Homotopy Classes for Stable Periodic and Chaotic Patterns in Fourth-Order Hamiltonian Systems

W.D. Kalies J. Kwapisz J.B. VandenBerg
R.C.A.M. VanderVorst*

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Abstract

We investigate periodic and chaotic solutions of Hamiltonian systems in $\mathbb{R}^4$ which arise in the study of stationary solutions of a class of bistable evolution equations. Under very mild hypotheses, variational techniques are used to show that, in the presence of two saddle-focus equilibria, minimizing solutions respect the topology of the configuration plane punctured at these points. By considering curves in appropriate covering spaces of this doubly punctured plane, we prove that minimizers of every homotopy type exist and characterize their topological properties.

1 Introduction

This work is a continuation of [5] where we developed a constrained minimization method to study heteroclinic and homoclinic local minimizers of the action functional

$$J_I[u] = \int_I j(u,u',u'')\,dt = \int_I \Big[\frac{\gamma}{2}|u''|^2 + \frac{\beta}{2}|u'|^2 + F(u)\Big]\,dt, \tag{1.1}$$

which are solutions of the equation

$$\gamma u'''' - \beta u'' + F'(u) = 0 \tag{1.2}$$

with $\gamma, \beta > 0$. This equation with a double-well potential F has been proposed in connection with certain models of phase transitions. For brevity we will omit a detailed background of this problem and refer only to those sources required in the proofs of the results. A more extensive history and reference list are provided in [5], to which we refer the interested reader.

The above equation is Hamiltonian with

$$H = -\gamma u'''u' + \frac{\gamma}{2}|u''|^2 + \frac{\beta}{2}|u'|^2 - F(u). \tag{1.3}$$

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The configuration space of the system is the (u, u')-plane, and solutions to (1.2) can be represented as curves in this plane. Initially these curves do not appear to be restricted in any way. However, the central idea presented here is that, when $(\pm 1, 0)$ are saddle-foci, the minimizers of J respect the topology of this plane punctured at these two points, which allows for a rich set of minimizers to exist. Using the topology of the doubly -punctured plane and its covering spaces, we describe the structure of all possible types of minimizers, including those which are periodic and chaotic. Since the action of the minimizers of these latter types is infinite, a different notion of minimizer is required that is reminiscent of the minimizing (Class A) geodesics of Morse [8]. Such minimizers have been intensively studied in the context of geodesic flows on compact manifolds or the Aubry-Mather theory (see e.g. [1] for an introduction). A crucial difference is that we are dealing with a non-mechanical system on a non-compact space. Nevertheless, we are able to emulate many of Morse's original arguments about how the minimizers can intersect with themselves and each other. For a precise statement of the main results we refer to Theorem 4.2 and Theorem 6.8. For related work on mechanical Hamiltonian systems we refer to [9, 2] and the references therein.

Another important aspect of the techniques employed here and in [5] is the mildness of the hypotheses. In particular, our approach requires no transversality or non-degeneracy conditions, such as those found in other variational methods and dynamical systems theory, see [5]. Specifically, we will assume the following hypothesis on F:

(H): $F \in C^2$, $F(\pm 1) = F'(\pm 1) = 0$, $F''(\pm 1) > 0$, *and* $F(u) > 0$ *for* $u \neq \pm 1$.
Moreover there are constants c_1 *and* c_2 *such that* $F(u) \geq -c_1 + c_2 u^2$.

We will also assume for simplicity of the formulation that F is even, but many analogous results will hold for nonsymmetric potentials, c.f. [5]. Finally, we assume that the parameters γ and β are such that $u = \pm 1$ are saddle-foci, i.e. $4\gamma/\beta^2 > 1/F''(\pm 1)$. An example of a nonlinearity satisfying these conditions is $F(u) = (u^2 - 1)^2/4$, in which case (1.2) is the stationary version of the so-called extended Fisher-Kolmogorov (EFK) equation.

In [5] we classify heteroclinic and homoclinic minimizers by a finite sequence of even integers which represent the number of times a minimizer crosses $u = \pm 1$. More general minimizers can be similarly classified by infinite and bi-infinite sequences, as described in Section 2. A more general notion of minimizer for these types is defined in Section 3, and in Section 4 we prove that such minimizers exist.

In Sections 5 and 6 we show that many properties of these symbol sequences such as symmetry and periodicity are reflected in the corresponding minimizers. In particular, we show that for any periodic type, there exists a periodic minimizer of that type. The classification of minimizers by symbol sequences has other properties in common with symbolic dynamics; for example, if a type is asymptotically periodic in both directions, then there exists a minimizer of that type which is a heteroclinic connection between two periodic minimizers.

The minimizers discussed here all lie in the 3-dimensional 'energy-manifold' $M_0 = \{(u, u', u'', u''') \mid H((u, u', u'', u''') = 0\}$. Exploiting certain properties of minimizers that are established in this paper, we can deduce various linking and knotting characteristics when they are represented as smooth curves in M_0. However, we will not address this issue in this paper. The minimizers found in this paper are also used in [13] to construct stable patterns for the evolutionary EFK equation on a bounded interval.

Some notation used in this paper was introduced previously in [5]. While we have attempted to present a self-contained analysis, we have avoided reproducing details (particularly in Section 5.1) which are not central to the ideas presented here, and which are thoroughly explained in [5].

2 Types and function classes

A function $u:\ \mathbb{R}\to\mathbb{R}$ can be represented as a curve in the $(u,u')-$plane, and the associated curve will be denoted by $\Gamma(u)$. Removing the equilibrium points $(\pm 1,0)$ from the $(u,u')-$plane (the configuration space) creates a space with nontrivial topology, denoted by $\mathcal{P}=\mathbb{R}^2\backslash\{(\pm 1,0)\}$. In $\mathcal{P}$ we can represent functions u which have the property that $u'\neq 0$ when $u=\pm 1$, and various equivalence classes of curves can be distinguished. For example, in [5] we considered classes of curves that terminate at the equilibrium points $(\pm 1,0)$. Another important class consists of closed curves in $\mathcal{P}$, which represent periodic functions. We now give a systematic description of all classes to be considered.

Definition 2.1 *A* **type** *is a sequence* $\mathbf{g}=(g_i)_{i\in\mathcal{I}}$ *with* $g_i\in 2\mathbb{N}\cup\{\infty\}$*, where* ∞ *acts as a terminator. To be precise,* $\mathbf{g}$ *satisfies one of the following conditions:*

i) $\mathcal{I}=\mathbb{Z}$*, and* $\mathbf{g}\in 2\mathbb{N}^{\mathbb{Z}}$ *is referred to as a* **bi-infinite type**.

ii) $\mathcal{I}=\{0\}\cup\mathbb{N}$*, and* $\mathbf{g}=(\infty,g_1,g_2,...)$ *with* $g_i\in 2\mathbb{N}$ *for all* $i\geq 1$*, or*
$\mathcal{I}=-\mathbb{N}\cup\{0\}$*, and* $\mathbf{g}=(...,g_{-2},g_{-1},\infty)$ *with* $g_i\in 2\mathbb{N}$ *for all* $i\leq -1$.
In these cases $\mathbf{g}$ *is referred to as a* **semi-terminated type**.

iii) $\mathcal{I}=\{0,...,N+1\}$ *with* $N\geq 0$*, and* $\mathbf{g}=(\infty,g_1,...,g_N,\infty)$ *with* $g_i\in 2\mathbb{N}$.
In this case $\mathbf{g}$ *is referred to as a* **terminated type**.

These types will define function classes using the vector $\mathbf{g}$ to count the crossings of u at the levels $u=\pm 1$. Since there are two equilibrium points, we introduce the notion of **parity** denoted by $\mathbf{p}$, which will be equal to either 0 or 1.

Definition 2.2 *A function* $u\in H^2_{\rm loc}(\mathbb{R})$ *is in the* **class** $M(\mathbf{g},\mathbf{p})$ *if there are nonempty sets* $\{A_i\}_{i\in\mathcal{I}}$ *such that*
i) $u^{-1}(\pm 1)=\bigcup_{i\in\mathcal{I}}A_i$,
ii) $\#A_i=g_i$ *for* $i\in\mathcal{I}$,
iii) $\max A_i<\min A_{i+1}$,
iv) $u(A_i)=(-1)^{i+\mathbf{p}+1}$*, and*
v) $\bigcup_{i\in\mathcal{I}}A_i$ *consists of transverse crossings of* ± 1*, i.e.,* $u'(x)\neq 0$ *for* $x\in A_i$.

Note that by Definition 2.1, a function u in any class $M(\mathbf{g},\mathbf{p})$ has infinitely many crossings of ± 1. Definition 2.2 is similar to the definition of the class $M(\mathbf{g})$ in [5] except that here it is assumed that all crossings of ± 1 are transverse. Only finitely many crossings were assumed to be transverse in [5] so that the classes $M(\mathbf{g})$ would be open subsets of $\mathcal{X}+H^2(\mathbb{R})$. Since we will not directly minimize over $M(\mathbf{g},\mathbf{p})$, we now require transversality of all crossings of ± 1 to guarantee that $\Gamma(u) \in \mathcal{P}$. However, note that the minimizers found in [5] are indeed contained in classes $M(\mathbf{g}, \mathbf{p})$ as defined above, where the types $\mathbf{g}$ are terminated.

The classes $M(\mathbf{g}, \mathbf{p})$ are nonempty for all pairs $(\mathbf{g}, \mathbf{p})$. Conversely, any function $u \in H^2_{\text{loc}}(\mathbb{R})$ is contained in the closure of some class $M(\mathbf{g}, \mathbf{p})$ with respect to the complete metric on $H^2_{\text{loc}}(\mathbb{R})$ given by $\rho(u, v) = \sum_i 2^{-i} \min\{1, \|u - v\|_{H^2(-i,i)}\}$, cf. [10]. That is, if we define $\overline{M(\mathbf{g}, \mathbf{p})} := \{u \in H^2_{\text{loc}}(\mathbb{R}) \mid \exists u_n \in M(\mathbf{g}, \mathbf{p}), \text{ with } u_n \to u \text{ in } H^2_{\text{loc}}(\mathbb{R})\}$, then $H^2_{\text{loc}}(\mathbb{R}) = \cup_{(\mathbf{g},\mathbf{p})} \overline{M(\mathbf{g}, \mathbf{p})}$. Note that the functions in $\partial M(\mathbf{g}, \mathbf{p}) := \overline{M(\mathbf{g}, \mathbf{p})} \setminus \text{int}(M(\mathbf{g}, \mathbf{p}))$ have tangencies at ± 1 and thus are limit points of more than one class. In the case of an infinite type, shifts of $\mathbf{g}$ can give rise to the same function class. Therefore certain infinite types need to be identified. Let σ be the shift map defined by $\sigma(\mathbf{g})_i = g_{i+1}$ and the map $\tau: \{0, 1\} \to \{0, 1\}$ be defined by $\tau(\mathbf{p}) = (\mathbf{p} + 1) \bmod 2 = |\mathbf{p} - 1|$. Two pairs (infinite types) $(\mathbf{g}, \mathbf{p})$ and $(\mathbf{g}', \mathbf{p}')$ are equivalent if $\mathbf{g}' = \sigma^n(\mathbf{g})$ and $\mathbf{p}' = \tau^n(\mathbf{p})$ for some $n \in \mathbb{Z}$, and this implies $M(\mathbf{g}, \mathbf{p}) = M(\mathbf{g}', \mathbf{p}')$.

