Hyperbolic pseudo-Anosov maps a. e. embed into a toral automorpism

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Abstract

Fathi and Franks showed that a pseudo-Anosov diffeomorphism f with orientable foliations and dilation coefficient λ with no conjugates (over $\mathbb{Q}$) in the unit circle factors onto a (homologically non-trivial) invariant subset of a hyperbolic toral automorphism. After recounting this result, we show that the factor map is either a.e. 1-to-1 or a.e. m-to-1 for some $m > 1$ and the pseudo-Anosov map f is an m-to-1 ramified covering of another pseudo-Anosov (or Anosov) map on a surface of smaller genus. As a corollary, any pseudo-Anosov diffeomorphism with orientable foliations and hyperbolic action on the first homologies a.e. embedds into a hyperbolic toral automorphism.

1 Introduction

At the end of the sixties, Hirsh asked about the nature of the compact sets invariant under hyperbolic toral automorphisms [8]. Symbolic dynamics (via a Markov partition) readily supplies uncountably many such sets, which ignites hope that some more exotic examples of dynamical systems can be unearthed in this way.

For instance, one naturally asks (see [5]) if an embedded submanifold other than a subtorus can be realized. That question, as most questions regarding the global nature of these invariant sets, remains open. Ostensibly, the difficulty lies in the fact that the embedding (if it exists) must be fractal: local results show that any one of a number of "smoothness" assumptions on the invariant set forces it to be a subtorus — see the references in [5]. To make it clear, we shall not answer the question here. Our goal is to merely point out that the answer is a rather easy *yes* if one allows for a.e. embeddings: many pseudo-Anosov maps factor onto an invariant set of a toral automorphisms via a mapping that is continuous and 1-1 on an open full measure set. The following theorem found in [5] serves as our departure point.

Theorem 1 (Fathi via Franks) *Suppose that $f : M \to M$ is a pseudo-Anosov diffeomorphism of a closed surface of genus $g(M)$ with orientable stable and unstable measured foliations, W_f^s and W_f^u, whose dilation coefficient λ has no conjugate over $\mathbb{Q}$ in the unit circle. Let Λ be any family of eigenvalues of the action $f_* : H_1(M, \mathbb{Z}) \to H_1(M, \mathbb{Z})$ induced on the first homology[1] such that Λ contains λ and $1/\lambda$, is closed under conjugation over $\mathbb{Q}$, and avoids the unit circle. If d is the number of eigenvalues in Λ (counted with multiplicity), then there is a continuous $h : M \to \mathbb{T}^d$ that is locally injective at every nonsingular point (of W_f^s and W_f^u) and such that the following diagram commutes*

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ h\downarrow & & h\downarrow \\ \mathbb{T}^d & \xrightarrow{f_A} & \mathbb{T}^d \end{array} \tag{1.1}$$

where f_A is a hyperbolic toral automorphism associated to a matrix[2] A with eigenvalues Λ. The map $h_* : H_1(M, \mathbb{Z}) \to H_1(\mathbb{T}^d, \mathbb{Z})$ is surjective.

If that is plenty to read, restrict attention to the simplest case when Λ consists of all the eigenvalues of f_* and A represents f_*; the number theoretic contortions are a result of finessing this case.

The existence of h is an instance of *the π_1-stability of hyperbolic toral automorphisms* due to Franks [6]. The local injectivity hinges on the natural local product structure given by W_f^u and W_f^s and the global stretching of W_f^u and W_f^s under f and f^{-1}, respectively. The following addendum unveils a rather benign nature of the identifications effected by h.

[1] f_* preserves the intersection form on $H_1(M, \mathbb{Z})$ which makes it symplectic and its characteristic polynomial p reciprocal; in particular, λ and $1/\lambda$ are both eigenvalues.

[2] In fact, A represents f_* restricted to the direct sum of all the eigenspaces of f_* corresponding to the eigenvalues in Λ.

Theorem 2 *In the context of Theorem 1, there is $m \geq 1$ such that f factors via a m-to-1 branched covering $\delta : M \to M_1$ to $f_1 : M_1 \to M_1$ that is pseudo-Anosov (or Anosov if M_1 equals $\mathbb{T}^2$). Moreover, h factors via δ to $h_1 : M_1 \to \mathbb{T}^d$ such that $h_1 \circ f_1 = f_A \circ h_1$ and h_1 is 1-to-1 on an open full measure f_1-invariant G subset of M_1; in fact, $\# h_1^{-1}(h(x)) = 1$ for all $x \in G$. (Here, the measure is the canonical invariant measure of the pseudo-Anosov map.)*

Note that m is uniquely determined by f as the cardinality of the fiber $h^{-1}(h(x))$ for a generic $x \in M$.

Theorem 2 gives the promised almost everywhere 1-to-1 embedding for f inducing a hyperbolic action on $H_1(M, \mathbb{R})$:

Corollary 3 *In the context of Theorem 1, if Λ consists of all the eigenvalues (so that $d = 2g(M)$), then the map h is 1-to-1 on a full measure residual subset $G \subset M$.*

A recent work by Band [3] analyzes an interesting infinite family of pseudo-Anosov maps with foliations having one singularity. For those maps, h fails to be locally injective at the singularity.

We also mention that the global (everywhere) injectivity of h is closely related to the issue of coincidence of *Nielsen classes* and *abelian Nielsen classes* for all periodic points of the pseudo-Anosov map. This is of independent interest in Nielsen-Thurston theory of surface diffeomorphisms (cf. [4]).

