1}, \qquad i = 1, \ldots, n, \quad (8.8)$$

and the vectors $w_i = \mathrm{pr}_s(v_i)$. By Theorem 7.1, the Ω_i are piecewise disjoint up to measure zero, as are the Ω_i^s (with respect to the $d-1$-dimensional measure).

Note that if kl is any two letter word then $\psi^n(kl) = k \cdots nl \cdots n$. In particular, the abelianization of $\psi^n(l)$, which is just $A^n(v_l)$, is among the return vectors in $\Theta(l)$. It follows that the return lattice Σ coincides with the lattice $\Gamma = \langle v_1, \ldots, v_n \rangle$. Note also that $v_1, \ldots, v_d$ are linearly independent (since $Av_i = v_{i+1}$ for $i = 1, \ldots, n-1$, and $V = \mathrm{lin}(v_1, \ldots, v_n)$ has dimension d) and so also are $w_1, \ldots, w_d$ (since E^u is irrational). Therefore, upon setting

$$\Lambda := \langle w_i - w_j : \ i, j = 1, \ldots, d \rangle \quad (8.9)$$

the translation $\tau : x + \Lambda \mapsto x + w_1 + \Lambda$ is transitive on $E^s/\Lambda \to E^s/\Lambda$.

Let $c_{m,i,j} \in \mathbb{Z}$, $M_m \in \mathbb{Z}^+$ be as in (8.7) with $M_m = 1$ and $c_{m,i,1} = 1$ for $i = 1, \ldots, d-1$. For each $m = 1, \ldots, n$, we take M_m *stable slices* of Ω_m: $\Omega_{m,0}^s := \Omega_m^s$ and

$$\Omega_{m,k}^s := \left(\Omega_m^s - \beta_1 \left(\sum_{j=1}^{k} \eta_{m,j}\omega \right) \right) + \sum_{j=1}^{k} \sum_{i=0}^{d-1} c_{m,i,j} v_{i+1}, \quad \text{for } \ k = 1, \ldots, M_m - 1. \quad (8.10)$$

Since

$$\mathrm{pr}_u \left(\sum_{i=0}^{d-1} c_{m,i,j} v_{i+1} \right) = \sum_{i=0}^{d-1} c_{m,i,j} \beta_1 \beta^i \omega = \beta_1 \eta_{m,j} \omega,$$

$\Omega_{m,k}^s \subset E^s$. Since the boxes Ω_m/Γ are disjoint (up to measure zero), so are the $\Omega_{m,k}^s$.

Now, for $m \in 1, \ldots, d$ and $x \in \Omega_m^s$, the domain exchange E_ψ is given by $E_\psi(x) = x + w_i$, which is congruent to $x + w_1 \bmod \Lambda$. For $m > d$ and $x \in \Omega_m^s$, we have

$$
\begin{aligned}
E_\psi(x) = x &\;+\; w_m \\
= x &\;-\; \beta_1\beta^{m-1}\omega + v_m \\
= x &\;-\; \beta_1\left(\sum_{j=1}^{M_m}\eta_{m,j}\right)\omega + v_m \\
= x &\;+\; \left(\sum_{i=0}^{d-1} c_{m,i,1}v_{i+1} - \beta_1\eta_{m,1}\omega\right) \\
&\;+\; \left(\sum_{i=0}^{d-1} c_{m,i,2}v_{i+1} - \beta_1\eta_{m,2}\omega\right) \\
&\;+\; \cdots + \left(\sum_{i=0}^{d-1} c_{m,i,M_m}v_{i+1} - \beta_1\eta_{m,M_m}\omega\right) \\
&\;+\; v_m - \sum_{j=1}^{M_m}\left(\sum_{i=0}^{d-1} c_{m,i,j}v_{i+1}\right).
\end{aligned}
$$

Each of the vectors $\sum_{i=0}^{d-1} c_{m,i,j}v_{i+1} - \beta_1\eta_{m,j}\omega$ lies in E^s (as its image under pr_u is zero), and so do x and $x - \beta_1\beta^{m-1}\omega + v_m$. Thus the vector $v_m - \sum_{j=1}^{M_m}\left(\sum_{i=0}^{d-1} c_{m,i,j}v_{i+1}\right)$, lying in both E^s and Γ, must be 0. Furthermore,

$$
\mathrm{pr}_s\left(\sum_{i=0}^{d-1} c_{m,i,j}v_{i+1} - \beta_1\eta_{m,j}\omega\right) = \sum_{i=0}^{d-1} c_{m,i,j}w_{i+1} \equiv w_1, \quad \mod \Lambda,
$$