3 Definition of minimizer

When the domain of integration is $\mathbb{R}$, the action $J[u]$ given in (1.1) is well-defined only for terminated types $\mathbf{g}$ and $u \in M(\mathbf{g}, \mathbf{p}) \cap \{\chi_{\mathbf{p}} + H^2(\mathbb{R})\}$, where $\chi_{\mathbf{p}}$ is a smooth function from $(-1)^{\mathbf{p}+1}$ to $(-1)^{\mathbf{p}}$. For semi-terminated types or infinite types the action J is infinite for every $u \in M(\mathbf{g}, \mathbf{p})$. We will define an alternative notion of minimizer in order to overcome this difficulty.

For every compact interval $I \subset \mathbb{R}$ the restricted action J_I is well-defined for all types. When we restrict u to an interval I, we can define its *type and parity relative to* I, which we denote by $(\mathbf{g}(u|_I), \mathbf{p}(u|_I))$. Namely, let $u \in M(\mathbf{g}, \mathbf{p})$. It is clear that $(u, u')|_{\partial I} \notin (\pm 1, 0)$ for any bounded interval I. Then $\mathbf{g}(u|_I)$ is defined to be the finite-dimensional vector which counts the consecutive instances of $u|_I = \pm 1$, and $\mathbf{p}(u|_I)$ is defined such that the first time $u|_I = \pm 1$ in I happens at $(-1)^{\mathbf{p}+1}$. Note that the components of $\mathbf{g}(u|_I)$ are not necessarily all even, since the first and the last entries may be odd. We are now ready to state the definition of a (global) minimizer in $M(\mathbf{g}, \mathbf{p})$.

Definition 3.1 *A function $u \in M(\mathbf{g}, \mathbf{p})$ is called a* **minimizer for J over** $M(\mathbf{g}, \mathbf{p})$ *if and only if for every compact interval I the number $J_I[u|_I]$ minimizes $J_{I'}[v|_{I'}]$ over all functions $v \in M(\mathbf{g}, \mathbf{p})$ and all compact intervals I' such that $(v, v')|_{\partial I'} = (u, u')|_{\partial I}$ and $(\mathbf{g}(v|_{I'}), \mathbf{p}(v|_{I'})) = (\mathbf{g}(u|_I), \mathbf{p}(u|_I))$.*

The pair $(\mathbf{g}(u|_I), \mathbf{p}(u|_I))$ defines a homotopy class of curves in $\mathcal{P}$ with fixed end points $(u, u')|_{\partial I}$. The above definition says that a function u, represented as a curve $\Gamma(u)$ in $\mathcal{P}$, is a minimizer if and only if for any two points P_1 and P_2 on $\Gamma(u)$, the segment $\Gamma(P_1, P_2) \subset \Gamma(u)$ connecting P_1 and P_2 is the most J-efficient among all connections $\widetilde{\Gamma}(P_1, P_2)$ between P_1 and P_2 that are induced by a function v and are of the same homotopy type as $\Gamma(P_1, P_2)$, regardless the length of the interval needed to parametrize the curve $\widetilde{\Gamma}(P_1, P_2)$. As we mentioned in the introduction, this is analogous to the length minimizing geodesics of Morse and Hedlund and the minimizers in the

Aubry-Mather theory. The set of all (global) minimizers in $M(\mathbf{g}, \mathbf{p})$ will be denoted by $CM(\mathbf{g}, \mathbf{p})$.

Lemma 3.2 *Let $u \in M(\mathbf{g}, \mathbf{p})$ be a minimizer, then $u \in C^4(\mathbb{R})$ and u satisfies equation* (1.2)*. Moreover, u satisfies the relation $H(u, u', u'', u''') = 0$, i.e. the associated orbit lies on the energy level $H = 0$.*

Proof. From the definition of $M(\mathbf{g}, \mathbf{p})$, on any bounded interval $I \subset \mathbb{R}$ there exists $\epsilon_0(I) > 0$ sufficiently small such that $u+\phi \in M(\mathbf{g}, \mathbf{p})$ for all $\phi \in H_0^2(I)$, with $||\phi||_{H^2} < \epsilon \le \epsilon_0$. Therefore $J_I[u + \phi] \ge J_I[u]$ for all such functions ϕ, which implies that $dJ_I[u] = 0$ for any bounded interval $I \subset \mathbb{R}$, and thus u satisfies (1.2).

To prove the second statement we argue as follows. Since $u \in M(\mathbf{g}, \mathbf{p})$, there exists a bounded interval I such that $u'|_{\partial I} = 0$. Introducing the rescaled variable $s = t/T$ with $T = |I|$ and $v(s) = u(t)$, we have

$$J_I[u] = J[T, v] \equiv \int_0^1 \left[\frac{1}{T^3}\frac{\gamma}{2}|v''|^2 + \frac{1}{T}\frac{\beta}{2}|v'|^2 + TF(v)\right] ds, \tag{3.1}$$

which decouples u and T. Since $u'|_{\partial I} = 0$ we see from Definition 3.1 that $J[T \pm \epsilon, v] \ge J_T[u] = J[T, v]$. The smoothness of J in the variable $T > 0$ implies that $\frac{\partial}{\partial\tau}J[\tau, v]\Big|_{\tau=T} = 0$. Differentiating yields

$$\begin{aligned}
\frac{\partial}{\partial\tau}J[\tau, v] &= \int_0^1 \left[-\tau^{-4}\frac{3}{2}\gamma|v''|^2 - \tau^{-2}\frac{\beta}{2}|v'|^2 + F(v)\right] ds \\
&= \tau^{-1}\int_0^\tau \left[-\frac{3}{2}\gamma|u''|^2 - \frac{\beta}{2}|u'|^2 + F(u)\right] dt \\
&= -\tau^{-1}\int_0^\tau H(u, u', u'', u''')dt \equiv -E,
\end{aligned}$$

Thus $E = 0$, and $H(u, u', u'', u''') = 0$ for $t \in I$. This immediately implies that $H = 0$ for all $t \in \mathbb{R}$. $\square$

The minimizers for J found in [5] also satisfy Definition 3.1, and we restate one of the main results of [5].

Proposition 3.3 *Suppose F is even and satisfies* (H)*, and $\beta, \gamma > 0$ are chosen such that ± 1 are saddle-focus equilibria. Then for any terminated type $\mathbf{g}$ with parity either 0 or 1 there exists a minimizer $u \in M(\mathbf{g}, \mathbf{p})$ of J.*

From Definition 2.2, the crossings of $u \in M(\mathbf{g}, \mathbf{p})$ with ± 1 are transverse and hence isolated. We adapt from [5], the notion of a normalized function with a few minor changes to reflect the fact that we now require every crossing of ± 1 to be transverse.

Definition 3.4 *A function $u \in M(\mathbf{g}, \mathbf{p})$ is* **normalized** *if, between each pair $u(a)$ and $u(b)$ of consecutive crossings of ± 1, the restriction $u|_{(a,b)}$ is either monotone or $u|_{(a,b)}$ has exactly one local extremum.*

Clearly, the case of $u|_{(a,b)}$ being monotone can occur only between two crossings at different levels ± 1, in which case we say that u has a **transition** on $[a, b]$.

Lemma 3.5 *If $u \in CM(\mathbf{g}, \mathbf{p})$, then u is normalized.*

Proof. Since $u \in M(\mathbf{g}, \mathbf{p})$, all crossings of $u = \pm 1$ are transverse, i.e. $u' \neq 0$. Thus for any critical point $t_0 \in \mathbb{R}$, $u(t_0) \neq \pm 1$, and the Hamiltonian relation from Lemma 3.4 implies that $\gamma u''(t_0)^2/2 = F(u(t_0)) > 0$. Therefore u is a Morse function, and between any two consecutive crossings of ± 1 there are only finitely many critical points. Now on any interval between consecutive crossings where u is not normalized, the clipping lemmas of Section 3 in [5] can be applied to obtain a more J-efficient function, which contradicts the fact that u is a minimizer. $\square$

4 Minimizers of arbitrary type

In this section we will introduce a notion of convergence of types which will be used in Section 6.2 to establish the existence of minimizers in every class $M(\mathbf{g}, \mathbf{p})$ by building on the results proved in [5].

Definition 4.1 *Consider a sequence of types $(\mathbf{g}^n, \mathbf{p}^n) = ((g_i^n)_{i \in I_n}, \mathbf{p}^n)$ and a type $(\mathbf{g}, \mathbf{p}) = ((g_i)_{i \in I}, \mathbf{p})$. The sequence $(\mathbf{g}^n, \mathbf{p}^n)$* **limits** *to the type $(\mathbf{g}, \mathbf{p})$ if and only if there exist numbers $N_n \in 2\mathbb{Z}$ such that $g^n_{i+N_n+\mathbf{p}^n-\mathbf{p}} \to g_i$ for all $i \in I$ as $n \to \infty$. We will abuse notation and write $(\mathbf{g}^n, \mathbf{p}^n) \to (\mathbf{g}, \mathbf{p})$.*

We should point out that a sequence of types can limit to more than one type. For example the sequence $(\mathbf{g}^n, 0) = ((\infty, 2, 2, n, 4, 4, 4, 4, n, 2, 2, 2, ...), 0)$ limits to the types $((\infty, 2, 2, \infty), 0)$, $((\infty, 4, 4, 4, 4, \infty), 1)$ and $((\infty, 2, 2, 2, ...), 0)$.

Theorem 4.2 *Let $(\mathbf{g}^n, \mathbf{p}^n) \to (\mathbf{g}, \mathbf{p})$ and $u_n \in CM(\mathbf{g}^n, \mathbf{p}^n)$ with $\|u_n\|_{1,\infty} \leq C$ for all n. Then there exists a subsequence u_{n_k} such that $u_{n_k} \to \widehat{u} \in M(\mathbf{g}, \mathbf{p})$ in $C^4_{\mathrm{loc}}(\mathbb{R})$, and $\widehat{u}$ is a minimizer in the sense of Definition* 3.1, *i.e. $\widehat{u} \in CM(\mathbf{g}, \mathbf{p})$.*

Proof. This proof requires arguments developed in [5] to which the reader is referred for certain details. The idea is to take the limit of u_n restricted to bounded intervals. We define the numbers N_n as in Definition 4.1, and we denote the convex hull of A_i by $I_i = \mathrm{conv}(A_i)$. Due to translation invariance we can pin the functions u_n so that $u_n(0) = (-1)^{\mathbf{p}+1}$, which is the beginning of the transition between $I^n_{N_n+\mathbf{p}^n-\mathbf{p}}$ and $I^n_{1+N_n+\mathbf{p}^n-\mathbf{p}}$. Due to the assumed a priori bound and interpolation estimates which can be found in the appendix to [7], there is enough regularity to yield a limit function $\widehat{u}$ as a C^4_{loc}–limit of u_n, after perhaps passing to a subsequence. Moreover $\widehat{u}$ satisfies the differential equation (1.2) on $\mathbb{R}$. The question that remains is whether $\widehat{u} \in M(\mathbf{g}, \mathbf{p})$.

To simplify notation we will now assume that $N_n = 0$ and $\mathbf{p}^n = \mathbf{p} = 0$. Fixing a small $\delta > 0$, we define $I_i^n(\delta) \supset I_i^n$ as the smallest interval containing I_i^n such that $u|_{\partial I_i^n(\delta)} = (-1)^{i+1} - (-1)^{i+1}\delta$. If $\mathbf{g}$ is a (semi-)terminated type then $I_i^n(\delta)$ is a half-line. The interval of transition between $I_i^n(\delta)$ and $I_{i+1}^n(\delta)$ is denoted by $L_i^n(\delta)$. To see that

$\widehat{u} \in M(\mathbf{g}, \mathbf{p})$, the goal is to eliminate the two possibilities that a priori may lead to the loss or creation of crossings in the limit so that $\widehat{u} \notin M(\mathbf{g}, \mathbf{p})$: the distance between two consecutive crossings in u_n could grow without bound or $\widehat{u}$ could posess tangencies at $u = \pm 1$.