Proof of Corollary 3 form Theorem 2: If $m = 1$ then we are done. Let us see that $m > 1$ leads to a contradiction. Since M is an m-to-1 branched covering of M_1, its Euler characteristics satisfies $\chi(M) = m\chi(M_1) - \delta$ where $\delta \geq 0$ is a correction due to the presence of the branch points. It follows[3] that $\dim(H_1(M, \mathbb{Q})) = -\chi(M) + 2 > -\chi(M_1) + 2 = \dim(H_1(M_1, \mathbb{Q}))$ and the map $\delta_* : H_1(M, \mathbb{Q}) \to H_1(M_1, \mathbb{Q})$ cannot be injective. Hence, $h_* = h_{1*} \circ \delta_* : H_1(M, \mathbb{Q}) \to H_1(\mathbb{T}^d, \mathbb{Q})$ is not an isomorphism, which contradicts its surjectivity. □

Applied with the minimal possible Λ that consists of λ, $1/\lambda$ and all their conjugates, Theorem 2 can be viewed as a dichotomy: f either collapses to a pseudo-Anosov (or Anosov) f_1 of a smaller genus[4] (and the same λ) or f a.e. embeds into a toral automorphism. We note that both parts of the dichotomy can be realized. In [2], Arnoux and Fathi give an example of a pseudo-Anosov diffeomorphism f with orientable foliations on a surface of genus 3 whose dilatation coefficient is algebraic of degree 4 and such that f does not factor via a branched covering to a pseudo-Anosov map with irreducible action on its rational homology.[5] A similar example is impossible for λ that is a quadratic irrationality; indeed, we recover the result (Theorem 2.3) from [7]:

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[3] Indeed, $-\chi(M) + 2 = -\chi(M_1) + 2 + (m - 1) \cdot (-\chi(M_1)) + \delta$ where either $-\chi(M_1) > 0$ or $M_1 = \mathbb{T}^2$ and then $\delta > 0$.

[4] Equal to the index of the field $\mathbb{Q}(\lambda, 1/\lambda)$ over $\mathbb{Q}$.

[5] This f must then a.e embed into a toral automorphism.

Corollary 4 (Franks-Rykken) *A pseudo-Anosov map with orientable foliations and a quadratic dilatation factors via a branched covering to an Anosov automorphism on* $\mathbb{T}^2$.

Proof: What makes $d = 2$ special is that $h_1 : M_1 \to \mathbb{T}^2$ is actually onto; indeed, $h_1(G)$ is open in $\mathbb{T}^2$ and thus dense by ergodicity of f_1. It remains to see that M_1 is a torus. Suppose not. Then $W^s_{f_1}$ has a singularity p. Now, $h_1 : M_1 \to \mathbb{T}^2$ respects the dynamics so it maps stable sets into stable sets. In particular, for each prong P_i of $W^s_{f_1}$ starting at p, $i = 1, \dots, l$, $h_1(P_i)$ contains a half-leaf Q_i of the stable foliation on $\mathbb{T}^2$ starting at $q := h_1(p)$. (Indeed, being dense in $\mathbb{T}^2$, $h_1(Q_i)$ cannot be a bounded segment of the stable leaf in $\mathbb{T}^2$.) Since $l \geq 4 \geq 2$, there must be $i \neq j$ with $Q_i = Q_j$. Thus, for $y \in Q_i$, $\#h^{-1}(y) \geq 2$, which is a contradiction since P_i is dense in M_1 and thus enters G. $\square$

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We prove Theorem 2 in Section 3. The basic idea is that, by ergodicity of f, h is m-to-1 on a generic full measure set G, and the space of fibers (treated as divisors on M) of the restriction $h|_G$ can be completed to form the *smaller* surface M_1.

For completeness, in Section 2, we give an account of Theorem 1. Here we adopt a very concrete point of view and exhibit explicit formulas for h. The basic idea is to first *statically* map M into $\mathbb{T}^d$ by using the integration of closed 1-forms on M (a la Jacobi) and then adjust that map to h that respects the dynamics by implementing global shadowing on the level of the homology cover of M. There is nothing really new here of course but a reader interested in Hirsh's question may appreciate having a self-contained picture at hand.

2 The map h and its local injectivity

Recall that $f : M \to M$ is a pseudo-Anosov diffeomorphism of an oriented closed surface M of genus $g(M) > 1$ and that the stable and unstable (measured) foliations of f, W^s_f and W^u_f, are orientable. This last assumption assures that W^s_f and W^u_f are induced as level sets of closed 1-forms ω^s and ω^u, respectively, where $f^*(\omega^s) = \lambda^{-1}\omega^s$ and $f^*(\omega^u) = \lambda\omega^u$. To fix attention, let us suppose that f preserves the orientation on M so that $\lambda > 1$.

Our goal is to give a very concrete construction of h satisfying the commutative diagram (1.1). The initial ingredient comes in the form of a collection of closed 1-forms on M, $\omega_1, \dots \omega_d$, $d \leq 2g(M)$, whose cohomology classes are linearly independent and generate a subgroup

$$\Omega := \{k_1[\omega_1] + \dots k_d[\omega_d] : \; k_j \in \mathbb{Z}\} \subset H^1(M, \mathbb{Z})$$

invariant under the cohomology action $f^* : H^1(M,\mathbb{Z}) \to H^1(M,\mathbb{Z})$. Thus

$$f^*([\omega_i]) = \sum_j a_{ij}[\omega_j], \quad i = 1, \dots, d, \tag{2.1}$$

for some integer matrix $A = (a_{ij})_{i,j=1}^d$. Note that A is invertible[6] over $\mathbb{Z}$ because f^* is an automorphism. We make the following hypotheses.

(H) A is hyperbolic, i.e., all its eigenvalues are off the unit circle.

(S) $[\omega^u]$ and $[\omega^s]$ are in the linear span of Ω, $[\omega^u], [\omega^s] \in \mathrm{lin}_{\mathbb{R}}(\Omega)$.

(The assumption (S) will only play a role in the local injectivity of h, not its existence.)

The choice of the cohomology classes $[\omega_i]$ is dictated by Λ. In the simplest case, when Λ contains all the eigenvalues and $d = 2g(M)$, one can take $[\omega_i]$ that form a basis of $H^1(M,\mathbb{Z})$. In the general case, Λ being closed under conjugation allows one to factor (over $\mathbb{Q}$) the characteristic polynomial $p(z)$ of f^* as $p(z) = q(z)r(z)$ so that Λ consists of the roots of $q(z)$. By the rational canonical form, there is then a subspace $V \subset H^1(M,\mathbb{Q})$ such that Λ makes up the eigenvalues of the restriction $f^*|V$. Suitable $[\omega_i]$ are obtained by taking any integral basis of V, i.e., a basis made of vectors in $H^1(M,\mathbb{Z}) \cap V$.

