On quotient-convergence factors.

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Introduction and statement of results.

Consider the following recurrence

$$\begin{cases} x_{n+1} = \lambda \cdot x_n + \Omega x_n, \ n = 0, 1, 2, ... \\ x_0 = x, \end{cases} \tag{1}$$

where x_n's belong to a certain linear space X over the field of real numbers $\mathbf{R}$, Ω is a linear operator on X and $\lambda \in \mathbf{R}$ is fixed. If Ω is assumed to be *small* (in the sense yet to be made precise), then one may think of (1) as a perturbation of the problem $y_{n+1} = \lambda \cdot y_n$, $y_0 = x$. It is expected then that the behavior of the perturbed sequence (x_n) resembles that of (y_n). Our main result, stated below and proved in Section 1, is a new instance of this familiar situation.

Theorem 1 *Suppose that (x_n) satisfies (1) and that $\psi : X \to \mathbf{R}$ is an arbitrary linear functional. If either*
(U) $\lim_{k\to\infty} \limsup_{n\to\infty} |\psi(\Omega^{k+1}x_n)/\psi(\Omega^k x_n)| = 0$,
or
(P) $\lim_{k\to\infty} \psi(\Omega^{k+1}x_0)/\psi(\Omega^k x_0) = 0$ *and* $\lambda > 0$, $\psi(\Omega^k x_0) > 0$, $k = 0, 1, 2...$,
then

$$\lim_{n\to\infty} \psi(x_{n+1})/\psi(x_n) = \lambda.$$

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The usefulness of Theorem 1 may be seen in the context of the waveform relaxation (WR) method as considered in [1]. While the reader should consult [1] for the details, let us recall that it deals with an initial value problem for a functional-differential system of the form $y'(t) = f(t, y(\cdot), y'(\cdot))$, where $t \in [a, b]$ and $y(t) \in \mathbf{R}^d$. The functional f takes as arguments functions defined over $[\alpha, b]$ for some $\alpha \leq a$ (thus allowing for delays) and is assumed to satisfy the following Lipshitz condition with $L \geq 0$, $1 > K \geq 0$ and $\|\cdot\|_t$ standing for the sup norm over $[a, t]$:

$$\|f(t, y(\cdot), z(\cdot)) - f(t, \tilde{y}(\cdot), \tilde{z}(\cdot))\| \leq L\|y - \tilde{y}\|_t + K\|z - \tilde{z}\|_t, \tag{2}$$

for any continuous functions $y, \tilde{y}, z, \tilde{z}$ and $t \in [a, b]$.

The idea of the WR method is in decoupling the system by exploiting an initial approximation to the solution — which allows for parallel numerical integration of the different components — and then bootstrapping this procedure to construct an iterative scheme that renders consecutive approximations to the true solution. For example, by totally decoupling the components of y the WR Gauss-Jacobi iteration is obtained:

$$\begin{aligned} z_i^{(n)}(t) = f_i(t, & y_1^{(n-1)}(\cdot), ..., y_{i-1}^{(n-1)}(\cdot), y_i^{(n)}(\cdot), y_{i+1}^{(n-1)}(\cdot), ..., y_d^{(n-1)}(\cdot), \\ & z_1^{(n-1)}(\cdot), ..., z_{i-1}^{(n-1)}(\cdot), z_i^{(n)}(\cdot), z_{i+1}^{(n-1)}(\cdot), ..., z_d^{(n-1)}(\cdot)), \end{aligned}$$

$i = 1, ..., d$, where $z^{(n)}(t) = \frac{d}{dt} y^{(n)}(t)$, $t \in [a, b]$ and $n = 0, 1, 2, ...$.