Due to the a priori estimates in $W^{1,\infty}$ we have the following bounds on J:

$$J[u_n|_{I_i^n(\delta)}] \leq C, \tag{4.1}$$

and

$$J[u_n|_{L_i^n(\delta)}] \leq C',$$

where C and C' are independent of n and i. Indeed, note that for n large enough the homotopy type of u_n on the intervals $I_i^n(\delta)$ is constant by the definition of convergence of types. Since the functions u_n are minimizers, $J[u_n|_{I_i^n(\delta)}]$ is less than the action of any test function of this homotopy type satisfying the a priori bounds on u and u' on $\partial I_i^n(\delta)$ (see [5], Section 6, for a similar test function argument). The estimate $|L_i^n(\delta)| \leq C(\delta)$ is immediately clear from Lemma 5.1 of [5]. We now need to show that the distance between two crossings of $(-1)^{i+1}$ within the interval $I_i^n(\delta)$ cannot tend to infinity.

First we will deal with the case when g_i^n is finite for all n. Suppose that the distance between consecutive crossings of $(-1)^{i+1}$ in $I_i^n(\delta)$ tends to infinity as $n \to \infty$. Due to Inequality (4.1) and Lemma 3.5, minimizers have exactly one extremum between crossing of $(-1)^{i+1}$ for any $\epsilon > 0$, and hence there exist subintervals $K_n \subset I_i^n(\delta)$ with $|K_n| \to \infty$, such that $0 < |u_n - (-1)^{q_n}| < \epsilon$ on K_n where $q_n \in \{0, 1\}$, and $|u'|_{\partial K_n}| < \epsilon$. Taking a subsequence we may assume that q_n is constant.

We begin by considering the case where $q_n = i + 1$. Now ϵ can be chosen small enough, so that the local theory in [5] is applicable in K_n. If $|K_n|$ becomes too large then u_n can be replaced by a function with lower action and with many crossings of $(-1)^{i+1}$. Subsequently, redundant crossings can be clipped out, thereby lowering the action. This implies that u_n is not a minimizer in the sense of Definition 3.1, a contradiction.

The case where $q_n = i$ must be dealt with in a different manner. First, there are unique points $t_n \in K_n$ such that $u_n'(t_n) = 0$, and for these points $u_n(t_n) \to (-1)^i$ as $|K_n| \to \infty$. Let $u_n(s_n)$ be the first crossing of $(-1)^{i+1}$ to the left of K_n. Taking the limit (along subsequences) of $u_n(t - s_n)$ we obtain a limit function $\widetilde{u}$ which is a solution of (1.2). If $|t_n - s_n|$ is bounded then $\widetilde{u}$ has a tangency to $u = (-1)^i$ at some $t_* \in \mathbb{R}$. All u_n lie in $\{H = 0\}$ (see (1.3)) and so does $\widetilde{u}$, hence $\widetilde{u}''(t_*) = 0$. Moreover $\widetilde{u}'''(t_*) = 0$, because $\widetilde{u}(t_*)$ is an extremum. By uniqueness of the initial value problem this implies that $\widetilde{u} \equiv (-1)^i$, contradicting the fact that $\widetilde{u}(0) = (-1)^{i+1}$. If $|t_n - s_n| \to \infty$, then $\widetilde{u}$ is a monotone function on $[0, \infty)$, tending to $(-1)^i$ as $x \to \infty$, and its derivatives tend to zero (see Lemma 3 in [11] or Lemma 1 part (ii) in [7] for details). This contradicts the saddle-focus character of the equilibrium point.

In the case that $g_i^n = \infty$ we remark that (4.1) also holds when I_i^n is a half-line. It follows from the estimates in Lemma 5.1 in [5] that $u_i^n \to (-1)^{i+1}$ as $x \to \infty$ or $x \to -\infty$ (whichever is applicable). From the local theory in Section 4 of [5] and the fact that u_n is a minimizer, it follows that the derivatives of u_n tend to zero. The analysis above concerning the intervals K_n and the clipping of redundant oscillations now goes on unchanged.

We have shown that the distance between two crossings of ± 1 is bounded from above. Next we have to show that the limit function has only transverse crossings of ± 1. This ensures that no crossings are lost in the limit. When $\widehat{u}$ would be tangent to $(-1)^{i+1}$ in I_i, then we can construct a function in $v \in M(\mathbf{g}, \mathbf{p})$ in the same way as demonstrated in [5] by replacing tangent pieces by more J-efficient local minimizers and by clipping. The function v still has a lower action than $\widehat{u}$ on a slightly larger interval (the limit function $\widehat{u}$ also obeys (4.1), so that the above clipping arguments still apply). Since $u_n \to \widehat{u}$ in C^4_{loc} it follows that $J_I[u_n] \to J_I[u]$ on bounded intervals I. This then implies that for n large enough the function u_n is not a minimizer in the sense of Definition 3.1, which is a contradiction.

The limit function $\widehat{u}$ could also be tangent to $(-1)^i$ for some $t_0 \in I_i$. As before, such tangencies satisfy $\widehat{u}(t_0) - (-1)^i = \widehat{u}'(t_0) = \widehat{u}''(t_0) = \widehat{u}'''(t_0) = 0$, which leads to a contradiction the uniqueness of the initial value problem.

Finally, crossings of ± 1 cannot accumulate since this would imply that at the accumulation point all derivatives would be zero, leading to the same contradiction as above. In particular, if $g_i^n \to \infty$ for some i, then $|I_i^n| \to \infty$ and the crossings in A_j^n for $j > i$ move off to infinity and do not show in $\widehat{u}$, which is compatabile with the convergence of types.

We have now proved that $\widehat{u} \in M(\mathbf{g}, \mathbf{p})$ and, since $\widehat{u}$ is the C^4_{loc}–limit of minimizers, $\widehat{u}$ is also a minimizer in the sense of Definition 3.1. $\square$

Remark 4.3 *It follows from the estimates in Theorem 3 of* [7] *that in the theorem above we in fact only need an L^∞-bound on the sequence u_n.*

Remark 4.4 *It follows from the proof of Theorem* 4.2 *that there exists a constant $\delta_0 > 0$ such that for all uniformly bounded minimizers $u(t)$ it holds that $|u(t) - (-1)^{i+\mathbf{p}}| > \delta$ for all $t \in I_i$ and all $i \in \mathcal{I}$. This means that the uniform seperation property discussed in* [5] *is uniformly satisfied by all minimizers.*

Remark 4.5 *In order to take a limit of the sequence $u_n \in CM(\mathbf{g}_n, \mathbf{p})$ in the above theorem, we need the a priori estimate $\|u_n\|_{1,\infty} \le C$ for all n. We will show in Section 6 that this estimate will be satisfied for many sequences $\mathbf{g}_n$, see Corollary 6.2 and Theorem 6.3 below. Note that for the special case where $F(u) \sim |u|^s$ as $|u| \to \infty$ for some $s > 2$, an a priori L^∞ bound on the set of all solutions of* (1.2) *with domain of existence $\mathbb{R}$ can be obtained* [4].

5 Periodic minimizers

An bi-infinite type $\mathbf{g}$ is **periodic** if there exists an integer n such that $\sigma^n(\mathbf{g}) = \mathbf{g}$. The (natural) definition of the period of $\mathbf{g}$ is $\min\{n \in 2\mathbb{N} \,|\, \sigma^n(\mathbf{g}) = \mathbf{g}\}$. We will write $\mathbf{g} = \langle \mathbf{r} \rangle$ where $\mathbf{r} = (g_1, ..., g_n)$ and n is even. Cyclic permutations of $\mathbf{r}$ with possibly a flip of $\mathbf{p}$ give rise to the same function class $M(\langle \mathbf{r} \rangle, \mathbf{p})$. In reference to the type $\langle \mathbf{r} \rangle$ with parity $\mathbf{p}$ we will use the notation $(\mathbf{r}, \mathbf{p})$. Any such type pair $(\mathbf{r}, \mathbf{p})$ can formally be associated with a homotopy class in $\pi_1(\mathcal{P}, \mathbf{0})$ in the following way. Let e_0 and e_1 be the clockwise oriented circles of radius one centered at $(1, 0)$ and $(-1, 0)$ respectively, so that $[e_0]$ and $[e_1]$ are generators for $\pi_1(\mathcal{P},\mathbf{0})$. Defining $\theta(\mathbf{r},\mathbf{p}) = e_{\tau^n(\mathbf{p})}^{r_n/2}\cdot\ldots\cdot e_{\mathbf{p}}^{r_1/2}$, the map $\theta : \cup_{k\geq 1}2\mathbb{N}^{2k}\times\{0,1\}\to\ \pi_1(\mathcal{P},\mathbf{0})$ is an injection, and we define $\pi_1^+(\mathcal{P},\mathbf{0})$ to be the image of θ in $\pi_1(\mathcal{P},\mathbf{0})$. Powers of a type pair $(\mathbf{r},\mathbf{p})^k$ for $k\geq 1$ are defined by concatenation of $\mathbf{r}$ with itself k times, which is equivalent to $(\mathbf{r},\mathbf{p})^k = \theta^{-1}((\theta(\mathbf{r},\mathbf{p}))^k)$.

Definition 5.1 *Two pairs* $(\mathbf{r},\mathbf{p})$ *and* $(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$ *are* **equivalent** *if there are numbers* $p,q\in\mathbb{N}$ *such that* $(\mathbf{r},\mathbf{p})^p = (\widehat{\mathbf{r}},\widehat{\mathbf{p}})^q$ *up to cyclic permutations. This relation,* $(\mathbf{r},\mathbf{p})\sim(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$*, is an equivalence relation.*

Example: if $(\mathbf{r},\mathbf{p}) = ((2,4,2,4),0)$ and $(\widehat{\mathbf{r}},\widehat{\mathbf{p}}) = ((4,2,4,2,4,2),1)$, then $\theta(\mathbf{r},\mathbf{p})^3 = \theta(\widehat{\mathbf{r}},\widehat{\mathbf{p}})^2$. The equivalence class of $(\mathbf{r},\mathbf{p})$ is denoted by $[\mathbf{r},\mathbf{p}]$. A type $(\mathbf{r},\mathbf{p})$ is a minimal representative for $[\mathbf{r},\mathbf{p}]$ if for each $(\widehat{\mathbf{r}},\widehat{\mathbf{p}})\in[\mathbf{r},\mathbf{p}]$ there is $k\geq 1$ such that $(\widehat{\mathbf{r}},\widehat{\mathbf{p}}) = (\mathbf{r},\mathbf{p})^k$ up to cyclic permutations. A minimal representative is unique up to cyclic permutations. It is clear that in the representation of a periodic type $\mathbf{g} = \langle\mathbf{r}\rangle$, the type $\mathbf{r}$ is minimal if the length of $\mathbf{r}$ is the minimal period. Due to the above equivalences we now have that

$$M(\langle\mathbf{r}\rangle,\mathbf{p}) = M(\langle\widehat{\mathbf{r}}\rangle,\widehat{\mathbf{p}}),\ \ \forall\ (\widehat{\mathbf{r}},\widehat{\mathbf{p}})\in[\mathbf{r},\mathbf{p}].$$

It is not a priori clear that minimizers in $M(\langle\mathbf{r}\rangle,\mathbf{p})$ are periodic. However, we will see that among these minimizers, periodic minimizers can always be found.