Denote by $L : H_1(M,\mathbb{R}) \to \mathbb{R}^d$ the map given by $[\gamma] \mapsto (\int_\gamma \omega_i)_{i=1}^d$. Clearly, $L(H_1(M,\mathbb{R})) = \mathbb{R}^d$ and $\Gamma := L(H_1(M,\mathbb{Z}))$ is a lattice in $\mathbb{R}^d$, i.e., a finitely generated rank d subgroup. (When $[\omega_i]$'s are a basis of $H^1(M,\mathbb{Z})$, we simply have $\Gamma = \mathbb{Z}^d$.) Acting on the torus $\mathbb{T}^d := \mathbb{R}^d/\Gamma$ is an automorphism $f_A : \mathbb{T}^d \to \mathbb{T}^d$ induced by the matrix A. Here A multiplies column vectors on the right so that it is a factor of $f_* : H_1(M,\mathbb{R}) \to H_1(M,\mathbb{R})$ via L, $L \circ f_* = A \circ L$.[7]

The relation between f and f_A is most conveniently established on the level of appropriate (abelian) covering spaces: $\mathbb{R}^d$ for $\mathbb{T}^d$ and *the homology cover* $\hat{M}$ for M. Recall that M is the quotient of $\hat{M}$ under the natural (deck) action of $H_1(M,\mathbb{Z})$, for which we adopt the additive notation $x \mapsto x + v$ where $v \in H_1(M,\mathbb{Z})$ and $x \in \hat{M}$. Lifted (i.e. pulled back) to $\hat{M}$, ω_i can be represented as a total differential $d\hat{\phi}_i$ of a function $\hat{\phi}_i : \hat{M} \to \mathbb{R}$, $\hat{\omega}_i = d\hat{\phi}_i$. Because $\hat{\phi}_i(x + [\gamma]) = \hat{\phi}_i(x) + \int_\gamma \omega_i$ for $x \in \hat{M}$ and $[\gamma] \in H_1(M,\mathbb{Z})$, the map $\Phi : \hat{M} \to \mathbb{R}^d$ given by

$$\Phi(x) := (\hat{\phi}_1(x), \dots \hat{\phi}_d(x)) \tag{2.2}$$

is equivariant:

$$\Phi(x + v) = \Phi(x) + Lv, \quad v \in H_1(M,\mathbb{Z}),\ x \in \hat{M}. \tag{2.3}$$

[6] From $f^*\Omega \subset \Omega$, since f^* is an automorphism, $(f^*)^{-1}\Omega \subset (f^*)^{-2}\Omega \subset \ldots \subset H^1(M,\mathbb{Z})$ so that $(f^*)^{-k}\Omega \subset (f^*)^{-k-1}\Omega$ for some k so that $(f^*)^{-1}\Omega = \Omega$.

[7] $L \circ f_*([\gamma]) = \left(\int_{f_*(\gamma)} \omega_i\right)_i = \left(\int_\gamma f^*(\omega_i)\right)_i = \left(\int_\gamma \sum_j a_{ij}\omega_j\right)_i = A \circ L([\gamma])$.

When $d = 2g(M)$, it is a classical device to arrange the ω_i so that Φ is an embedding and $\hat{M}$ can be imagined as a *periodic surface in* $\mathbb{R}^d$ with its $\mathbb{Z}^d$-quotient being a copy of M embedded into $\mathbb{T}^d$. However, for our purposes, we only need that, if we lift f to $\hat{f} : \hat{M} \to \hat{M}$, then the diagram

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\hat{f}} & \hat{M} \\ {\scriptstyle\Phi}\downarrow & & {\scriptstyle\Phi}\downarrow \\ \mathbb{R}^d & \xrightarrow{A} & \mathbb{R}^d \end{array} \tag{2.4}$$

commutes up to a uniform constant $C > 0$ (i.e. $\sup_{x\in\hat{M}} |\Phi \circ \hat{f}(x) - A \circ \Phi(x)| \leq C < +\infty$). This is because $\Phi \circ \hat{f} - A \circ \Phi$ a lift of a function on M; indeed, (2.3) and $A \circ L = L \circ f_*$ yield the deck invariance: $(\Phi \circ \hat{f} - A \circ \Phi)(x+v) = (\Phi \circ \hat{f} - A \circ \Phi)(x)$.[8]

Now, improving a nearly commuting diagram (2.4) to a commuting one is a standard fare of hyperbolic dynamics. Given $x \in \hat{M}$, set $x_n := \hat{f}^n(x)$, $y_n := \Phi(x_n)$, and $\delta_n := \Phi \circ \hat{f}(x_n) - A \circ \Phi(x_n)$ so that

$$y_{n+1} = Ay_n + \delta_n, \quad n \in \mathbb{Z}. \tag{2.5}$$

Because A is hyperbolic and the δ_n are uniformly bounded, there exists a unique $z_0 \in \mathbb{R}^d$ such that the sequence $(z_n) \subset \mathbb{R}^d$ determined by

$$z_{n+1} = Az_n, \quad n \in \mathbb{Z} \tag{2.6}$$

globally shadows (y_n), i.e. $\sup_{n\in\mathbb{Z}} |y_n - z_n| < +\infty$. Explicitly, let E^u and E^s be the stable and unstable spaces of A so that $\mathbb{R}^d = E^s \oplus E^u$; and decompose $x_n = x_n^s + x_n^u$ with $x_n^u \in E^u$ and $x_n^s \in E^s$, etc. Then

$$z_0^u = y_0^u + \sum_{k=0}^{\infty} A^{-k-1}\delta_k^u \tag{2.7}$$

with a similar formula for z_0^s. One easily checks that $\hat{h} : \hat{M} \to \mathbb{R}^d$ given by $x_0 \mapsto z_0$ is (Hölder) continuous, equivariant (i.e. $h(x + v) = h(x) + Lv$ for $v \in H_1(M, \mathbb{Z})$, $x \in \hat{M}$), a bounded perturbation of Φ (i.e. $\sup_{x\in\hat{M}} |h(x) - \Phi(x)| < +\infty$), and makes the following diagram commute

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\hat{f}} & \hat{M} \\ {\scriptstyle\hat{h}}\downarrow & & {\scriptstyle\hat{h}}\downarrow \\ \mathbb{R}^d & \xrightarrow{A} & \mathbb{R}^d \end{array} \tag{2.8}$$

[8] Precisely, $(\Phi\circ\hat{f} - A\circ\Phi)(x+v) = \Phi(\hat{f}(x) + f_*(v)) - A\circ\Phi(x+v) = \Phi\circ\hat{f}(x) + L\circ f_*(v) - A(\Phi(x) + Lv) = \Phi\circ\hat{f}(x) - A\circ\Phi(x)$ where we used $\hat{f}(v+v) = \hat{f}(x) + f_*(v)$, (2.3), and $A \circ L = L \circ f_*$, in that order. Alternatively, from (2.1), $f^*\omega_i - \sum_{j=1}^d a_{ij}\omega_j$ is a coboundary making its antiderivative $\hat{\phi}_i \circ \hat{f} - \sum_{j=1}^d a_{ij}\hat{\phi}_j$ a lift of a function on M.