From (2), the error $v^{(n)}(t) := \|z^{(n)} - z\|_t := \sup_{a \leq s \leq t} \|z^{(n)}(s) - z(s)\|$ satisfies a bound $v^{(n)}(t) \leq \overline{u}^{(n)}(t) \exp(A(t-a))$ where $\overline{u}^{(n)}$ is defined via iteration (see (2.4) in [1])

$$\overline{u}^{(n)}(t) = K\overline{u}^{(n-1)}(t) + M \int_a^t \overline{u}^{(n-1)}(s) ds, \ n \in \mathbf{N}, \tag{3}$$

with the initial value $\overline{u}^{(0)}(t) = v^{(0)}(t) \exp(-A(t-a))$ and $A, M > 0$ some constants. In further reduction, the sequence

$$\sigma^{(n)}(t) = \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array} \right) \frac{(M(t-a))^i K^{n-i}}{i!}, \tag{4}$$

which is exactly the solution of (3) for the initial data equal to 1, is used for the majorization $\overline{u}^{(n)}(t) \leq Z_0 \sigma^{(n)}(t)$, $Z_0 := \sup_{a \leq t \leq b} v^{(0)}(t)$; and the linear convergence of the WR method derives from the following theorem.

Theorem 2 (Th. 2 in [1]) *If $\sigma^{(n)}(t)$ is defined by (4), then, for any $t \in [a,b]$,*

$$\lim_{n\to\infty} \sigma^{(n+1)}(t)/\sigma^{(n)}(t) = K.$$

The key iteration (3) is clearly an instance of (1) with $\lambda = K$ and $\Omega = \Omega^{(\mathrm{Vol})}$,

$$\Omega^{(\mathrm{Vol})}x(t) = \int_a^t x(s)ds, \ t \geq a,$$

and $\sigma^{(n)} = x_n$ under the initial condition $x_0(t) = 1$, $t \in [a,b]$. By taking for ψ the functional of evaluation at a fixed instant t, Theorem 1 yields Theorem 2 provided we can verify, say, hypothesis (P). For this we calculate:

$$\Omega^{k+1}(1)(t)/\Omega^k(1)(t) = \frac{(t-a)^{k+1}/(k+1)!}{(t-a)^k/k!} = (t-a)/(k+1) \to 0.$$

In this way Theorem 2 is a manifestation of a more general phenomenon regarding *quotient-convergence factors* — as the limits involved are often referred to, see Definition 9.1.1 in [3]. One advantage of this approach may be that it applies to the iteration (3) with less stringent initial conditions deprived of an explicit formula like (4). Indeed, the following fact, proved in Section 2, verifies the hypothesis of Theorem 1 (in the WR context) under more general circumstances.

Fact 1 *If $x_n(t)$ is defined by (1) with $\Omega = \Omega^{(\mathrm{Vol})}$ and $\lambda > 0$, then*

$$\lim_{k\to\infty} \sup_{n\in\mathbf{N}} \left\{ |\Omega^{k+1}x_n(t)/\Omega^k x_n(t)| \right\} = 0,$$

under either of the two following assumptions on the initial condition
(i) $x_0(t)$ is non-decreasing and positive for $t \in (a,b)$;
(ii) $x_0(0) \neq 0$ and $x_0(t)$ is continuous for $t \in [a,b)$.

Finally, let us note here that it is standard and much easier to show a weaker form of Theorem 2 concerning the *root-convergence factors* (see Definition 9.2.1 in [3]). Namely, for $x_n = \sigma^{(n)}$ and $\lambda = K$, we have

$$\lim_{n\to\infty} \sqrt[n]{x_n(t)} = \lambda.$$

The limit on the left-hand side is, in fact, equal to the spectral radius $\rho(T^{(\mathrm{Vol})})$ of $T^{(\mathrm{Vol})} := \lambda \cdot \mathrm{Id} + \Omega^{(\mathrm{Vol})}$ with respect to the sup-norm over the interval $[a, t]$. (This follows for example from Theorem 9.1 in [2].) Thus, we recognize the above equality as an instance of the following well known theorem: *if λ is a scalar and Ω is a linear operator with $\rho(\Omega) = 0$, then $\rho(\lambda \cdot Id + \Omega) = \lambda$.* One may think of Theorem 1 as an analog of the above result. Actually, for the spectral radius a more general statement holds : *if Λ and Ω are two commuting linear operators and $\rho(\Omega) = 0$, then $\rho(\Lambda + \Omega) = \rho(\Lambda)$.* This is an immediate consequence of sub-additivity of the spectral radius on commuting operators (see e.g. chapt. 2, equ. 5.2 in [2]).