For a given periodic type $\langle\mathbf{r}\rangle$ we consider the subset of periodic functions in $M(\langle\mathbf{r}\rangle,\mathbf{p})$,

$$M_{\mathrm{per}}(\langle\mathbf{r}\rangle,\mathbf{p}) = \{u\in M(\langle\mathbf{r}\rangle,\mathbf{p})\mid u \text{ is periodic}\}.$$

For any $u\in M_{\mathrm{per}}(\langle\mathbf{r}\rangle,\mathbf{p})$ and a period T of u, $\Gamma(u|_{[0,T]})$ is a closed loop in $\mathcal{P}$ whose homotopy type corresponds to a nontrivial element of $\pi_1^+(\mathcal{P},\mathbf{0})$. In this correspondence there is no natural choice of a basepoint. For specificity, we will describe how to make the correspondence with the origin as the basepoint and thereafter omit it from the notation. Translate u so that $u(0) = 0$. Let $\gamma : [0,1]\to\mathcal{P}$ be the line from $\mathbf{0}$ to $(0,u'(0))$, and let $\gamma^*(t) = \gamma(1-t)$. Then $\widetilde{\Gamma}(u|_{[0,T]}) = \gamma^*\circ\Gamma(u|_{[0,T]})\circ\gamma$, and $[\widetilde{\Gamma}(u|_{[0,T]})]\in\pi_1^+(\mathcal{P},\mathbf{0})$. Now define $[\Gamma(u|_{[0,T]})]\equiv[\widetilde{\Gamma}(u|_{[0,T]})]$. Thus there exists a pair $\theta^{-1}[\Gamma(u|_{[0,T]})] = (\widehat{\mathbf{r}},\widehat{\mathbf{p}})\in[\mathbf{r},\mathbf{p}]$, with $\widehat{\mathbf{r}} = \mathbf{r}^k$ for some $k\geq 1$. Therefore we define for any $(\widehat{\mathbf{r}},\widehat{\mathbf{p}})\in[\mathbf{r},\mathbf{p}]$

$$M_{\mathrm{per}}(\widehat{\mathbf{r}},\widehat{\mathbf{p}}) = \{u\in M_{\mathrm{per}}(\langle\mathbf{r}\rangle,\mathbf{p})\mid[\Gamma(u|_{[0,T]})]\sim\theta(\widehat{\mathbf{r}},\widehat{\mathbf{p}})\in\pi_1(\mathcal{P})\text{ for a period } T \text{ of } u\}.$$

The type $\widehat{\mathbf{r}} = \mathbf{g}(u|_{[0,T]})$, with $\mathbf{g} = \langle\mathbf{r}\rangle$, is the homotopy type of u relative to a period T. This type has an even number of entries. It follows that $M_{\mathrm{per}}(\mathbf{r},\mathbf{p})\subset M_{\mathrm{per}}(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$ for all $(\widehat{\mathbf{r}},\widehat{\mathbf{p}}) = (\mathbf{r},\mathbf{p})^k$, $k\geq 1$. Furthermore $M_{\mathrm{per}}(\langle\mathbf{r}\rangle,\mathbf{p}) = \cup_{(\widehat{\mathbf{r}},\widehat{\mathbf{p}})\in[\mathbf{r},\mathbf{p}]}M_{\mathrm{per}}(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$. In order to get a better understanding of periodic minimizers in $M(\langle\mathbf{r}\rangle,\mathbf{p})$ we consider the following minimization problem:

$$\mathcal{J}_{\mathrm{per}}(\mathbf{r},\mathbf{p}) = \inf_{u\in M_{\mathrm{per}}(\mathbf{r},\mathbf{p})} J_T[u] = \inf_{\substack{M^T_{\mathrm{per}}(\mathbf{r},\mathbf{p})\\ T\in\mathbb{R}^+}} J_T[u], \tag{5.1}$$

where J_T is action given in (1.1) integrated over one period of length T, and $M^T_{\mathrm{per}}(\mathbf{r},\mathbf{p})$ is the set of T-periodic functions $u\in M_{\mathrm{per}}(\mathbf{r},\mathbf{p})$ for which $\mathbf{g}(u|_{[0,T]}) = \mathbf{r}$. Note that

T is not necessarily the minimal period, unless $\mathbf{r}$ is a minimal representative for $[\mathbf{r}]$. It is clear that for $\gamma, \beta > 0$ the infima $\mathcal{J}_{\text{per}}(\mathbf{r}, \mathbf{p})$ are well-defined and are nonnegative for any homotopy type $\mathbf{r}$. At this point it is not clear, however, that the infima $\mathcal{J}_{\text{per}}(\mathbf{r}, \mathbf{p})$ are attained for all homotopy types $\mathbf{r}$. We will prove in Section 6 that existence of minimizers for (5.1) can be obtained using the existence of homoclinic and heteroclinic minimizers already established in [5].

Lemma 5.2 *If $\mathcal{J}_{\text{per}}(\mathbf{r}, \mathbf{p})$ is attained for some $u \in M_{\text{per}}(\mathbf{r}, \mathbf{p})$ then $u \in C^4(\mathbb{R})$ and satisfies* (1.2). *Moreover, since u is minimal with respect to T we have $H(u, u', u'', u''') = 0$, i.e. the associated periodic orbit lies in the energy surface $H = 0$.*

Proof. Since $\mathcal{J}_{\text{per}}(\mathbf{r}, \mathbf{p})$ is attained by some $u \in M_{\text{per}}(\mathbf{r}, \mathbf{p})$ for some period T, we have that $J_T[u + \phi] - J_T[u] \geq 0$ for all $\phi \in H^2(S^1, T)$ with $\|\phi\|_{H^2} \leq \epsilon$, sufficiently small. This implies that $dJ_T[u] = 0$, and thus u satisfies (1.2). The second part of this proof is analogous to the proof of Lemma 3.2. $\square$

We now introduce the following notation:

$$
\begin{aligned}
&CM(\langle \mathbf{r} \rangle, \mathbf{p}) = \{u \in M(\langle \mathbf{r} \rangle, \mathbf{p}) \mid u \text{ is a minimizer according to Definition 3.1}\},\\
&CM_{\text{per}}(\langle \mathbf{r} \rangle, \mathbf{p}) = \{u \in CM(\langle \mathbf{r} \rangle, \mathbf{p}) \mid u \text{ is periodic}\},\\
&CM_{\text{per}}(\mathbf{r}, \mathbf{p}) = \{u \in M_{\text{per}}(\mathbf{r}, \mathbf{p}) \mid u \text{ is a minimizer for } \mathcal{J}_{\text{per}}(\mathbf{r}, \mathbf{p})\}.
\end{aligned}
$$

5.1 Existence of periodic minimizers of type $\mathbf{r} = (2, 2)^k$

If we seek periodic minimizers of type $\mathbf{r} = (2, 2)^k$, the uniform separation property for minimizing sequences (see Section 5 in [5]) is satisfied in the class $M_{\text{per}}(\mathbf{r})$. Note that the parity is omitted because it does not distinguish different homotopy types here. The uniform separation property as defined in [5] prevents minimizing sequences from crossing the boundary of the given homotopy class. For any other periodic type the uniform separation property is not a priori satisfied. For the sake of simplicity we begin with periodic minimizers of type $(2, 2)$ and minimize J in the class $M_{\text{per}}((2, 2))$.

Minimizing sequences can be chosen to be normalized due to the following lemma, which we state without proof. The proof is analogous to Lemma 3.5 in [5].

Lemma 5.3 *Let $u \in M_{\text{per}}((2, 2))$ and T be a period of u. Then for every $\epsilon > 0$ there exists a normalized function $w \in M_{\text{per}}((2, 2))$ with period $T' \leq T$ such that $J_{T'}[w] \leq J_T[u] + \epsilon$.*

The goal of this subsection is to prove that when F satisfies (H) and $\beta, \gamma > 0$ are such that ± 1 are saddle-foci, then $\mathcal{J}_{\text{per}}((2, 2))$ is attained, Theorem 5.5 below. The proof relies on the local structure of the saddle-focus equilibria ± 1 and is a modification of arguments in [5]; hence we will provide only a brief argument. The reader is referred to [5] for further details.

In preparation for the proof of Theorem 5.5, we fix $\tau_0 > 0, \epsilon_0 > 0$, and $\delta > 0$ so that the conclusion of Theorem 4.2 of [5] holds, i.e. the characterization of the oscillatory behavior of solutions near the saddle-focus equilibria ± 1 holds. Let $u \in M^T_{\text{per}}((2, 2))$

be normalized, and let t_0 be such that $u(t_0) = 0$. Then t_0 is part of a transition from ∓ 1 to ± 1. Assume without loss of generality that this transition is from -1 to 1. Define $t_- = \sup\{t < t_0 : \ |u(t) + 1| < \delta\}$ and $t_+ = \inf\{t > t_0 : \ |u(t) - 1| < \delta\}$. Then let $S(u) = \{t : \ |u(t) \pm 1| < \delta\}$ and $B[u,T] = |S(u) \cap [t_+, t_- + T]|$, and note that $[t_0, t_0 + T] = \{S(u) \cap [t_+, t_- + T]\} \cup \{S(u)^c \cap [t_0, t_0 + T]\}$. With these definitions we can establish the following estimate (c.f. Lemma 5.4 in [5]). For all $u \in M_{\rm per}((2,2))$ with $J_T[u] \le \mathcal{J}_{\rm per}((2,2)) + \epsilon_0$

$$\|u\|_{H^2}^2 \le C(1 + \mathcal{J}_{\rm per}((2,2)) + B[u,T]). \tag{5.2}$$

First, $\|u'\|_{H^1}^2 \le C(\mathcal{J}_{\rm per}((2,2)) + \epsilon_0)$, and second if $|u \pm 1| > \delta$ then $F(u) \ge \eta^2 u^2$, which implies that $\|u\|_{L^2}^2 \le 1/\eta^2 \int_{t_0}^{t_0+T} F(u)\,dt + (1+\delta)^2 B[u,T] \le C(J_T[u] + B[u,T])$. Combining these two estimates proves (5.2).

For functions $u \in M_{\rm per}^T((2,2))$ that satisfy $J_T[u] \le \mathcal{J}_{\rm per}((2,2)) + 1$, it follows from Lemma 5.1 of [5] that there exist (uniform in u) constants T_1 and T_2 such that $T_2 \ge |S(u)^c \cap [t_0, t_0 + T]| \ge T_1 > 0$ and thus $T > T_1$. The next step is to give an a priori upper bound on T by considering the minimization problem (c.f. Section 5 in [5])

$$\begin{aligned} B_\epsilon = \inf\{\, B[u,T] \quad | \quad & u \in M_{\rm per}^T((2,2)) \text{ normalized, } T \in \mathbb{R}^+, \\ & \text{and } J_T[u] \le \mathcal{J}_{\rm per}((2,2)) + \epsilon\}. \end{aligned}$$

Lemma 5.4 *There exists a constant $K = K(\tau_0) > 0$ such that $B_\epsilon \le K$ for all $0 < \epsilon < \epsilon_0$. Moreover, if $T_0 \equiv K + T_2$, then for any $0 < \epsilon < \epsilon_0$, there is a normalized $u \in M_{\rm per}^T((2,2))$ with $J_T[u] \le \mathcal{J}_{\rm per}((2,2)) + 2\epsilon$ and $T_1 < T \le T_0$.*