Thus the quotient $h : M \to \mathbb{T}^d$ of $\hat{h}$ factors f onto an invariant subset of f_A. This is a special case of the map constructed by Franks in [6]. Incidentally, the resemblance of (2.7) to the classical Weierstrass function underpins the nowhere differentiability of h. [9]

Remark 1 *h depends on the choice of lift $\hat{f}$. Upon replacing $\hat{f}$ by $\hat{f}_v := \hat{f} + v$ for some $v \in H_1(M,\mathbb{Z})$, $\hat{h}$ becomes $\hat{h}_v = \hat{h} - w$ where $w = (I - A)^{-1}Lv$.[10] It is easy to see that any h in Theorem 1 arises through global shadowing and thus is uniquely determined by $L := h_* : H_1(M,\mathbb{Z}) \to H_1(\mathbb{T}^d,\mathbb{Z})$ up to post-composition with a translation by a vector from the finite subgroup $(I - A)^{-1}\Gamma/\Gamma \subset \mathbb{T}^d$ (c.f. [6]).*

Our next task is to show, following Fathi [5], the local injectivity of h near non-singular points (i.e. regular points of W^u_f and W^s_f). Let $\hat{\phi}^u, \hat{\phi}^s : \hat{M} \to \mathbb{R}$ be such that $d\hat{\phi}^{s/u} = \hat{\omega}^{s/u}$; and set $\hat{\Phi}^{su} := \hat{\phi}^u \oplus \hat{\phi}^s : \hat{M} \to \mathbb{R}^2$. Since $\hat{\Phi}^{su}$ is manifestly locally injective near non-singular points it suffices to show that, for $x, y \in \hat{M}$,

$$\hat{h}(x) = \hat{h}(y) \Longrightarrow \hat{\Phi}^{su}(x) = \hat{\Phi}^{su}(y). \tag{2.9}$$

To prove that, suppose that $\hat{h}(x) = \hat{h}(y)$. From the definition of $\hat{h}$, there is $C > 0$ so that

$$|\Phi(\hat{f}^n(x)) - \Phi(\hat{f}^n(y))| \leq 2C, \quad n \in \mathbb{Z}.$$

From hypothesis (S), $[d\hat{\phi}^u] = \sum_{i=1}^d b_i[d\hat{\phi}_i]$ for some $b_i \in \mathbb{R}$; and we can integrate to get $\hat{\phi}^u = \sum_{i=1}^d b_i\hat{\phi}_i + \epsilon$ where $\epsilon : \hat{M} \to \mathbb{R}$ is bounded. By using $f^*\omega^u = \lambda\omega^u$ and $|\hat{\phi}_i| \leq |\Phi|$, the above inequality yields

$$|\lambda^n\hat{\phi}^u(x) - \lambda^n\hat{\phi}^u(y)| = |\hat{\phi}^u \circ \hat{f}^n(x) - \hat{\phi}^u \circ \hat{f}^n(y)| \leq \sum_{i=1}^d |b_i|2C + \max|\epsilon|, \quad n \in \mathbb{Z}.$$

That forces $\hat{\phi}^u(x) = \hat{\phi}^u(y)$. An analogous argument shows $\hat{\phi}^s(x) = \hat{\phi}^s(y)$.

Before leaving this section, let us attend to a certain trivial redundancy of $\hat{h}$ that we have chosen to ignore so far. Set

$$\Omega^\perp := \left\{ v \in H_1(M,\mathbb{Z}) : \int_v \omega_i = 0, \ i = 1,\ldots,d \right\} = \{v \in H_1(M,\mathbb{Z}) : Lv = 0\}.$$

For $\hat{x} \in \hat{M}$ and $v \in \Omega^\perp$, we have $\Phi(\hat{x} + v) = \Phi(\hat{x})$ and thus also $\hat{h}(\hat{x} + v) = \hat{h}(\hat{x})$. (From (S), $\hat{\Phi}^{su}(\hat{x}+v) = \hat{\Phi}^{su}(\hat{x})$ as well.) That is, throughout this section, $\hat{M}$ could have been replaced by a smaller abelian covering $\hat{M}/\Omega^\perp$. For future reference we record the following.

[9] By global shadowing, leaves of the stable foliation for $\hat{f}$ must map into cosets of E^s. The resulting graphs look like the Weierstrass function, see e.g. the figure in [3].

[10] Indeed, x under $\hat{f}$ shadows y under A iff x under $\hat{f}_v$ shadows y under $A + Lv$; and y under $A + Lv$ is uniquely shadowed by $y - w$ under A because $(T_{Lv} \circ A) \circ T_w = T_w \circ A$. Thus x under $\hat{f}_v$ shadows $y - w$ under A.

Fact 2 *For $x, y \in M$, $h(x) = h(y)$ iff $\hat{h}(\hat{x}) = \hat{h}(\hat{y})$ for some lifts $\hat{x}, \hat{y} \in \hat{M}$. Moreover, such $\hat{x}$ and $\hat{y}$ are unique modulo $\Omega^\perp$ and a common deck translation, i.e., if $\hat{h}(\hat{x}+v) = \hat{h}(\hat{y}+u)$ for some $u, v \in H_1(M, \mathbb{Z})$ then $u - v \in \Omega^\perp$. Also, $\hat{\Phi}^{su}(\hat{x}) = \hat{\Phi}^{su}(\hat{y})$.*

The map $\overline{h} : \hat{M}/\Omega^\perp \to \mathbb{R}^d$ induced by $\hat{h}$ has therefore an advantage over $\hat{h}$ in that, given $p \in \mathbb{T}^d$ and its lift $\hat{p} \in \mathbb{R}^d$, $h^{-1}(p)$ and $\overline{h}^{-1}(\hat{p})$ are in natural bijective correspondence.