Nevertheless, the analogous statement for the *quotient-convergence factors* is most likely not true. More specifically, there should be a linear space X, two commuting linear operators Λ and Ω, a linear functional ψ and $x \in X$, such that, for $x_n := (\Lambda + \Omega)^n x$, the quotients $\psi(\Omega^{k+1} x_n)/\psi(\Omega^k x_n)$ and $\psi(\Lambda^{k+1} x_n)/\psi(\Lambda^k x_n)$ converge uniformly in n to λ and 0 respectively, but $\psi(x_{n+1})/\psi(x_n)$'s does not converge to λ. The author does not know any such examples.

Section 1: Proof of Theorem 1.

Let us first argue that hypothesis (P) implies hypothesis (U). By the binomial formula, we have

$$x_n = (\lambda + \Omega)^n x_0 = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} \Omega^j x_0. \tag{5}$$

Inserting this formula into $\psi(\Omega^{k+1}(x_n))/\psi(\Omega^k(x_n))$ we get a quotient of two sums. Since all the terms are positive, we can dominate it by the maximum of quotients of corresponding terms i.e. $\max_{j=0}^{n}\{\psi(\Omega^{j+k+1} x_0)/\psi(\Omega^{j+k} x_0)\}$. Now, hypothesis (P) implies that these maxima converge to 0 as k tends to infinity — (U) follows.

In what follows we will prove the assertion of Theorem 1 under hypothesis (U). Note that we may assume that $\lambda = 1$. Indeed, otherwise one can force this condition by substitution $\tilde{x}_n := \lambda^{-n} \cdot x_n$. Set

$$x_{n,k} := \Omega^k x_n, \ k, n = 0, 1, 2, \ldots .$$

From (1) we have

$$x_{n+1,k} = x_{n,k} + x_{n,k+1}, \ k, n = 0, 1, 2, \dots . \tag{6}$$

So, to investigate the quotients $\psi(x_{n+1,k})/\psi(x_{n,k})$ we may represent them as $1 + \delta_{n,k}$, where

$$\delta_{n,k} := \psi(x_{n,k+1})/\psi(x_{n,k}).$$

We have to prove that $\delta_{n,0}$ converges to zero as $n \to \infty$.

Using (6) one can see that $\delta_{n,k}$'s satisfy the following recurrence equation

$$\delta_{n+1,k} = \delta_{n,k} \cdot \frac{1 + \delta_{n,k+1}}{1 + \delta_{n,k}}. \tag{7}$$

This kind of recurrence relation guarantees strong ties between the limit behavior of the two sequences involved. The following lemma, which we prove later, is critical for our argument.

Lemma 1 *If $a_n > -1$, $n = 1, 2, 3, \dots$ and reals z_n satisfy*

$$z_{n+1} = z_n \cdot \frac{1 + a_n}{1 + z_n}, \tag{8}$$

then $\min\{0, \liminf a_n\} \le \liminf z_n \le \limsup z_n \le \max\{0, \limsup a_n\}$.