Proof. Let $(u_n, T_n) \in M_{\rm per}^{T_n}((2,2)) \times \mathbb{R}^+$ be a minimizing sequence for B_ϵ, with normalized functions u_n. As in the proof of Theorem 5.5 of [5], in the weak limit this yields a pair $(\widehat{u}, \widehat{T})$ such that $B[\widehat{u}, \widehat{T}] \le B_\epsilon$. We now define $K((2,2),\tau_0) = 8((2\tau_0 + 2) + 2)$. This gives two possibilities for $B[\widehat{u}, \widehat{T}]$, either $B[\widehat{u}, \widehat{T}] > K$ or $B[\widehat{u}, \widehat{T}] \le K$. If the former is true then we can construct (see Theorem 5.5 of [5]) a pair $(\widehat{v}, \widehat{T}') \in M_{\rm per}^{\widehat{T}'}((2,2)) \times \mathbb{R}^+$, with $\widehat{v}$ normalized, such that

$$J_{\widehat{T}'}[\widehat{v}] < J_{\widehat{T}}[\widehat{u}] \le \mathcal{J}_{\rm per}((2,2)) + \epsilon \quad \text{and} \quad B[\widehat{v}, \widehat{T}'] < B[\widehat{u}, \widehat{T}] \le B_\epsilon,$$

which is a contradiction excluding the first possibility. In the second case, where $B[\widehat{u}, \widehat{T}] \le K$, we can construct a pair $(\widehat{v}, \widehat{T}')$ with $\widehat{v}$ normalized such that

$$J_{\widehat{T}'}[\widehat{v}] < J_{\widehat{T}}[\widehat{u}] + \epsilon \le J_{\rm per}((2,2)) + 2\epsilon, \quad \text{and} \quad B[\widehat{v}, \widehat{T}'] < B[\widehat{u}, \widehat{T}] \le K,$$

which implies that $T_1 < \widehat{T}' < \widehat{T} \le K + T_2 = T_0$ and concludes the proof. For details concerning these constructions, see Theorem 5.5 in [5]. $\square$

Theorem 5.5 *Suppose that F satisfies* (H) *and $\beta, \gamma > 0$ are such that ± 1 are saddle-foci, then $\mathcal{J}_{\rm per}((2,2)^k)$ is attained for any $k \ge 1$. Moreover, the projection of any minimizer in $CM_{\rm per}((2,2))$ onto the (u,u')–plane is a simple closed curve.*

Proof. By Lemma 5.4, we can choose a minimizing sequence $(u_n, T_n) \in M^{T_n}_{\mathrm{per}}((2,2)) \times \mathbb{R}^+$, with u_n normalized and with the additional properties that $\|u_n\|_{H^2} \le C$ and $T_1 < T_n \le T_0$. Since the uniform separation property is satisfied for the type $(2,2)$ this leads to a minimizing pair $(\widehat{u}, \widehat{T})$ for (5.1) by following the proof of Theorem 2.2 in [5]. As for the existence of periodic minimizers of type $\mathbf{r} = (2,2)^k$ the uniform separation property is automatically satisfied and the above steps are identical.

Lemma 3.5 yields that minimizers are normalized functions and the projection of a normalized function in $M_{\mathrm{per}}((2,2))$ is a simple closed curve in the (u, u')–plane. $\square$

We would like to have the same theorem for arbitrary periodic types $\langle \mathbf{r} \rangle$. For homotopy types that satisfy the uniform separation property the analog of Theorem 5.5 can be proved. However, in Section 5 we will prove a more general result using the information about the minimizers with terminated types (homoclinic and heteroclinic minimizers) which was obtained in [5].

Remark 5.6 *The existence of a $(2,2)$-type minimizer is proved here in order to obtain a priori $W^{1,\infty}$-estimates for all minimizers (Section 6). However, if F satisfies the additional hypothesis that $F(u) \sim |u|^s$, $s > 2$ as $|u| \to \infty$, then such estimates are automatic (c.f. [7], [4]). In that case the existence of a minimizer of type $(2,2)$ follows from Theorem 5.14 below. To prove existence of minimizers of arbitrary type $\mathbf{r}$ we will use an analogue of Theorem 5.14 (see Lemma 6.7 and Theorem 6.8 below).*

5.2 Characterization of minimizers of type $\mathbf{g} = \langle (2,2) \rangle$

Periodic minimizers associated with $[e_0]$ or $[e_1]$ are the constant solutions $u = -1$ and $u = 1$ respectively. The simplest nontrivial periodic minimizers are those of type $\mathbf{r} = (2,2)^k$, i.e. $\mathbf{r} \in [(2,2)]$. These minimizers are crucial to the further analysis of the general case. The type $\mathbf{r} = (2,2)$ is a minimal type (associated with $[e_1 e_0]$), and we want to investigate the relation between minimizers in $M(\langle (2,2) \rangle)$ and periodic minimizers of type $(2,2)^k$.

Considering curves in the configuration space $\mathcal{P}$ is a convenient method for studying minimizers of type $(2,2)$. For example, minimizers in $CM(\langle (2,2) \rangle)$ and $CM_{\mathrm{per}}((2,2))$ all satisfy the property that they do not intersect the line segment $L = (-1,1) \times \{0\}$ in $\mathcal{P}$. If other homotopy types $\mathbf{r}$ are considered, i.e. $\mathbf{r} \notin [(2,2)]$, then minimizers represented as curves in $\mathcal{P}$ necessarily have self-intersections and they must intersect the segment L, which makes their comparison more complicated. We will come back to this problem in Section 6. Note that for a C^1-function u the associated curve $\Gamma(u)$ is a closed loop if and only if u is a periodic function.

Lemma 5.7 *For any non-periodic minimizer $u \in CM(\langle (2,2) \rangle)$ and any bounded interval I the curve $\Gamma[u|_I]$ has only a finite number of self-intersections. For periodic minimizers $u \in CM_{\mathrm{per}}(\langle (2,2) \rangle)$ this property holds when the length of I is smaller than the minimal period.*

Proof. Fix a time interval $I = [0, T]$. If u is periodic, T should be chosen smaller than the minimal period of u. Let $P = (u_0, u_0')$ be an accumulation point of self-intersections of $u|_I$. Then P is a self intersection point, and there exists a monotone sequence of times $\tau_n \in I$ converging to t_0 such that $\Gamma(u(\tau_n))$ are self-intersection points and $\Gamma(u(t_0)) = P$. Also there exists a corresponding sequence $\sigma_n \in I$ with $\sigma_n \neq \tau_n$ such that $\Gamma(u(\tau_n)) = \Gamma(u(\sigma_n))$. Choosing a subsequence if necessary, $\sigma_n \to s_0$ monotonically. Since u is a minimizer in $CM(\langle(2,2)\rangle)$, the intervals $[\sigma_n, \tau_n]$ must contain a transition, and hence $|\tau_n - \sigma_n| > T_0 > 0$. Therefore, $s_0 \neq t_0$, and we will assume that $s_0 < t_0$ (otherwise change labels). The homotopy type of $\Gamma(u|_{[s_0,t_0]})$ is $(2,2)^k$ for some $k \geq 1$ (since I is bounded).

Assume that σ_n and τ_n are increasing; the other case is similar. Using the times $\sigma_n < s_0 < \tau_n < t_0$, the curve $\Gamma_* = \Gamma[u|_{[\sigma_n-\delta,t_0+\delta]}]$, for δ sufficiently small, can be decomposed as $\Gamma_* = a \circ \sigma_2 \circ \gamma \circ \sigma_1 \circ b$ where $b = \Gamma(u|_{[\sigma_n-\delta,\sigma_n]})$, $\gamma_1 = \Gamma(u|_{[\sigma_n,s_0]})$, $\gamma = \Gamma(u|_{[s_0,\tau_n]})$, $\gamma_2 = \Gamma(u|_{[\tau_n,t_0]})$, and $a = \Gamma(u|_{[t_0,t_0+\delta]})$. For n sufficiently large, γ_1 and γ_2 have the same homotopy type, and $\gamma_1 \neq \gamma_2$, since otherwise u would be periodic with period smaller than $t_0 - \sigma_n < T$. We can now construct two more paths

$$\Gamma_1 = a \circ \gamma_1 \circ \gamma \circ \gamma_1 \circ b \quad \text{and} \quad \Gamma_2 = a \circ \gamma_2 \circ \gamma \circ \gamma_2 \circ b$$

which have the same homotopy type for n sufficiently large. Since $J[\Gamma_*]$ is minimal, $J[\Gamma_1] \geq J[\Gamma_*]$ and $J[\Gamma_1] \geq J[\Gamma_*]$, and thus $J[\gamma_1] \geq J[\gamma_2]$ and $J[\gamma_2] \geq J[\gamma_1]$ which implies that $J[\gamma_1] = J[\gamma_2]$. Therefore $J[\Gamma_*] = J[\Gamma_1] = J[\Gamma_2]$, and Γ_1, Γ_2 and Γ_* are all distinct minimizers with the same homotopy type and same boundary conditions. Since these curves all coincide along γ, the uniqueness of the initial value problem is contradicted. An argument very similar to the one above is also used in the proof of Lemma 5.12 and demonstrated in Figure 5.1. $\square$

Lemma 5.8 *If* $\mathbf{r} = (2,2)^k$ *with* $k > 1$, *then* $CM_{\rm per}(\mathbf{r}) = CM_{\rm per}((2,2))$ *and* $\mathcal{J}_{\rm per}(\mathbf{r}) = k \cdot \mathcal{J}_{\rm per}((2,2))$.

Proof. Let $u \in CM_{\rm per}(\mathbf{r})$ with $\mathbf{r} = (2,2)^k$ for $k > 1$, and let T be the period such that the associated curve in $\mathcal{P}$, $\Gamma(u|_{[0,T]})$, has the homotopy class of $\theta((2,2)^k)$. First we will prove that $\Gamma(u|_{[0,T]})$ is a simple closed curve in $\mathcal{P}$, and hence $u \in M_{\rm per}((2,2))$. Suppose not, then by Lemma 5.7 the curve $\Gamma(u|_{[0,T]})$ can be fully decomposed into k distinct simple closed curves Γ_i for $i = 1, \ldots, k$ (just call the inner loop Γ_1, cut it out, and call the new inner loop Γ_2, and so on). Denote by J_i the action associated with loop Γ_i, then $\sum_i J_i = J_T[u]$. Let $v_i \in M_{\rm per}((2,2)^k)$ be the function obtained by pasting together k copies of u restricted to the loop Γ_i. If v_i were a minimizer in $M_{\rm per}((2,2)^k)$, then by Lemma 5.2 the functions u and v_i would be distinct solutions to the differential equation (1.2) which coincide over an interval. This would contradict the uniqueness of solutions of the initial value problem, and hence v_i is not a minimizer, i.e. $J_{\widehat{T}}[v_i] = k \cdot J_i > \mathcal{J}_{\rm per}((2,2)^k)$. Consequently $\mathcal{J}_{\rm per}((2,2)^k) = \sum_i J_i > \mathcal{J}_{\rm per}((2,2)^k)$, which is a contradiction. Thus $u \in M_{\rm per}((2,2))$ and $\Gamma(u|_{[0,T]})$ is a simple loop traversed k times.

Now we will show that $u \in CM_{\rm per}((2,2))$. Since $\Gamma(u)$ is the projection of a function into the (u,u')–plane, u traverses the loop once over the interval $[0, T/k]$, and $\mathcal{J}_{\rm per}((2,2)^k) = k \cdot J_{T/k}[u]$. Suppose $J_{T/k} > \mathcal{J}_{\rm per}((2,2))$. Then we can construct a function in $M_{\rm per}((2,2)^k)$ with action less than $J[u] = \mathcal{J}_{\rm per}((2,2)^k)$ by gluing together k copies of a minimizer in $M_{\rm per}((2,2))$, which is a contradiction. $\square$

Lemma 5.9 *For any* $k \geq 1,\ CM_{\rm per}((2,2)^k) = CM_{\rm per}((2,2)) = CM_{\rm per}(\langle(2,2)\rangle)$.