for $j = 1, \ldots, M_n$, since $\sum_{i=0}^{d-1} c_{m,i,j} = 1$. Thus, for $x \in \Omega^s_m$, $E_\psi(x) = x + M_m w_1 \mod \Lambda$. Furthermore, we see that, if $x \in \Omega^s_{m,0} = \Omega^s_m$, then $\tau(x+\Lambda) \in \Omega^s_{m,1} + \Lambda$, $\tau^2(x+\Lambda) \in \Omega^s_{m,2} + \Lambda$, $\ldots$, $\tau^{M_m-1}(x+\Lambda) \in \Omega^s_{m,M_m-1} + \Lambda$, and $\tau^{M_m}(x+\Lambda) = E_\psi(x) + \Lambda \in \Omega^s + \Lambda$.

Thus $E_\psi : \Omega^s \to \Omega^s$ coincides (a.e.) with the first return to Ω^s under translation by w_1, mod Λ, provided we can show that the $\Omega^s_{m,j} + \Lambda$ are pairwise disjoint (up to measure zero). (At this point we only know that the $\Omega^s_{m,j}$ are pairwise disjoint.)

If the disjointness were to fail we would have $x \in \Omega^s_{m,i}$ and $y \in \Omega^s_{l,j}$ with $(m,i) \neq (l,j)$ so that: $x + \Gamma$ and $y + \Gamma$ lie in the full measure set G^u_ψ (see Theorem 4.2) on which h_ψ^{-1} is single valued, and there is a vector $w \in \Delta := \langle v_i - v_j : \; i,j = 1,\ldots,d\rangle$ so that $y - x = \mathrm{pr}_s(w)$. First suppose that $i = 0 = j$ (that is, x and y are in the original Rauzy pieces Ω^s_m and Ω^s_l). Since $x \in \Omega^s_m \cap G^u_\psi$, $h_\psi^{-1}(x+\Gamma) = (\alpha_k)$ is such that x is a vertex of strand α_0 and the edge I of α_0 with $\min I = x$ is labeled m. Let $h_\psi^{-1}(y+\Gamma) = (\beta_k)$ with y a vertex of β_0. Then $x + w$ is a vertex of the strand $\beta_0 + \mathrm{pr}_u(w) =: \gamma_0$ with $(\gamma_k) \in \mathcal{F}^{\leftarrow}_\psi$ and $h_\psi((\gamma_k)) = x + w + \Gamma = h_\psi((\alpha_k))$. Hence, $\gamma_0 = \alpha_0$ so that $x' := x + w$ must be a vertex of α_0. Now, given $u, v \in \Gamma$, we shall say that u *is less than* v, denoted $u \prec v$, provided $u - v = \sum_{i=1}^{d} c_i v_i$ with $\sum_{i=1}^{d} c_i < 0$[11]. Since $0 \prec v_i$ for $i = 1, \ldots, n$ (from (8.7)) and $x' - x = w$ is neither less than nor greater than 0, x' can neither precede nor follow x. Thus $x = x'$, $w = 0$, and $y = x$, a contradiction.

Suppose now that $(m, i) \neq (l, j)$ without the extra assumption $i = 0 = j$. Let $x' \in \Omega_m$ and $y' \in \Omega_l$ and $u, v \in \Gamma$ be such that $x' + u = x$ and $y' + v = y$. Then $x'' := x' + \mathrm{pr}_u(u) \in \Omega^s_{m,0}$, $y'' := y' + \mathrm{pr}_u(v) \in \Omega^s_{l,0}$ and $y'' - x'' = \mathrm{pr}_s(w - u + v)$. If $u - v \in \Delta$, then $y'' = x''$ from the above proof, so that $y = x$. If $u - v \notin \Delta$, then, without loss of generality, $u \prec v$. Let $\tilde{y} = y' + \mathrm{pr}_u(w)$ so that $\tilde{y} - x' \in \Gamma$. Now $h_\psi^{-1}(x' + \Gamma) = (\alpha_k)$ is a singleton, $x' = \min I$, I an edge of α_0 labeled m, and, since $y' \in \Omega_l$, y' is a vertex of β_0, $(\beta_k) \in \mathcal{F}_\psi^{\leftarrow}$, so that $\tilde{y}$ is a vertex of $\gamma_0 := \beta_0 + \mathrm{pr}_u(w)$, $(\gamma_k) \in \mathcal{F}_\psi^{\leftarrow}$. As $h_\psi((\gamma_k)) = \tilde{y} + \Gamma = x' + \Gamma = h_\psi((\alpha_k))$ and $x' + \Gamma \in G^u_\psi$, $\gamma_0 = \alpha_0$ and $\tilde{y}$ is a vertex of α_0. Note that $u \prec v$ implies that $x' - \tilde{y} = u - w - v \prec 0$ so that $\tilde{y}$ must follow x' on α_0. On the other hand, $x \in \Omega^s_{m,i}$ means that $v \prec v_m$. Since the edge following x' on α_0 is labeled m, the vertex following x' is $x' + v_m$ and $\tilde{y} - (x' + v_m) = -v_m + v + w - u \prec 0$. That is, $\tilde{y}$ must come before $x' + v_m$ on α_0 (as well as after x') and this is not possible. This concludes the proof that the $\Omega^s_{m,i} + \Lambda$ are pairwise disjoint up to the measure zero.