Denote $\limsup_{n\to\infty} \delta_{n,k}$ and $\liminf_{n\to\infty} \delta_{n,k}$ by δ_k^+ and δ_k^- respectively. Assumption (U) of Theorem 1 guarantees that $\lim_{k\to\infty} \delta_k^+ = \lim_{k\to\infty} \delta_k^- = 0$. In particular, for sufficiently large k, $\delta_{k+1}^- > -1$, and we can use Lemma 1 with $a_n := \delta_{n,k+1}$ and $z_n := \delta_{n,k}$ to conclude that

$$\min\{\delta_{k+1}^-, 0\} \le \min\{\delta_k^-, 0\} \le \max\{\delta_k^+, 0\} \le \max\{\delta_{k+1}^+, 0\}.$$

Note that necessarily $\delta_k^- > -1$, so we can get analogous inequalities with k replaced by $k - 1$ by invoking Lemma 1 again with $a_n := \delta_{n,k}$, $z_n := \delta_{n,k-1}$. Repeating this k times leads to

$$\min\{\delta_k^-, 0\} \le \min\{\delta_0^-, 0\} \le \max\{\delta_0^+, 0\} \le \max\{\delta_k^+, 0\}.$$

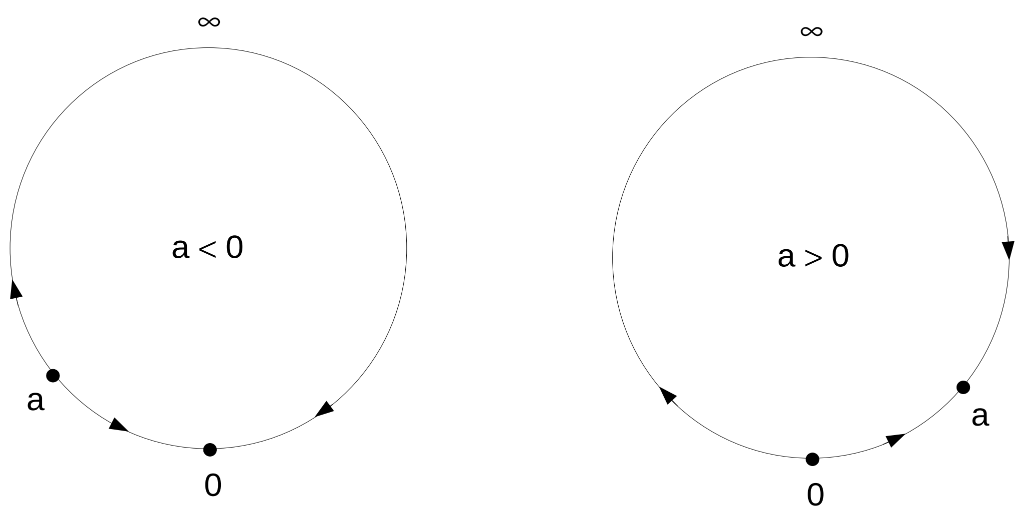
In the limit $k \to \infty$, we get $\delta_0^+ = \lim_{k\to\infty} \delta_k^+ = 0$ and $\delta_0^- = \lim_{k\to\infty} \delta_k^- = 0$. This ends the proof of Theorem 1. □

Proof of Lemma 1. First let us make a technical observation that, due to $a_n > -1$, we have $z_n > -1$ for large enough n. Indeed, if for some n_0 we have $z_{n_0} < -1$, then $z_{n_0+1} = z_{n_0} \cdot \frac{1+a_{n_0}}{1+z_{n_0}} \geq 1 + a_{n_0} > 0$. On the other hand, again from (8), we see that once $z_n > 0$, then $z_{n+k} > 0$ for $k = 1, 2, 3, \ldots$.

If z_n converged, then, by passing to the limit in (8), we would see that so does a_n and $\lim a_n = \lim z_n$ — we would be done.

Denote by I_n the interval with the endpoints $a_n, 0$ and let I be an arbitrary interval such that $\min I < \liminf \min I_n \leq \limsup \max I_n < \max I$. Clearly $I_n \subset I$ for large n. We would be done if we could see that almost all z_n sit in I. Let us assume for now the following claim.

Claim 1 *The following alternative holds for $z_n > -1$:*
(*) if $z_n \notin I_n$, then $z_{n+1} \leq z_n$;
(**) if $z_n \in I_n$, then $z_n \leq z_{n+1} \leq \max I_{n+1}$.



