Asymptotic Entropy, Periodic orbits, and Pseudo-Anosov Maps

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Abstract

In this paper we derive some properties of a variety of entropy that measures rotational complexity of annulus homeomorphisms, called asymptotic or rotational entropy (as it is sometimes called). In previous work ([KS]) the authors showed that positive asymptotic entropy implies the existence of infinitely many periodic orbits corresponding to an interval of rotation numbers. In our main result, we show that a Hölder C^1 diffeomorphism with nonvanishing asymptotic entropy is isotopic rel a finite set to a pseudo-Anosov map. We also prove that the closure of the set of recurrent points supports positive asymptotic entropy for a (C^0) homeomorphism with nonzero asymptotic entropy.

1 Introduction, definitions and main results

Some horseshoes are associated with trivial rotation sets, notwithstanding their complicated dynamics. Others have rotation intervals of positive length, and are sometimes called rotary horseshoes. Handel [Ha] found a deeper relation between complicated dynamics and rotation. When rotation numbers of periodic orbits vary discontinuously in a certain sense, one can infer the existence of pseudo-Anosov orbits: the original annulus map is pseudo-Anosov rel a finite invariant set. Maps of the annulus that are pseudo-Anosov rel a finite set have a nontrivial rotation interval. In this article we build on Handel's lemma to conclude the existence of pseudo-Anosov behavior, in the Hölder

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smooth case, from nonvanishing asymptotic entropy. More rigorously, we show there is a finite invariant set F such that the original map is homotopic rel F to a pseudo-Anosov map. Because asymptotic entropy was formulated independently by each of the authors, it has been called "rotational entropy" elsewhere ([Sw]). The switch to the present notation not only makes for consistent terminology but separates our formulation from earlier less satisfactory versions (e.g. [Bot]).

We should caution the reader, who may be familiar with the work of A. Katok, that this result does not immediately follow from Katok's theorem that nonvanishing topological entropy for Hölder smooth maps implies the existence of transverse homoclinic points ([Ka]). Indeed, we do not know if positive asymptotic entropy of a smooth map implies the existence of rotary homoclinic points, although we can find a semiconjugacy from such a map to a map with rotary horseshoes. However, in [Ka] it is proved that hyperbolic measures are supported in the closure of the periodic points, and that fact will prove to be critical. Our eventual tack is to exhibit an infinite sequence of hyperbolic measures whose supports are associated with asymptotic entropy and manifest positive ϵ topological entropy, with ϵ arbitrarily large. This depends on the scale sensitivity of asymptotic entropy.

The assumption of smoothness may be unnecessary, but we cannot prove this yet. For the $\mathcal{C}^0$ case, we have proved that nonvanishing asymptotic entropy forces the existence of infinitely many periodic points corresponding to an interval of rotation numbers ([KS]). This is certainly suggestive of a stronger result.

Notation. For a set A we will use A^c to denote its complement. For the cardinality of A, we write card$A$. The r-neighborhood of a set is $B_r(A) := \{x : dist(x, A) < r\}$, and $Conv(A)$ is the convex hull of A. What metric or affine structure is used will be always clear from the context. The sets involved are always assumed to be Borel measurable. Given two families of sets $\mathcal{A}, \mathcal{B}$ we denote by $\mathcal{A} \vee \mathcal{B}$ the join $\{A \cap B : A \cap B \neq \emptyset,\ A \in \mathcal{A}, B \in \mathcal{B}\}$. Note that $\mathrm{card}(\mathcal{A} \vee \mathcal{B}) \leq \mathrm{card}\mathcal{A} \cdot \mathrm{card}\mathcal{B}$. We will, by abuse of language, call the supremum of diameters of sets in a given family $\mathcal{A}$, *diameter of* $\mathcal{A}$, abbreviated to $diam(\mathcal{A})$.

Let $\mathbb{A}$ be the annulus $\mathbf{S} \times [0,1]$ i.e. the quotient of $\widetilde{\mathbb{A}} = \mathbb{R} \times [0,1]$ under the action of $\mathbb{Z}$ by integer translation along $\mathbb{R}$. We will refer to the quotient map as $\pi : \widetilde{\mathbb{A}} \to \mathbb{A}$. On the strip $\widetilde{\mathbb{A}}$ we have the Euclidean metric $\tilde{d}$. There is a unique corresponding metric d on $\mathbb{A}$.

From now on it will be assumed that we are given a homeomorphism $f : \mathbb{A} \to \mathbb{A}$, isotopic to the identity transformation. We also fix a lift $F : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ of f. This lift determines the *displacement function* of F, $\phi_F : \mathbb{A} \to \mathbb{R}^2$, obtained as the factor of $(j \circ F - j \circ id) : \tilde{\mathbb{A}} \to \mathbb{R}^2$ where j is the embedding of $\tilde{\mathbb{A}}$ into $\mathbb{R}^2$.

Definition 1.1 *For $n \in \mathbb{N}$. Define $\tilde{d}_n$ a metric on $\tilde{\mathbb{A}}$ by the formula:*

$$\tilde{d}_n(x, y) := \max\{\tilde{d}(F^i(x), F^i(y)) : 0 \leq i \leq n-1\}.$$

Define d_n a metric on $\mathbb{A}$ by the formula:

$$d_n(x, y) := \max\{d(f^i(x), f^i(y)) : 0 \leq i \leq n-1\}.$$

We will say that a set E is $(\tilde{d}_n, R)$-*separated* (or (d_n, R)-*separated*) if the distance between any distinct points in E is at least R with respect to $\tilde{d}_n$ metric (or with respect to d_n correspondingly).