Proof. We have already shown in Lemma 5.8 that $CM_{\rm per}((2,2)^k) = CM_{\rm per}((2,2))$. We now first prove that $CM_{\rm per}((2,2)) \subset CM_{\rm per}(\langle(2,2)\rangle)$. Let $u \in CM_{\rm per}((2,2))$ have period T. Suppose $u \notin CM_{\rm per}(\langle(2,2)\rangle)$. Then there exist two points $\Gamma(u(t_1)) = P_1$ and $\Gamma(u(t_2)) = P_2$ on $\Gamma(u)$ such that the curve γ between P_1 and P_2 obtained by following $\Gamma(u)$ is not minimal. Replacing γ by a curve with smaller action and the same homotopy type yields a function $v \in M_{\rm per}(\langle(2,2)\rangle)$ for which $J_{[t_1,t_2]}[v] \leq J_{[t_1,t_2]}[u]$. Choose $k \geq 0$ such that $kT > t_2 - t_1$. Then u is a minimizer in $CM_{\rm per}((2,2)^k) = CM_{\rm per}((2,2))$ which is a contradiction.

To finish the proof of the lemma we show that $CM_{\rm per}(\langle(2,2)\rangle) \subset CM_{\rm per}((2,2))$. Let $u \in CM_{\rm per}(\langle(2,2)\rangle)$ have period T. Let $\Gamma(u|_{[0,T]})$ be the associated closed curve in $\mathcal{P}$ and ω its winding number with respect to the segment L. Suppose $J_T[u] > \mathcal{J}_{\rm per}((2,2)^\omega) = \omega \cdot \mathcal{J}_{\rm per}(2,2)$. This implies the existence of a function $v \in M_{\rm per}((2,2)^\omega)$ and a period $\widehat{T}$ such that $J_{\widehat{T}}[v] < J_T[u]$. Choose a time $t_0 \in [0,T]$ such that $u(t_0) = 1$ and $u'(t_0) > 0$. Let $P_0 = (1, u'(t_0)) \in \mathcal{P}$. There exists a $\delta > 0$ sufficiently small such that $u(t_0 \pm \delta) > 0,\ u'(t_0 \pm \delta) > 0$, and u does not cross ± 1 in $[t_0 - \delta, t_0 + \delta]$ except at t_0. Let P_1 and P_2 denote the points $(u(t_0 \mp \delta), u'(t_0 \mp \delta))$ respectively. Let γ denote the piece of the curve $\Gamma(u)$ from P_1 to P_2 and γ^* the curve tracing $\Gamma(u)$ backward in time from P_2 to P_1. Now choose a point P_3 on $\Gamma(v)$ for which $v = 1$ and $v' > 0$. We can easily construct cubic polynomials p_1 and p_2 for which the curve $\Gamma(p_1)$ connects P_1 to P_3 and the curve $\Gamma(p_2)$ connects P_3 to P_2 in $\mathcal{P}$. These curves $\Gamma(p_i)$ are monotone functions, and hence the loop $\Gamma(p_1) \circ \Gamma(p_2) \circ \gamma^*$ has trivial homotopy type in $\mathcal{P}$. Therefore $\Gamma(u|_{[0,T]})^k \circ \gamma \sim \Gamma(p_2) \circ \Gamma(v|_{[0,\widehat{T}]})^k \circ \Gamma(p_1)$ in $\mathcal{P}$ for any $k \geq 1$, and from Definition 3.1 $J[\Gamma(u|_{[0,T]})^k \circ \gamma] \leq J[\Gamma(p_2) \circ \Gamma(v|_{[0,\widehat{T}]})^k \circ \Gamma(p_1)]$. Thus,

$$k \cdot J_T[u] + J[\gamma] \leq J[p_1] + J[p_2] + k \cdot J_{\widehat{T}}[v]$$

which implies

$$0 \leq k(J_T[u] - J_{\widehat{T}}[v]) \leq J[p_1] + J[p_2] - J[\gamma]$$

These estimates lead to a contradiction for k sufficiently large. $\square$

Lemma 5.10 *For any two distinct minimizers* u_1 *and* u_2 *in* $CM_{\rm per}((2,2))$*, the associated curves* $\Gamma(u_i)$ *do not intersect.*

Proof. Suppose $\Gamma(u_1)$ and $\Gamma(u_2)$ intersect at a point $P \in \mathcal{P}$. Translate u_1 and u_2 so that $\Gamma(u_1(0)) = \Gamma(u_2(0)) = P$. Define the function $u \in M_{\rm per}((2,2)^2)$ as the periodic extension of

$$u(t) = \begin{cases} u_1(t) & \text{for } t \in [0, T_1], \\ u_2(t - T_1) & \text{for } t \in [T_1, T_1 + T_2], \end{cases}$$

where T_i is the minimal period of u_i. Then $J_{T_1+T_2}[u] = 2\mathcal{J}_{\rm per}((2,2)) = \mathcal{J}_{\rm per}((2,2)^2)$. By Lemma 5.8 we have $u \in CM_{\rm per}((2,2))$, which contradicts the fact that u_1 and u_2 are distinct minimizers with $\Gamma(u_1) \neq \Gamma(u_2)$. $\square$

As a direct consequence of this lemma, the periodic orbits in $M_{\rm per}((2,2))$ are ordered in the sense that $\Gamma(u_1)$ lies either strictly inside or outside the region enclosed by $\Gamma(u_2)$. The ordering will be denoted by $>$.

Theorem 5.11 *There exists a largest and a smallest periodic orbit in $CM_{\rm per}((2,2))$ in the sense of the above ordering, which we will denote by $u_{\max}$ and $u_{\min}$ respectively. Moreover $1 < \|u_{\min}\|_{1,\infty} \le \|u_{\max}\|_{1,\infty} \le C_0$, and $u_{\min} < u < u_{\max}$ for every $u \in CM_{\rm per}((2,2))$. In particular the set $CM_{\rm per}((2,2))$ is compact.*

Proof. Either the number of periodic minimizers is finite, in which case there is nothing to prove, or the set of minimizers is infinite. Let $U = \bigcup\{\Gamma(u) \mid u \in CM_{\rm per}((2,2))\} \subset \mathcal{P}$, and let $A = U \cap \{(u,u') \mid u' = 0, u > 0\}$. Every minimizer in $CM_{\rm per}((2,2))$ intersects the positive u–axis transversely exactly once. Moreover distinct minimizers cross this axis at distinct points by Lemma 5.10. Thus we can use A as an index set and label the minimizers as u_α for $\alpha \in A$. Due to the a priori upper bound on minimizers (Lemma 5.1 in [5]), A is a bounded set. The set A is contained in the u-axis and hence has an ordering induced by the real numbers. This order corresponds to the order on minimizers, i.e. $\alpha < \beta$ in A if and only if $u_\alpha < u_\beta$ as minimizers.

Suppose α_* is an accumulation point of A. Then there exists a sequence α_n converging to α_*. From Theorem 4.2 (the a priori L^∞-bound on u_{α_n} is sufficient by Remark 4.3) we see that there exists $\widehat{u} \in CM(\langle(2,2)\rangle)$ which is a solution to Equation (1.2) such that $u_{\alpha_n} \to \widehat{u}$ in $C^1_{\rm loc}(\mathbb{R})$. Since u_{α_n} is periodic and the $C^1_{\rm loc}$–limit of a sequence of periodic functions with uniformly bounded periods (compare with the proof of Theorem 4.2 to find a uniform bound on the periods) is periodic, $\widehat{u} \in CM_{\rm per}(\langle(2,2)\rangle)$. By Lemma 5.9, $\widehat{u} \in CM_{\rm per}((2,2))$. Furthermore $\widehat{u}$ corresponds to u_{α_*}, and hence A is compact.

Consequently A contains maximal and minimal elements. Let $u_{\max}$ and $u_{\min}$ be the periodic minimizers through the maximal and minimal points of A respectively. This proves the theorem. $\square$

The above lemmas characterize periodic minimizers in $CM(\langle(2,2)\rangle)$. Now we turn our attention to non-periodic minimizers. We conclude this subsection with a theorem that gives a complete description of the set $CM(\langle(2,2)\rangle)$.

Let $u \in CM(\langle(2,2)\rangle)$ be non-periodic. Suppose that P is a self-intersection point of $\Gamma(u)$. Then there exist times $t_1 < t_2$ such that $\Gamma(u(t_1)) = \Gamma(u(t_2)) = P$, and $\Gamma(u|_{[t_1,t_2]})$ is a closed loop. By Lemma 5.7 there are only finitely many self-intersections on $[t_1,t_2]$. Without loss of generality we may therefore assume that γ is a simple closed loop, i.e, we need only consider the case where $P = \Gamma(u(t_1)) = \Gamma(u(t_2))$ and $\Gamma(u|_{[t_1,t_2]})$ is a simple closed loop. We now define $\Gamma_+ = \Gamma(u|_{(t_1,\infty)})$ and $\Gamma_- = \Gamma(u|_{(-\infty,t_2)})$. We will refer to $\Gamma_\pm$ as the forward and backward orbits of u relative to P.

Lemma 5.12 *Let $u \in CM(\langle(2,2)\rangle)$ be a non-periodic minimizer with at least one self-intersection. Let P and $\Gamma_\pm$ be defined as above. Then the forward and backward orbits $\Gamma_\pm$ relative to P do not intersect themselves. Furthermore, P and $\Gamma_\pm$ are unique, and the curve $\Gamma(u)$ passes through any point in $\mathcal{P}$ at most twice.*

Proof. We will prove the result for Γ_+; the argument for Γ_- is similar. Suppose that Γ_+ has self-intersections. Define

$$t_* = \min\{t > t_1 \mid \Gamma(u(t)) = \Gamma(u(\tau)) \text{ for some } \tau \in (t_1,t)\}.$$

The minimum t_* is attained by Lemma 5.7, and $t_* > t_2$ since $\gamma \equiv \Gamma(u|_{[t_1,t_2]})$ is a simple closed loop. Let $t_0 \in (t_1, t_*)$ be the point such that $\Gamma(u(t_0)) = \Gamma(u(t_*))$. This point is unique by the definition of t_*, and $\tilde{\gamma} \equiv \Gamma(u|_{[t_0,t_*]})$ is a simple closed loop. For small positive δ we define $Q = \Gamma(u(t_*))$, $B = \Gamma(u(t_1 - \delta))$, $E = \Gamma(u(t_* + \delta))$ and $\Gamma_* = \Gamma(u|_{[t_1-\delta,t_*+\delta]})$, see Figure 5.1. We can decompose this curve into five parts; $\Gamma_* = \sigma_3 \circ \tilde{\gamma} \circ \sigma_2 \circ \gamma \circ \sigma_1$ where σ_1 joins B to P, σ_2 joins P to Q, σ_3 joins Q to E, and γ and $\tilde{\gamma}$ are simple closed loops based at P and Q respectively, see Figure 5.1. The simple closed curves γ and $\tilde{\gamma}$ go around L exactly once and thus have the same homotopy type. Moreover, $\gamma \neq \tilde{\gamma}$ since u is non-periodic.

Besides Γ_* we can construct two other distinct paths from B to E:

$$\Gamma_1 = \sigma_3 \circ \sigma_2 \circ \gamma \circ \gamma \circ \sigma_1 \quad \text{and} \quad \Gamma_2 = \sigma_3 \circ \tilde{\gamma} \circ \tilde{\gamma} \circ \sigma_2 \circ \sigma_1.$$

It is not difficult to see that Γ_1, Γ_2 and Γ_* all have the same homotopy type. Since $J[\Gamma_*]$ is minimal in the sense of Definition 3.1 we have, by the same reasoning as in Lemma 5.7, that $J[\Gamma_1] \geq J[\Gamma_*]$ and $J[\Gamma_2] \geq J[\Gamma_*]$, which implies that $J[\tilde{\gamma}] \geq J[\gamma]$ and $J[\gamma] \geq J[\tilde{\gamma}]$. Hence $J[\gamma] = J[\tilde{\gamma}]$. Therefore $J[\Gamma_1] = J[\Gamma_2] = J[\Gamma_*]$ which implies that Γ_1, Γ_2 and Γ_* are all distinct minimizers of the same type as curves joining B to E. Since these curves all contain the paths σ_1, σ_2 and σ_3, and are solutions to (1.2), the uniqueness to the initial value problem is contradicted.