Action of iterates of f_a on the points of the circle of reals.

Observe that, if for some large n_0, $z_{n_0} \in I$, then either $z_{n_0+1} < \min I$ or $z_{n_0+1} \in I$. Indeed, otherwise $z_{n_0+1} \geq \min I$ and $z_{n_0+1} \notin I$, which means that $z_{n_0+1} > \max I$. This however is incompatabile with neither (*) nor (**) of Claim 1. Moreover, once $z_n < \min I$ then by (*) we have $z_{n+1} \leq z_n \leq \min I$, so, by induction, $z_n < \min I$ for almost all z_n's.

In this way, if z_n hits I and subsequently leaves it, then z_n never comes back to I again. Thus, either almost all z_n sit in I and we are done, or almost all z_n lie outside I. In the later case, by (*), we see that z_n is non-increasing, so it is convergent — and we are done as well. □

Proof of Claim 1. According to (8) we have $z_{n+1} = f_{a_n}(z_n)$, where $f_a(z) := (1+a)z/(1+z)$. For $a > -1$, this map has two fixed points $z = 0$ and $z = a$. If I is the interval with endpoints at these fixed points and $z > -1$, then $f_a(z) \geq z$ for $z \in I$, and $f_a(z) \leq z$ for $z \notin I$. Simple verification is left for the reader. □

The proof of Lemma 1 is technical. One can gain an intuitive picture of what is going on by thinking of z_n as lying on the circle of real numbers with ∞ that is acted upon by the Möbius transformations $f_a(z) := (1+a)z/(1+z)$, $a = a_n > -1$. Each of these transformations has dynamics depicted in the figure. Of course the fact that a_n's are not constant adds difficulty to understanding how points z_n move along the circle, but one thing which should be fairly clear (at least for small a_n's) is the following: as n increases, z_n either goes once around the circle and then must stay near $\max\{0, a_n\}$, or it persists near $\min\{0, a_n\}$. This is, very roughly, the idea of Lemma 1.

Let us indicate here that, if one is interested only in the case where a_n's and z_n's are positive (as encountered in hypothesis (ii) or [1]), then Lemma 1 has the following stronger analogue with a more esthetic proof.

Lemma 2 (positive case) *If $a_n > 0, n = 1, 2, 3, ...$ and $z_n > 0$ satisfy*

$$z_{n+1} = z_n \cdot \frac{1+a_n}{1+z_n}, \tag{9}$$

then $\liminf a_n \leq \liminf z_n \leq \limsup z_n \leq \limsup a_n$.

Proof of Lemma 2. Choose a_+, a_- arbitrarily so that $\limsup a_n < a_+$, $\liminf a_- > a_-$. If the sequence (z_n) had a limit, (9) would yield

$$\lim z_n \frac{1+\liminf a_n}{1+\lim z_n} \leq \lim z_n \leq \lim z_n \frac{1+\limsup a_n}{1+\lim z_n},$$

and we would be done by comparison of the numerators and denominators above. Dividing term by term in (9), we get

$$\min\{a_n, z_n\} \leq z_{n+1} \leq \max\{a_n, z_n\}.$$

Let us deal with lim sup-part of our lemma — for the lim inf-part the analogous argument works. If $z_n \le a_+$ for some large n, then $z_{n+1} \le \max\{a_n, z_n\} \le a_+$. Repeating this argument, we see that $z_{n+k} \le a_+$, $k = 1, 2, 3, ...$, so $\limsup z_n \le a_+$, as required. If $z_n \ge a_+$ for all large n, then in particular $z_n \ge a_n$ and, by the right inequality, we see that $z_{n+1} \le z_n$. Thus z_n is eventually non-increasing, so it has a limit — we are done. □

Section 2: Proof of Fact 1.

Since there is no danger of ambiguity, we will abbreviate $\Omega^{(\mathrm{Vol})}$ to Ω. Also, setting $a = 0$ and $b = \infty$ will simplify the notation without compromising generality. Let us first deal with (i). It is easy to see that all of $x_n(t)$ are positive and non-decreasing in t. Hence, we will be done by showing the following fact.