Proof of Fact 2: Fix lifts $\hat{x}$ and $\hat{y}$ of x and y. If $\hat{h}(\hat{x}) = \hat{h}(\hat{y})$ then $h(x) = h(y)$ by definition of h. Also, if $h(x) = h(y)$ then $\hat{h}(\hat{x}) = \hat{h}(\hat{y}) + v$ for some $v \in \Gamma$, and so $\hat{h}(\hat{x}) = \hat{h}(\hat{y}+a)$ upon choosing $a \in H_1(M, \mathbb{Z})$ with $La = v$ (made possible by $\Gamma = L(H_1(M, \mathbb{Z}))$).

As for the uniqueness, from $\hat{h}(\hat{x}+v) = \hat{h}(\hat{y}+u)$, $|\Phi \circ \hat{f}^n(\hat{x}+v) - \Phi \circ \hat{f}^n(\hat{y}+u)| = |\Phi \circ \hat{f}^n(\hat{x}) + A^n Lv - \Phi \circ \hat{f}^n(\hat{y}) - A^n Lu|$ is bounded uniformly in $n \in \mathbb{Z}$. Since so is $|\Phi \circ \hat{f}^n(\hat{x}) - \Phi \circ \hat{f}^n(\hat{y})|$, we must have $A^n(Lu - Lv)$ uniformly bounded. Hyperbolicity of A forces that $L(u - v) = 0$, i.e., $u - v \in \Omega^\perp$.

That $\hat{\Phi}^{su}(\hat{x}) = \hat{\Phi}^{su}(\hat{y})$ was already observed in (2.9). □

3 Proof of Theorem 2

The idea is to construct M_1 abstractly as a space of certain divisors on M (obtained from the generic fibers of h). In order to show that M_1 inherits from M a structure of a 2-dimensional surface as well as pseudo-Anosov dynamics, we have to attend more carefully to the relevant structures on M. Hence, to a simply connected open neighborhood U of $x \in M$ we associate a unique function $\phi : U \to \mathbb{C} \simeq \mathbb{R} + i\mathbb{R}$ given by $\phi(x) := \phi^s(x) + i\phi^u(x)$ where $\phi^s, \phi^u : U \to \mathbb{R}$ are such that $d\phi^s = \omega^s$, $d\phi^u = \omega^u$ and $\phi^s(x) = \phi^u(x) = 0$. We shall refer to such ϕ as *an ω-chart centered at x*. Of course, if x is a singularity then ϕ fails to be 1-1; however, if x is a regular point of the foliations and U is small enough, then ϕ is a bijection onto a open subset of $\mathbb{C}$. The set of all such *regular ω-charts* forms an atlas with transition maps that are translations of $\mathbb{C}$ and thus defines a flat Euclidean structure on $M \setminus S$ where S is the set of singularities of W^s_f and W^u_f. This Euclidean structure becomes singular at the points of S — which are *conical singularities* (with trivial holonomy) — but the associated conformal structure on $M \setminus S$ has removable singularities at S and thus extends uniquely to all of M. In fact, for our purposes, it is best to view *an orientation preserving pseudo-Anosov map $f : M \to M$ with orientable foliations* as a homeomorphism of a Riemann surface M of genus $g(M) > 0$ such that there exists a holomorphic closed 1-form ω and $\lambda > 1$ such that $f^*\omega = \lambda\omega^u + i\lambda^{-1}\omega^s$ where ω^s and ω^u are the real and imaginary parts of ω.

This analytic definition of a pseudo-Anosov map can be found, for instance, in

[1] (see page 100 in 3.4. of the Russian edition)[11]. Still, for the reader accustomed to the geometric definition of a pseudo-Anosov map, let us outline the process of extending the conformal structure across the puncture at a singularity x — we shall use this classical device later in the proof. If x is a singularity with $2s$ prongs, then the ω-chart ϕ is a branched s-to-1 covering onto its image with x serving as the unique branch point. For a suitable choice of U, $U \setminus \{x\}$ is a topological annulus and $\phi(U \setminus \{x\}) = D_r \setminus \{0\}$ where $D_r := \{z \in \mathbb{C} : \ |z| < r\}$, for some $r > 0$. Classification of the coverings of an annulus asserts that the restriction $\phi : U \setminus \{x\} \to D_r \setminus \{0\}$ is isomorphic to the standard s-to-1 covering $D_{r^{1/s}} \setminus \{0\} \to D_r \setminus \{0\}$, $z \mapsto z^s$, which is to say that there is a (unique up to rotation by $2\pi/s$) 1-to-1 map $\tilde{\phi} : U \setminus \{x\} \to D_{r^{1/s}} \setminus \{0\}$ such that $\tilde{\phi}^s = \phi$. Being holomorphic and 1-1, $\tilde{\phi}$ is biholomorphic. The conformal structure near $x \in M$ is obtained by admitting $\tilde{\phi}$, extended by $\tilde{\phi}(x) := 0$, into the atlas. (That ϕ is analytic and thus ϕ_s and ϕ_u are harmonic with respect to that structure is now clear.)

We shall use the following lemma extracted from the construction of h in the previous section.

Lemma 1 *There is $r_0 > 0$ such that if $h(x) = h(y)$ and $\phi : U \to \mathbb{C}$ and $\psi : V \to \mathbb{C}$ are ω-charts centered at x and y, respectively, with $U, V \subset M$ of diameter less than r_0, then $\phi(p) = \psi(q)$ whenever $h(p) = h(q)$ for $p \in U$ and $q \in V$.*

Proof: Let $\hat{x}$ and $\hat{y}$ be the lifts of x and y to $\hat{M}$ with $\hat{h}(\hat{x}) = \hat{h}(\hat{y})$ as supplied by Fact 2. Let $\hat{p}$ and $\hat{q}$ be the lifts to $\hat{M}$ with $\hat{h}(\hat{p}) = \hat{h}(\hat{q})$ and $\hat{p}$ chosen near $\hat{x}$ so that $\text{dist}(\hat{x}, \hat{p}) \leq r_0$. Of course, for some $v \in H_1(M, \mathbb{Z})$, $\hat{q}$ is near $\hat{y} + v$ so that $\text{dist}(\hat{y} + v, \hat{q}) \leq r_0$. We claim that, if $r_0 > 0$ is small enough, then necessarily $v \in \Omega^{\perp}$. From that claim, $\hat{\Phi}^{su}(\hat{x}) = \hat{\Phi}^{su}(\hat{y}) = \hat{\Phi}^{su}(\hat{y} + v)$ and $\hat{\Phi}^{su}(\hat{p}) = \hat{\Phi}^{su}(\hat{q})$ so that $\phi(p) = \hat{\Phi}^{su}(\hat{p}) - \hat{\Phi}^{su}(\hat{x}) = \hat{\Phi}^{su}(\hat{q}) - \hat{\Phi}^{su}(\hat{y} + v) = \psi(q)$, and we are done.