We conclude the proof of Proposition 8.5 by translating the above construction back to terms of ϕ via the shift equivalence (Fact 7.2) $\sigma \circ \rho = \phi^{n-1}$, $\rho \circ \sigma = \psi^{n-1}$. Let S and P be the abelianizations of the morphisms σ and ρ. Then $SP = A_\phi^{n-1}$ and $PS = A_\psi^{n-1}$. It follows that the linear maps $P|_{E^s_\phi} : E^s_\phi \to E^s_\psi$, $P|_{E^u_\phi} : E^u_\phi \to E^u_\psi$, $S|_{E^s_\psi} : E^s_\psi \to E^s_\phi$, and $S|_{E^u_\psi} : E^u_\psi \to E^u_\phi$ are isomorphisms. Moreover, P and S restrict to isomorphisms between the lattices $\Sigma_\phi = \Gamma_\phi$ and $\Sigma_\psi = \Gamma_\psi$. (That $\Sigma_\phi = \Gamma_\phi$ is proved as was $\Sigma_\psi = \Gamma_\psi$.)

Let

$$\Lambda_\phi := P^{-1}\Lambda_\psi = \left\langle P^{-1}w_i(\psi) - P^{-1}w_j(\psi) : \; i, j = 1, \ldots, d \right\rangle \subset E^s_\phi \tag{8.11}$$

and let $\tilde{w} := P^{-1}w_1(\psi)$.

Let $\Omega^s_i(\phi)$ and $\Omega_i(\phi)$ be the Rauzy pieces and the corresponding Markov boxes for ϕ, let $\Omega_{m,0}(\psi) := \Omega_m(\psi)$, $m = 1, \ldots, d$, and let

$$\Omega_{m,j}(\psi) = \bigcup \left\{ \Omega^s_m(\psi) - t\omega : \; \beta_1 \sum_{l=1}^{j} \eta_{m,l} \le t \le \beta_1 \sum_{l=1}^{j+1} \eta_{m,l} \right\} \tag{8.12}$$

for $m = d+1, \ldots, n$, $j = 1, \ldots, M_m - 1$. Then P takes each $\Omega_i(\phi)$ in a Markovian way across the $\Omega_{m,j}(\psi)$: let

$$\Omega_i(\phi) = \bigcup_{k=1}^{K_i} \Omega_{i,k}(\phi) \tag{8.13}$$

[11]This is the order on V lifted from that on the line $V/\mathrm{lin}(\Delta)$.

be the corresponding decomposition of $\Omega_i(\phi)$, $i = 1, \ldots, n$. (Thus $P(\Omega_{i,k}(\phi))$ runs entirely across some $\Omega_{m,j}(\phi)$). The domain exchange E_ϕ restricted to $\Omega_i^s(\phi)$ is then a composition of K_i translations, each congruent to a translation by $\tilde{w}$ modulo Λ_ϕ. Thus $E_\phi : \Omega^s(\phi) \to \Omega^s(\phi)$, modulo Λ_ϕ, coincides (a.e.) with the first return to $\Omega^s(\phi) + \Lambda_\phi$ under the toral translation by $\tilde{w} + \Lambda_\phi$ on $E_\phi^s/\Lambda_\phi \simeq \mathbb{T}^{d-1}$. □

Remark 8.6 *Since the translation $\tau : E^s/\Lambda \to E^s/\Lambda$ given by Proposition 8.5 is transitive and the (generalized Rauzy pieces) $\Omega_{m,j}^s$ (see the proof of Proposition 8.5) are closed with nonempty and pairwise disjoint interiors, and $\bigcup_{m,j} \Omega_{m,j}^s + \Lambda$ is invariant under τ, it must be that $\bigcup_{m,j} \Omega_{m,j}^s + \Lambda = E^s + \Lambda$. That is, $\bigcup_{m,j} \Omega_{m,j}^s$ is a fundamental domain for the $d-1$ torus E^s/Λ, as noted in the examples of [11], which all satisfy the hypotheses of Proposition 8.5.*

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