Finally, the curve $\Gamma(u)$ can pass through a point at most twice because it is a union of Γ_+ and Γ_-, each visiting a point at most once. Moreover, points in $\Gamma(u|_{(t_1,t_2)})$, common to both Γ_+ and Γ_-, are passed exactly once. It now follows that if there is another self-intersection besides P, say at $R = \Gamma(u(s_1)) = \Gamma(u(s_2))$, then $s_1 < t_1$ and $t_2 < s_2$. We conclude that the curve $\Gamma(u|_{(s_1,s_2)})$ contains $\Gamma(u|_{[t_1,t_2]})$ and therefore it is not a simple closed curve. Thus P is a unique self-intersection that cuts off a simple loop. $\square$

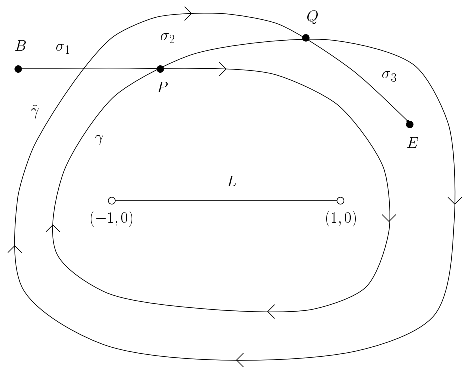

Figure 5.1: The forward orbit Γ_+ starting at P with a self-intersection at the point Q. Lemma 5.12 implies that this cannot happen for non-periodic $u \in CM(\langle(2,2)\rangle)$.

Lemma 5.13 *Let $u \in CM(\langle(2,2)\rangle)$ be non-periodic. Suppose that $u \in L^\infty(\mathbb{R})$. Then u is a connecting orbit between two periodic minimizers $u_-, u_+ \in CM_{\rm per}((2,2))$, i.e. there are sequences $t_n^-, t_n^+ \to \infty$ such that $u(t - t_n^-) \to u_-(t)$ and $u(t + t_n^+) \to u_+(t)$ in $C^4_{\rm loc}(\mathbb{R})$.*

Proof. Lemma 5.12 implies that Γ_+ is a spiral which intersects the positive u–axis at a bounded, monotone sequence of points $(\alpha_n, 0)$ in $\mathcal{P}$ converging to a point $(\alpha_*, 0)$. Let t_n be the sequence of consecutive times such that $u(t_n) = \alpha_n$. Consider the sequence of minimizers in $CM(\langle(2,2)\rangle)$ defined by $u_n(t) = u(t+t_n)$. By Theorem 4.2 there exist a $C^1_{\rm loc}$–limit $u_+ \in CM(\langle(2,2)\rangle)$. If u_+ is periodic, there is nothing more to prove. Thus suppose u_+ is non-periodic. Then the curve $\Gamma(u_+)$ crosses the u–axis infinitely many times. On the other hand, from the $C^1_{\rm loc}$ convergence $\Gamma(u_+)$ crosses this axis only at α_*. By Lemma 5.12, $\Gamma(u_+)$ can intersect α_* at most twice, which is a contradiction. The $C^4_{\rm loc}$–convergence follows from regularity (as in the proof of Theorem 4.2). The proof of the existence of u_- is similar. $\square$

Theorem 5.14 *Let $u \in CM(\langle(2,2)\rangle)$. Either u is unbounded, u is periodic and $u \in CM_{\rm per}((2,2))$, or u is a connecting orbit between periodic minimizers in $CM_{\rm per}((2,2))$.*

Proof. Let $u \in CM(\langle(2,2)\rangle)$ be bounded, then u is either periodic or non-periodic. In the case that u is periodic it follows from Lemma 5.9 that $u \in CM_{\rm per}((2,2))$. Otherwise if u is not periodic it follows from Lemma 5.13 that u is a connecting orbit between two minimizers $u_-, u_+ \in CM_{\rm per}((2,2))$. $\square$

In Section 6.2 we give analogues of the above theorems for arbitrary homotopy types $\mathbf{r}$.

Notice that the option of $u \in CM(\langle(2,2)\rangle)$ being unbounded in the above theorem does not occur when $F(u) \sim |u|^s$, $s > 2$ as $|u| \to \infty$.

6 Properties of minimizers

In Section 5, we proved the existence of minimizers in $M_{\rm per}((2,2))$, which will provide a priori bounds on the minimizers of arbitrary type. These bounds and Theorem 4.2 will establish the existence of such minimizers. In this section we will also prove that certain properties of a type $\mathbf{g}$ are often reflected in the associated minimizers. The most important examples are the periodic types $\mathbf{g} = \langle\mathbf{r}\rangle$. Although there are minimizers in every class $M(\langle\mathbf{r}\rangle, \mathbf{p})$, it is not clear a priori that among these minimizers there are also periodic minimizers. In order to prove existence of periodic minimizers for every periodic type $\langle\mathbf{r}\rangle$ we use the theory of covering spaces.

6.1 Existence

The periodic minimizers of type $(2,2)$ are special for the following reason. For a normalized $u \in M_{\rm per}((2,2))$, define $D(u)$ to be the closed disk in $\mathbb{R}^2$ such that $\partial D(u) = \Gamma(u)$.

Theorem 6.1 *i) If $u \in CM(\langle\mathbf{r}\rangle, \mathbf{p})$ then $\Gamma(u) \subset D(u_{\rm min})$ for any periodic type $\langle\mathbf{r}\rangle \neq \langle(2,2)\rangle$. ii) If $u \in CM(\mathbf{g}, \mathbf{p})$ then $\Gamma(u) \subset D(u_{\rm min})$ for any terminated type $\mathbf{g}$.*

Proof. i) If $\langle\mathbf{r}\rangle \neq \langle(2,2)\rangle$ then every $u \in CM(\langle\mathbf{r}\rangle, \mathbf{p})$ has the property that $\Gamma(u)$ intersects the u-axis between $u = \pm 1$. Suppose that $\Gamma(u)$ does not lie inside $D(u_{\rm min})$. Then $\Gamma(u)$ must intersect $\Gamma(u_{\rm min})$ at least twice, and let P_1 and P_2 be distinct intersection points with the property that the curve Γ_1 obtained by following $\Gamma(u)$ from P_1 to P_2 lies entirely outside of $D(u_{\min})$. Let $\Gamma_2 \subset \Gamma(u_{\min})$ be the curve from P_1 to P_2 following $u_{\min}$, such that Γ_1 and Γ_2 are homotopic (traversing the loop $\Gamma(u_{\min})$ as many times as necessary) and thus $J[\Gamma_1] = J[\Gamma_2]$ is minimal. Replacing Γ_1 by Γ_2 leads to a minimizer in $CM(\langle \mathbf{r} \rangle, \mathbf{p})$ which partially agrees with u. This contradicts the uniqueness of the initial value problem for (1.2).

ii) As in the previous case the associated curve $\Gamma(u)$ either intersects $\Gamma(u_{\min})$ at least twice or lies completely inside $D(u_{\min})$, and the proof is identical. $\square$

Corollary 6.2 *For all minimizers in the above theorem,* $\|u\|_{1,\infty} \leq \|u_{\min}\|_{1,\infty} \leq C_0$.

In order to prove existence of minimizers in every class we now use the above theorem in combination with an existence result from [5].

Theorem 6.3 *For any given type* $\mathbf{g}$ *and parity* $\mathbf{p}$ *there exists a (bounded) minimizer* $u \in CM(\mathbf{g}, \mathbf{p})$. *Moreover* $\|u\|_{1,\infty} \leq C_0$, *independent of* $(\mathbf{g}, \mathbf{p})$.

Proof. Given a type $\mathbf{g}$ we can construct a sequence $\mathbf{g}_n$ of terminated types such that $\mathbf{g}_n \to \mathbf{g}$ as $n \to \infty$. For any terminated type $\mathbf{g}_n$ there exists a minimizer $u_n \in CM(\mathbf{g}_n, \mathbf{p})$ by Proposition 3.3 (Theorem 1.3 of [5]). Clearly such a sequence u_n satisfies $\|u_n\|_{1,\infty} \leq C$ by Corollary 6.2. Applying Theorem 4.2 completes the proof. $\square$

6.2 Covering spaces and the action of the fundamental group

The fundamental group of $\mathcal{P}$ is isomorphic to the free group on two generators e_0 and e_1 which represent loops (traversed clockwise) around $(1,0)$ and $(-1,0)$ respectively with basepoint $(0,0)$. Indeed, $\mathcal{P}$ is homotopic to a bouquet of two circles $X = S_1 \vee S_1$. The universal covering of X denoted by $\widetilde{X}$ can be represented by an infinite tree whose edges cover either e_0 or e_1 in X, see Figure 6.1. The universal covering of $\mathcal{P}$ denoted by $\wp : \widetilde{\mathcal{P}} \to \mathcal{P}$ can then be viewed by thickening the tree $\widetilde{X}$ so that $\widetilde{\mathcal{P}}$ is homeomorphic to an open disk in $\mathbb{R}^2$.

An important property of the universal covering is that the fundamental group $\pi_1(\mathcal{P})$ induces a left group action on $\widetilde{\mathcal{P}}$ in a natural way, via the lifting of paths in $\mathcal{P}$ to paths in $\widetilde{\mathcal{P}}$. This action will be denoted by $\theta \cdot p$ for $\theta \in \pi_1(\mathcal{P})$ and $p \in \widetilde{\mathcal{P}}$. We will not reproduce the construction of this action here, and the reader is referred to an introductory book on algebraic topology such as [3]. However, we will utilize the structure of the quotient spaces of $\widetilde{\mathcal{P}}$ obtained from this action, which are again coverings of $\mathcal{P}$. These quotient spaces will be the natural spaces in which to consider the lifts of curves $\Gamma(u)$ which lie in more complicated homotopy classes than those in the case of $u \in M_{\mathrm{per}}((2,2))$.

A periodic type $\mathbf{g} = \langle \mathbf{r} \rangle$ is generated by a finite type $\mathbf{r}$, which together with the parity $\mathbf{p}$ determines an element of $\pi_1(\mathcal{P})$ of the form $\theta(\mathbf{r}) = e_{|\mathbf{p}-1|}^{r_{2n}} \cdot \ldots \cdot e_{\mathbf{p}}^{r_1}$. Since we only consider curves in $\mathcal{P}$ which are of the form $\Gamma(u) = (u(t), u'(t))$, the numbers r_i are all positive. The infinite cyclic subgroup generated by any such element θ will be denoted by $\langle \theta \rangle \subset \pi_1(\mathcal{P})$. The quotient space $\widetilde{\mathcal{P}}_\theta = \widetilde{\mathcal{P}} / \langle \theta \rangle$ is obtained by identifying points p and q in $\widetilde{\mathcal{P}}$ for which $q = \theta^k \cdot p$ for some $k \in \mathbb{Z}$. The resulting space $\widetilde{\mathcal{P}}_\theta$ is homotopic