It is left to prove the claim. Take $C > 0$ as in the diagram (2.4). The hyperbolicity of A and the fact that $\Gamma = L(H_1(M, \mathbb{Z}))$ is discrete in $\mathbb{R}^d$ assure existence of $N > 0$ such that if $v \in H_1(M, \mathbb{Z})$ and $|A^n L v| \leq 4C + 2 \cdot 2004$ for all $-N \leq n \leq N$ then $Lv = 0$. By using the uniform continuity of $\hat{f}$, one can pick $r_0 > 0$ small enough (and independent of x, y, p, q) that,

$$
\begin{aligned}
|A^n L v| &= |\Phi \circ \hat{f}^n(\hat{y}) + A^n L v - \Phi \circ \hat{f}^n(\hat{y})| \\
&= |\Phi \circ \hat{f}^n(\hat{y} + v) - \Phi \circ \hat{f}^n(\hat{y})| \\
&\leq |\Phi \circ \hat{f}^n(\hat{y} + v) - \Phi \circ \hat{f}^n(\hat{q}))| \\
&+ |\Phi \circ \hat{f}^n(\hat{q}) - \Phi \circ \hat{f}^n(\hat{p}))| \\
&+ |\Phi \circ \hat{f}^n(\hat{p}) - \Phi \circ \hat{f}^n(\hat{x}))| \\
&+ |\Phi \circ \hat{f}^n(\hat{x}) - \Phi \circ \hat{f}^n(\hat{y}))| \\
&\leq 2004 + 2C + 2004 + 2C = 4C + 2 \cdot 2004
\end{aligned}
$$

[11]In fact, [1] is more general in allowing for non-orientable foliations, which brings in quadratic differentials in the place of harmonic 1-forms.

for all $-N \leq n \leq N$. For this r_0, $Lv = 0$, i.e., $v \in \Omega^{\perp}$, as claimed. □

Corollary 2 *h is finite to one, i.e.,* $\sup_{x\in M} \# h^{-1}(x) < \infty$.

Proof: Cover M with finitely many ω-charts as in the lemma. The points of $h^{-1}(x)$ within each ω-chart ϕ are in the fiber of ϕ; and ϕ is finite to one. □

Let E be the exceptional set of h,

$$E := \{x \in M : \; h^{-1}(h(x)) \text{ contains a singularity } \}, \tag{3.1}$$

and let m be the minimal degree of h restricted to the complement of E,

$$m := \min\{\# h^{-1}(h(x)) : \; x \in M \setminus E\}. \tag{3.2}$$

Consider

$$G := \{x \in M \setminus E : \; \# h^{-1}(h(x)) = m\}. \tag{3.3}$$

Fact 3 *G is an open dense f-invariant full measure subset and $G \ni x \mapsto h^{-1}(h(x))$ is a continuous mapping into the space of compact subsets of M (taken with the usual Hausdorff distance).*

Proof: Consider $x_n \in M$ with $x := \lim_{n\to\infty} x_n \in G$. Suppose that $h^{-1}(h(x_n))$ fails to converge to $h^{-1}(h(x))$. Since h is univalent on a neighborhood of every point of $h^{-1}(h(x))$ and $\# h^{-1}(h(x_n)) \geq \# h^{-1}(h(x)) = m$, one finds $y_n \in h^{-1}(h(x_n))$ with $\text{dist}(y_n, h^{-1}(h(x))) > \epsilon > 0$ along a subsequence of $n \to \infty$. For any limit point y of such y_n's, $h(y) = \lim_{n\to\infty} h(x_n) = h(x)$, which contradicts $\text{dist}(y, h^{-1}(h(x))) \geq \epsilon > 0$.

Thus, at $x \in G$, $M \ni x \mapsto h^{-1}(h(x))$ is continuous. This being a mapping into $\mathbb{N}$ forces G to be open. Since G is f-invariant and f is ergodic, G must be of full measure and thus also dense. □

By using the dynamics of f, one can of course learn more about G. At this point, all that we need is the following *metric* path connectedness condition. For convenience let us fix the path metric on M given by measuring the total variation of the forms ω^s and ω^u along paths:

$$\text{dist}(x, y) := \inf \{|\gamma| : \quad \gamma \text{ is a smooth path connecting } x \text{ to } y\}, \tag{3.4}$$

$$|\gamma| := \int_\gamma |\omega^u| + \int_\gamma |\omega^s|. \tag{3.5}$$

Fact 4 *For $x, y \in G$, one can find a smooth curve $\gamma \subset G$ connecting x to y and such that*

$$|\gamma| \leq 4 \, \text{dist}(x, y). \tag{3.6}$$

Proof: Fix $x, y \in G$. First, assemble finitely many segments of leaves of the W_f^s and W_f^u foliations into a piecewise-smooth curve $\eta \subset M$ connecting x to y with $|\eta| \le 2\,\mathrm{dist}(x,y)$. (The set of (x,y) for which this can be done is easily seen to be open and closed and thus equals $M \times M$.)

Now, being open, G must contain some open segment J^u of a leaf of W^u and some open segment J^s of a leaf of W^s. By the minimality of the foliations, $f^n(J^u) \subset G$ and $f^{-n}(J^s) \subset G$ become increasingly dense in M as $n \to \infty$. Thus, taking n large enough, one can easily replace each segment of η by a curve made of (say three) segments contained in $f^n(J^u) \cup f^{-n}(J^s) \subset G$ so that the length is at most doubled in the process. The resulting curve γ from x to y satisfies $|\gamma| \le 2\,|\eta| \le 2 \cdot 2\,\mathrm{dist}(x,y)$. □