Fact 2 *If $x(t)$ is positive and non-decreasing for $t \ge 0$, then*

$$(\Omega^k x)(t)/(\Omega^{k-1}x)(t) \le \frac{t}{k}.$$

Proof of Fact 2. Define operators A_k by

$$(A_k x)(t) := k/t^k \cdot \int_0^t x(s)s^{k-1}ds,\ t \ge 0,\ k = 1, 2, 3, ...$$

Note that the right hand side above is an average. Thus we have

$$\sup_{s \le t} |(A_k x)(s)| \le \sup_{s \le t} |x(s)|. \tag{10}$$

In particular, A_k sends non-decreasing functions to non-decreasing ones. Also, a straight-forward verification confirms that

$$\Omega^k = \frac{t^k}{k!} \cdot A_k \circ A_{k-1} \circ ... \circ A_1. \tag{11}$$

In this way, we can write

$$(\Omega^k x)(t)/(\Omega^{k-1}x)(t) = \frac{t}{k}(A_k \circ A_{k-1} \circ ... \circ A_1 x)(t)/(A_{k-1} \circ ... \circ A_1 x)(t).$$

Inequality (10) yields $(A_k \circ A_{k-1} \circ ... \circ A_1 x)(t) \leq (A_{k-1} \circ ... \circ A_1 x)(t)$, and we conclude that

$$(\Omega^k x)(t)/(\Omega^{k-1}x)(t) \leq \frac{t}{k}. \ \square$$

We will deal with (ii) now. With no loss of generality we can assume that $x_0(0) > 0$. It is a routine to see that

$$(\Omega^k x)(t) = \frac{1}{(k-1)!}\int_0^t x(s)(t-s)^{k-1}ds = \frac{t^k}{k!}\cdot\frac{k}{t^k}\int_0^t x(s)(t-s)^{k-1}ds.$$

For fixed t, the kernels $\frac{k}{t^k}(t-s)^k$, $s \in [0,t]$, approximate the delta-mass distribution concentrated at 0, so

$$\lim_{k\to\infty}(\Omega^k x_0)(t)/\left(\frac{t^k}{k!}\cdot x_0(0)\right) = 1. \tag{12}$$

Hence, we have

$$\lim_{k\to\infty}(\Omega^{k+1}x_0)(t)/(\Omega^k x_0)(t) = \lim_{k\to\infty}\left(\frac{t^{k+1}}{(k+1)!}\cdot x_0(0)\right)/\left(\frac{t^k}{k!}\cdot x_0(0)\right) = 0. \tag{13}$$

Now, using binomial formula (5), we can write

$$(\Omega^{k+1}x_n)(t)/(\Omega^k x_n)(t) = \frac{\sum_{j=0}^n \lambda^{n-j}\begin{pmatrix} n \\ j\end{pmatrix}(\Omega^{j+k+1}x_0)(t)}{\sum_{j=0}^n \lambda^{n-j}\begin{pmatrix} n \\ j\end{pmatrix}(\Omega^{j+k}x_0)(t)}.$$

Note that, due to (12), for large enough k, all terms in the numerator and the denominator are positive. This makes it possible to divide term by term to get the following estimates

$$\min_{j=0}^{n}\left\{\frac{(\Omega^{j+k+1}x_0)(t)}{(\Omega^{j+k}x_0)(t)}\right\} \leq (\Omega^{k+1}x_n)(t)/(\Omega^k x_n)(t) \leq \max_{j=0}^{n}\left\{\frac{(\Omega^{j+k+1}x_0)(t)}{(\Omega^{j+k}x_0)(t)}\right\}.$$

It follows by (13) that $(\Omega^{k+1}x_n)(t)/(\Omega^k x_n)(t)$ converges to 0 uniformly in n, as $k \to \infty$. $\square$

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