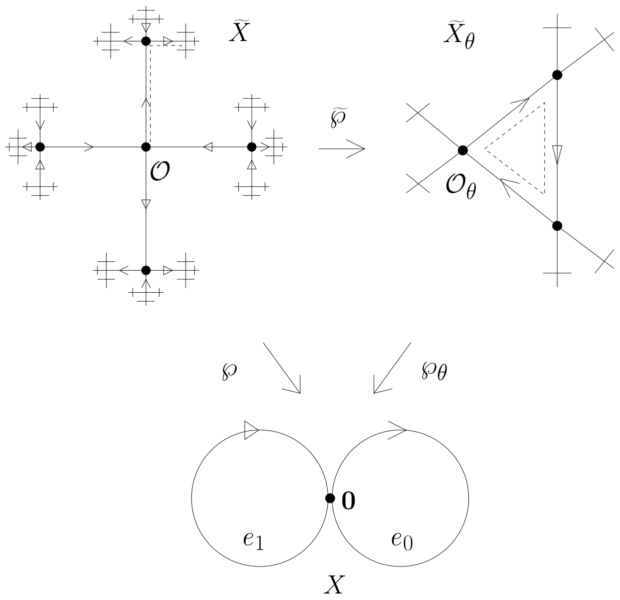

Figure 6.1: The universal cover $\widetilde{X}$ of X is a tree. Its origin is denoted by $\mathcal{O}$. For $\theta = e_0e_1e_0$, the quotient space $\widetilde{X}_\theta = \widetilde{X}/\langle\theta\rangle$ is also a covering space over X, and $\widetilde{X}_\theta \sim S^1$.

to an annulus, and $\wp_\theta : \widetilde{\mathcal{P}}_\theta \to \mathcal{P}$ is a covering space. Figure 6.1 illustrates the situation for X, since it is easier to draw, and for $\mathcal{P}$ the reader should imagine that the edges in the picture are thin strips. The lift of the path $\theta = e_0e_1e_0$ to $\widetilde{X}$ based at $\mathcal{O}$ is shown by the dashed line. This piece of the tree becomes a circle in the quotient space $\widetilde{X}_\theta$. Note that infinitely many edges in $\widetilde{X}$ are identified with this circle. The dashed lines in both $\widetilde{X}$ and $\widetilde{X}_\theta$ are strong deformation retracts of $\widetilde{X}$ and $\widetilde{X}_\theta$ respectively, and hence $\widetilde{X}_\theta$ is homotopic to a circle. Thickening $\widetilde{X}_\theta$ gives that $\widetilde{\mathcal{P}}_\theta$ is homotopic to an annulus. Thus $\pi_1(\widetilde{\mathcal{P}}_\theta)$ is a generated by a simple closed loop in $\widetilde{\mathcal{P}}_\theta$ which will be denoted by $\zeta(\mathbf{r})$. Note that for convenience we suppress the dependence of θ and ζ on the parity $\mathbf{p}$.

Remark 6.4 *If we define the shift operator σ on finite types $\mathbf{r}$ to be a cyclic permutation, then $M_{\text{per}}(\mathbf{r},\mathbf{p}) = M_{\text{per}}(\sigma^k(\mathbf{r}),\tau^k(\mathbf{p}))$ for all $k \in \mathbb{Z}$. Functions in $M_{\text{per}}(\mathbf{r},\mathbf{p})$ have a unique lift to simple closed curve in $\widetilde{\mathcal{P}}_\theta$, $\theta = \theta(\mathbf{r})$. However, functions in the shifted class $M_{\text{per}}(\sigma^k(\mathbf{r}),\tau^k(\mathbf{p}))$ are not simple closed curves in $\widetilde{\mathcal{P}}_\theta$. In order for such functions to be lifted to a unique simple closed curve we need to consider the covering space $\widetilde{\mathcal{P}}_{\theta_k}$, where $\theta_k = \theta(\sigma^k(\mathbf{r}),\tau^k(\mathbf{p}))$.*

6.3 Characterization of minimizers of type $\langle\mathbf{r}\rangle$

In Section 5.2, we characterized minimizers in $CM(\langle(2,2)\rangle)$ by studying the properties of their projections into $\mathcal{P}$. What was special about the types $(2,2)^k$ was that the projected curves were a priori contained in $\mathcal{P}\setminus L$, which is topologically an annulus. The J-efficiency of minimizing curves restricts the possibilities for their self and mutual intersections. In particular, we showed that all periodic minimizers in $CM(\langle(2,2)\rangle)$

project onto simple closed curves in $\mathcal{P} \setminus L$ and that no two such minimizing curves intersect. These two properties, coupled with the simple topology of the annulus, already force the minimizing periodic curves to have a structure of a family of nested simple loops.

Such a simple picture in the configuration plane $\mathcal{P}$ cannot be expected for minimizers in $CM((\langle \mathbf{r} \rangle, \mathbf{p}))$ with $\mathbf{r} \neq (2,2)$. The simple intersection properties (of Lemma 5.9 and 5.11) no longer hold; in fact, periodic minimizing curves must have self-intersections in $\mathcal{P}$ as do any curves in $\mathcal{P}$ representing the homotopy class of $(\langle \mathbf{r} \rangle, \mathbf{p})$. However, by lifting minimizing curves into the annulus $\widetilde{\mathcal{P}}_\theta$, we can remove exactly these necessary self-intersections and put us in a position to emulate the discussion for the types $(2,2)^k$. More precisely, for a minimal type $(\mathbf{r}, \mathbf{p})$, any $u \in M_{\mathrm{per}}((\mathbf{r}, \mathbf{p})^k)$ with period T such that $\theta^{-1}[\Gamma(u|_{[0,T]})] = (\mathbf{r}, \mathbf{p})^k$, there are infinitely many lifts of the closed loop $\Gamma(u|_{[0,T]})$ into $\widetilde{\mathcal{P}}_\theta(\mathbf{r})$ (see above remark) but there is exactly one lift, denoted $\Gamma_\theta(u|_{[0,T]})$, that is a closed loop homotopic to $\zeta^k(\mathbf{r})$ in $\widetilde{\mathcal{P}}_\theta(\mathbf{r})$. We can repeat all of the arguments in Section 5 by identifying intersections between the curves $\Gamma_\theta(u|_{[0,T]})$ in $\widetilde{\mathcal{P}}_\theta(\mathbf{r})$ instead of intersections between the curves $\Gamma(u|_{[0,T]})$ in $\mathcal{P} \setminus L$. Of course, when gluing together pieces of curves, the values of u and u' come from the projections into $\mathcal{P}$. In particular, the arguments of Lemma 5.9 show that $\Gamma_\theta(u|_{[0,T]})$ must be a simple loop traced k-times, which leads to the following:

Lemma 6.5 *For any periodic type $\langle \mathbf{r} \rangle$ and any $k \geq 1$ it holds that $CM_{\mathrm{per}}((\mathbf{r}, \mathbf{p})^k) = CM_{\mathrm{per}}(\mathbf{r}, \mathbf{p}) = CM_{\mathrm{per}}(\langle \mathbf{r} \rangle, \mathbf{p})$.*

The proof of the next theorem is a slight modification of Theorem 5.11.

Theorem 6.6 *For any periodic type $\langle \mathbf{r} \rangle$ the set $CM_{\mathrm{per}}(\mathbf{r}, \mathbf{p})$ is compact and totally ordered (in $\widetilde{\mathcal{P}}_\theta$).*

The following lemma is analogous to Lemma 5.13. Note however that by Theorem 6.1 we do not need to assume that the minimizer is uniformly bounded.

Lemma 6.7 *Let $u \in CM(\langle \mathbf{r} \rangle, \mathbf{p})$ for some periodic type $\langle \mathbf{r} \rangle \neq \langle (2,2) \rangle$. Either u is periodic and $u \in CM_{\mathrm{per}}(\mathbf{r}, \mathbf{p})$, or u is a connecting orbit between two periodic minimizers $u_-, u_+ \in CM_{\mathrm{per}}(\mathbf{r}, \mathbf{p})$, i.e. there are sequences $t_n^-, t_n^+ \to \infty$ such that $u(t - t_n^-) \to u_-(t)$ and $u(t + t_n^+) \to u_+(t)$ in $C^4_{\mathrm{loc}}(\mathbb{R})$.*

Combining Theorem 6.3 and Lemma 6.7 we obtain the existence of periodic minimizers in every class with a periodic type (this result can also be obtained in a way analogous to Theorem 5.5).

Theorem 6.8 *For any periodic type $\langle \mathbf{r} \rangle$ the set $CM_{\mathrm{per}}(\mathbf{r}, \mathbf{p})$ is nonempty.*

The classification of functions by type has some properties in common with symbolic dynamics. For example, if a type $\mathbf{g}$ is asymptotic to two different periodic types, i.e. $\sigma^n(\mathbf{g}) \to \mathbf{r}_+$ and $\sigma^{-n}(\mathbf{g}) \to \mathbf{r}_-$ as $n \to \infty$, with $\mathbf{r}_+ \neq \mathbf{r}_-$, then any minimizer $u \in CM(\mathbf{g}, \mathbf{p})$ is a connecting orbit between two periodic minimizers $u_- \in CM_{\mathrm{per}(\mathbf{r}_-, \mathbf{p})}$ and

$u_+ \in CM_{\rm per}(\mathbf{r}_+, \mathbf{p})$, i.e. there exist sequences $t_n^-, t_n^+ \to \infty$ such that $u(t - t_n^-) \to u_-(t)$ and $u(t + t_n^+) \to u_+(t)$ in $C^4_{\rm loc}(\mathbb{R})$. This result follows from Cantor's diagonal argument using Theorems 4.2 and 6.7, and hence we have used the symbol sequences to conclude the existence of heteroclinic and homoclinic orbits connecting any two types of periodic orbits.

Symmetry properties of types $\mathbf{g}$ are also often reflected in the corresponding minimizers. For example, define the map Ψ_{i_0} on infinite types by $\Psi_{i_0}(\mathbf{g}) = (g_{2i_0-i})_{i\in\mathbb{Z}}$, and consider types that satisfy $\Psi_{i_0}(\mathbf{g}) = \mathbf{g}$ for some i_0. Moreover assume that $\mathbf{g}$ is periodic. In this case we can prove that the corresponding periodic minimizers are symmetric and satisfy Neumann boundary conditions.

Theorem 6.9 *Let* $\mathbf{g} = \langle \mathbf{r} \rangle$ *satisfy* $\Psi_{i_0}(\langle \mathbf{r} \rangle) = \langle \mathbf{r} \rangle$ *for some* i_0. *Then for any* $u \in CM_{\rm per}(\mathbf{r}, \mathbf{p})$ *there exists a shift* τ *such that* $u_\tau(x) = u(x - \tau)$ *satisfies*
i) $u_\tau(x) = u_\tau(T - x)$ *for all* $x \in [0, T]$ *where* T *is the period of* u*,*
ii) $u'_\tau(0) = u'''_\tau(0) = 0$ *and* $u'_\tau(T) = u'''_\tau(T) = 0$*, and*
iii) u_τ *is a local minimizer for the functional* $J_T[u]$ *on the Sobolev space* $H^2_{\mathbf{n}}(0, T) = \{u \in H^2(0, T) \mid u'(0) = u'(T) = 0\}$.

Proof. Without loss of generality we may assume that $i_0 = 1$ and that $\mathbf{g} = \langle (g_1, \ldots, g_N) \rangle$ for some $N \in 2\mathbb{N}$. We can choose a point t_0 in the convex hull of A_1 such that $u'(t_0) = u'(t_0 + T) = 0$ and $\mathbf{g}(u|_{[t_0,t_0+T]}) = (g_1/2, g_2, \ldots, g_N, g_1/2)$. We now define $v(t) = u(t_0 + T - t)$. Then by the symmetry assumptions on $\mathbf{g}$ we have that $\mathbf{g}(v|_{[t_0,t_0+T]}) = \mathbf{g}(u|_{[t_0,t_0+T]})$. Since $J_{[t_0,t_0+T]}(v) = J_{[t_0,t_0+T]}(u)$ and $\Gamma(u(t_0)) = \Gamma(u(t_0+T)) = \Gamma(v(t_0)) = \Gamma(v(t_0+T))$, we conclude from the uniqueness of the initial value problem that $u(t) = v(t)$ for all $t \in [t_0, t_0 + T]$, which proves the first statement. The second statement follows immediately from *i)*. The third property follows from the definition of minimizer. $\square$

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