Recall that ***a (positive) divisor on*** M is a function $\delta : M \to \{0,1,2,\ldots\}$ taking value zero at all but finitely many points $y_1, \ldots y_r$ on M. If $a_i := \delta(y_i)$, then it is customary to express δ as a formal sum $a_1 y_1 + \ldots + a_r y_r$. By ***an*** m***-divisor on*** M we understand δ as above of *degree* m, i.e., $a_1 + \ldots + a_r = m$. By ***a marked m-divisor*** we understand an m-divisor with a distinguished element of the underlying set $\{y_1, \ldots, y_m\}$. The sets of all m-divisors, $\mathcal{D}iv_m(M) := \{x_1 + \ldots + x_m : \ x_i \in M\}$, and of all marked m-divisors, $\mathcal{D}iv_m^{\mathrm{mark}}(M)$, have Hausdorff topologies induced from M. (A compatible metric on $\mathcal{D}iv_m(M)$ is given by $\mathrm{dist}(x_1 + \cdots + x_m, y_1 + \cdots + y_m) := \min_{\sigma \in \Sigma_m} \max_{i=1}^m \mathrm{dist}(x_i, y_{\sigma(i)})$, where Σ_m is the set of all permutations of $\{1, \ldots, m\}$.) Note that h induces a natural map

$$\delta_G^{\mathrm{mark}} : G \to \mathcal{D}iv_m^{\mathrm{mark}}(M), \quad \delta_G^{\mathrm{mark}}(x) := x_1 + \ldots x_m \tag{3.7}$$

where $\{x_1, \ldots, x_m\} = h^{-1} \circ h(x)$ and $x_1 = x$ is taken as the distinguished point.

Proposition 5 *The mapping δ_G^{mark} uniquely extends to a continuous mapping $\delta^{mark} : M \to \mathcal{D}iv_m^{mark}(M)$. Moreover, denoting by $\delta(x)$ the unmarked divisor associated to $\delta^{mark}(x)$, for any $x, y \in M$, we have*

(i) $\delta(x)$ is contained in the fiber of h, i.e., $\#h(\delta(x)) = 1$.

(ii) $\delta(x) \cap \delta(y) \ne \emptyset \implies \delta(x) = \delta(y)$.

(iii) $\delta(f(x)) = f(\delta(x))$.

(iv) x belongs to $\delta(x)$.

Proof: The main task is to establish uniform continuity of δ_G^{mark}. Let $r_0 > 0$ be as in Lemma 1. Fix $z \in M$ and let $0 < \epsilon < r_0$ be small enough so that, if $\{z_1, \ldots, z_l\} = h^{-1} \circ h(z)$, then the balls $B_\epsilon(z_j)$ are pairwise disjoint and support ω-charts $\phi_i : B_\epsilon(z_j) \to \mathbb{C}$ centered at z_j.

Consider arbitrary $x, y \in B_{\epsilon/8}(z) \cap G$. Let $\gamma : [0,1] \to B_\epsilon(z) \cap G$ be a curve from $x = \gamma(0)$ to $y = \gamma(1)$ with $|\gamma| \le 4\,\mathrm{dist}(x,y)$ as provided by Fact 4. Also, for $k = 1, \ldots, m$, let $\gamma_k : [0,1] \to M$ be the curve starting at x_k and *ω-related* to γ, i.e., γ_k is the lift of $\phi_1(\gamma)$ via the covering $\phi_j : B_\epsilon(z_j) \setminus \{z_j\} \to \mathbb{C} \setminus \{0\}$ where j is taken so that $x_k \in B_\epsilon(z_j)$.

Because there are univalent ω-charts centered at $x, x_k \in G$ and $t \mapsto h^{-1}(h(\gamma(t))$ is continuous at $t = 0$ (by Fact 3), Lemma 1 assures that $\gamma_k(t) \in h^{-1}(h(\gamma(t))$ for all small enough $t \geq 0$. (Precisely, by continuity of h^{-1}, there is a point $x_k(t) \in h^{-1}(h(\gamma(t))$ near x_k and by using Lemma 1 and the univalence of the charts we conlude that $x_k(t) = \gamma_k(t)$.)

By the same token, the set of t for which $\{\gamma(t), \gamma_2(t), \ldots, \gamma_m(t)\} = h^{-1} \circ h(\gamma(t))$ is open and thus equals $[0, 1]$ (as it is also manifestly closed). In particular, $\{\gamma(1), \gamma_2(1), \ldots, \gamma_m(1)\} = h^{-1} \circ h(y)$. The lengths of the paths being equal: $|\gamma| = |\gamma_2| = \ldots = |\gamma_m|$, we have the following bound on the Hausdorff distance

$$\operatorname{dist}(h^{-1} \circ h(x), h^{-1} \circ h(y)) \leq |\gamma| \leq 4 \operatorname{dist}(x, y), \quad x, y \in G. \tag{3.8}$$

What is more, by construction, for $j = 1, \ldots, l$ and $k = 1, \ldots, m$, we have $\gamma_k(0) \in B_\epsilon(z_j)$ iff $\gamma_k(1) \in B_\epsilon(z_j)$. Hence, in each $B_\epsilon(z_j)$, $h^{-1} \circ h(x)$ and $h^{-1} \circ h(y)$ have the same number of points — denote it by $a_j \geq 0$. [12]

This last observation and (3.8) show that the distance in $\mathcal{D}iv_m^{\text{mark}}(M)$ between $\delta_G^{\text{mark}}(x)$ and $\delta_G^{\text{mark}}(y)$ shrinks to 0 as $\epsilon \to 0$ uniformly for $x, y \in G \cap B_{\epsilon/8}(z)$. Since M can be covered by finitely many balls $B_{\epsilon/8}(z)$ as above, δ_G^{mark} is uniformly continuous and thus has a unique continuous extension $\delta^{\text{mark}} : M \to \mathcal{D}iv_m^{\text{mark}}(M)$.

(i): is immediate by continuity of h.

(ii): It suffices to show that $y \in \delta(x) \implies \delta(y) = \delta(x)$. (Indeed, $z \in \delta(x) \cap \delta(y)$ implies then $\delta(x) = \delta(z) = \delta(y)$.) The implication is certainly true for $x, y \in G$, and the extension to all of $x, y \in M$ is afforded by continuity[13].

(iii): For $x \in G$, this is a manifestation of $h \circ f = f_A \circ h$. The extension to all $x \in M$ is afforded by continuity, as before.

(iv): Again, this is clear for $x \in G$ and thus holds for all x. □

We define M_1 as the set of unmarked divisors associated to the points of M,

$$M_1 := \{\delta(x) : \ x \in M\}. \tag{3.9}$$

Proposition 6 *The set M_1 has a natural structure of Riemannian surface such that $\delta : M \to M_1$ (as defined in Proposition 5) is a branched covering.*

Proof: Let r_0 be as in Lemma 1. Consider first $M_1^{\text{reg}} := \{\delta(x) : \ x \in M \setminus E\}$. For $\delta(x) = x_1 + \ldots + x_m \in M_1^{\text{reg}}$, there is $\epsilon \in (0, r_0)$ such that we have univalent disjoint ω-charts $\phi_i : B_\epsilon(x_i) \to \mathbb{C}$ for $i = 1, \ldots, m$. All the divisors $\delta(y) \in M_1^{\text{reg}}$ contained in $\bigcup_{i=1}^m B_\epsilon(x_i)$ form an open (relative to M_1) neighborhood of $\delta(x)$, denoted $B_\epsilon(\delta(x))$.

[12] This does not mean that there is a canonical bijection between $h^{-1} \circ h(x))$ and $h^{-1} \circ h(y))$ as there are many choices of the path γ; some of them may circumvent the singularity in different ways.

[13] Take $G \ni x_n \to x$ so that $[x_n] \to \delta(x)$. There are $y_n \in [x_n]$ so that $y_n \to y$. By passing to a limit in $x_n \in [y_n]$, we get $x \in \delta(y)$.

Any $\delta(y) \in B_\epsilon(\delta(x))$ can be uniquely written as $\delta(y) = y_1 + \ldots + y_m$ where $y_i \in B_\epsilon(x_i)$. Let us declare any one of the bijections $\Phi_i : \delta(y) \mapsto \phi_i(y_i)$ as a chart on $B_\epsilon(\delta(x))$, $i = 1, \ldots, m$. Thus defined charts cover M_1^{reg} and, by using Lemma 1, one readily verifies that the transition maps are (restrictions of) Euclidean translations. In this way we defined on M_1^{reg} a flat Euclidean structure, which in turn determines a conformal structure on M_1^{reg}. To extend the complex structure from M_1^{reg} to all of M_1, we shall argue that any one of the finitely many $\delta(x) \in M_1 \setminus M_1^{\text{reg}}$ has a punctured neighborhood in M_1^{reg} that is biholomorphic to a punctured disk in $\mathbb{C}$. Fix then $\delta(x) = a_1x_1 + \ldots + a_lx_l \in M_1 \setminus M_1^{\text{reg}}$; here the x_j are meant to be distinct and $a_j \neq 0$ with $a_1 + \ldots + a_l = m$. As before, we can take $\epsilon \in (0, r_0)$ so that $B_\epsilon(x_i)$, $i = 1, \ldots, l$, are disjoint and support ω-charts $\phi_i : B_\epsilon(x_i) \to \mathbb{C}$. Fix $i \in \{1, \ldots, m\}$. The map $\Phi_i : M_1^{\text{reg}} \supset B_\epsilon(\delta(x)) \setminus \{\delta(x)\} \to \mathbb{C}$ given by $\delta(y) \mapsto \phi_i(z)$ where $z \in \delta(y) \cap B_\epsilon(x_i)$ is well defined (i.e. independent on the choice of z). We claim that Φ_i is a covering onto the punctured disk $D_i := \phi_i(B_\epsilon(x_i) \setminus \{x_i\})$. It suffices to show that it is onto, locally injective and finite-to-one.

As for local injectivity, Φ_i is analytic with nonsingular differential $d\Phi_i \neq 0$. As for "onto", for any $z \in B_\epsilon(x_i) \setminus \{x_i\}$, there is $\delta(z) \in M_1$ containing z (see (iv) of Proposition 5). Therefore, $\Phi_i(B_\epsilon(\delta(x)) \setminus \{\delta(x)\}) = D_i$. Finally, any such z uniquely determines $\delta(z)$ (per (ii) of Proposition 5), so $\#\Phi_i^{-1}(\phi_i(z)) \leq \#\phi_i^{-1}(\phi_i(z))$, i.e., the degree of Φ_i does not exceed that of ϕ_i.

Being a finite-to-one analytic covering of a punctured disk, $B_\epsilon(\delta(x)) \setminus \{\delta(x)\}$ is biholomorphic to a punctured disk and the conformal structure extends uniquely to all of $B_\epsilon(\delta(x))$ by the process already explained before for a singularity on M.

This extension does not depend on the i used. Indeed, from Lemma 1, $\Phi_i = \Phi_{i'}$ for any $i, i' \in \{1, \ldots, m\}$. Thus any two charts on $B_\epsilon(\delta(x))$ constructed above coincide up to a rotation by a multiple of $2\pi/S$ where S is the degree of Φ_i. □

Conclusion of the proof of Theorem 2: On the domain of every chart Φ_i on M_1^{reg}, we have a 1-form $d\Phi_i$. By construction of Φ_i, $\phi_i = \Phi_i \circ \delta$ so that $d\phi_i$ is a pull-back via the covering δ of $d\Phi_i$, i.e., $\delta_*(d\Phi_i) = d\phi_i$. Much like the $d\phi_i$ glue together into a global form ω on M, the $d\Phi_i$ agree on overlaps of charts and thus define a global 1-form on M_1^{reg}, call it η. Also, from $\delta_*(d\Phi_i) = d\phi_i$, η is analytic and bounded, and thus uniquely extends across the punctures to a holomorphic 1-form on all of M_1. Of course, $\delta_*(\eta) = \omega$.

Now, given a divisor $x_1 + \ldots + x_m \in M_1$, neither $h(x_i)$ nor $\delta(f(x_i))$ depends on the choice of i by (i) and (iii) of Proposition 5. Thus we can define $h_1 : M_1 \to \mathbb{T}^d$ by $h_1(\delta(x)) := h(x)$ and $f_1 : M_1 \to M_1$ by $f_1(\delta(x)) := \delta(f(x))$ for $x \in M$. The commutation relations in Theorem 2 are then tautological.

That $f_1^*(\eta^s + i\eta^u) = (\lambda^{-1}\eta^s + i\lambda\eta^u)$ follows immediately from $f^*(\omega^s + i\omega^u) = (\lambda^{-1}\omega^s + i\lambda\omega^u)$ via $\delta^*(\eta) = \omega$. In this way $f_1 : M_1 \to M_1$ is a pseudo-Anosov map with orientable foliations (if η has zeros) or an Anosov map of $\mathbb{T}^2$ (if η has no zeros). □

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