Definition 1.2 *For a subset X of $\mathbb{A}$ and $R > 0$ we define :*

$$\tilde{s}_X(d_n, R) = max\ card\ of\ (\tilde{d}_n, R)\text{-}separated\ subset\ of\ \pi^{-1}(X) \cap [0,1]^2;$$

$$s_X(d_n, R) = max\ card\ of\ (d_n, R)\text{-}separated\ subset\ of\ X;$$

and the two corresponding entropies at scale R:

$$\tilde{h}_X(R) := \limsup_{n\to\infty} \frac{1}{n} \log(\tilde{s}_X(d_n, R));$$

$$h_X(R) := \limsup_{n\to\infty} \frac{1}{n} \log(s_X(d_n, R)).$$

Definition 1.3 *For a subset X of $\mathbb{A}$ topological entropy h_X^{top} is defined as*

$$h_X^{top} = \lim_{R\to 0} h_X(R)$$

and asymptotic entropy as

$$h_X^{as} = \limsup_{R\to\infty} R \cdot \tilde{h}_X(R).$$

Remark 1.1 *Both entropies are independent of the choice of lift F and of fundamental domain. However, asymptotic entropy depends on the lift metric, much like rotation numbers. The condition $h_X^{as} > 0$ does not depend on the metric, and that is the key property studied in this paper. The reader may be more accustomed to defining entropy using minimal spanning sets, which is equivalent for topological entropy but generates a sometimes smaller, but still commensurate, invariant for asymptotic entropy.*

Our goal in the next section, which deals with annulus diffeomorphisms, is to prove the following result:

Theorem 1.1 *Let f be a diffeomorphism of the annulus with Hölder continuous derivative which is isotopic to the identity. Nonvanishing of the asymptotic entropy for f implies that f satisfies the pA-hypothesis [Handel [Ha]].*

Our version of Handel's pA-hypothesis (defined below) asserts that some map restriction of f is semiconjugate to a pseudo-Anosov map on the finitely punctured annulus. Whether this, when f is smooth, implies the existence of a smoothly transversal rotary homoclinic point is unknown to the authors. Thus, this theorem is not the exact analogue of Katok's theorem [Ka], but captures the idea of a rotary horseshoe quite closely.

In the last section we prove some theorems for annulus homeomorphisms without additional smoothness requirements. The main result (Theorem 3.1) is the following:

Theorem 1.2 *The nonvanishing of asymptotic entropy implies the existence of a chain transitive component with nontrivial rotation.*

Corollary 1.1 *Nonvanishing of asymptotic entropy implies the existence of infinitely many periodic points corresponding to an interval of rotation.*

The corollary follows from a theorem of J. Franks ([Fr]) and suggests that Hölder smoothness may be superfluous in the results of the next section. One of the implications of the technique of the next section is that asymptotic entropy for homeomorphisms — like topological entropy — is zero if it is zero on the closure of the recurrent points. Ultimately, one would like a variational theorem for asymptotic entropy, but that has proved to be elusive so far because of the global character of this invariant.

2 Hölder smoothness and a rotational version of Katok's theorem

Our main result (Theorem 1.1) is a consequence of the following two theorems.

Theorem 2.1 *[Theorem A.] Suppose that f is an annulus homeomorphism isotopic to the identity. If $h^{as}_{cl(\mathcal{P})} > 0$, then f satisfies the pA-hypothesis.*

Theorem 2.2 *[Theorem B.] If f is a $C^{1+\epsilon}$-smooth annulus diffeomorphism then*

$$h^{as}_{cl(\mathcal{P})} \geq \frac{1}{2} \cdot h^{as}.$$

Proof of Theorem A.

Denote by $\phi_F^{(k)}$ the displacement function for the k-th iterate of f. In other words,

$$\phi_F^{(k)}(x) := F^k(\tilde{x}) - \tilde{x} = \sum_{j=0}^{k-1} \phi_F(f^j(x))$$

where $\tilde{x}$ is any lift of $x \in \mathbb{A}$.

Definition 2.1 *Let $\mathcal{M}$ be the space of all invariant Borel probability measures of f. For $\mu \in \mathcal{M}$ we define its* rotation number

$$\rho_\mu := \int \phi_F \, d\mu.$$

Given $X \subset \mathbb{A}$ we define its rotation set ρ_X

$$\rho_X := \bigcap_{n \in \mathbb{N}} cl \bigcup_{k > n} \left\{ \frac{\phi_F^{(k)}(x)}{k} : x \in X \right\}.$$

and its measure theoretic rotation set ρ_X^{meas}

$$\rho_X^{meas} := \{\rho_\mu : \ \mu \in \mathcal{M}, \ supp(\mu) \subset X\};$$

Note that we do not require above that X is invariant. However, we have the following consequence of the Birkhoff theorem ([MZ]).

Fact 2.1 *If $X \subset \mathbb{A}$ is compact and invariant, then*

$$\rho_X^{meas} = Conv(\rho_X).$$

Lemma 2.1 *Let X be a subset of $\mathbb{A}$. Fix arbitrary $\epsilon > 0$.*

(i) *For any $R > 0$ there exists $Y \subset X$ with $diam(Y) < \epsilon$ and $\tilde{h}_Y(R) = \tilde{h}_X(R)$.*

(ii) *There exists $Y \subset X$ with $diam(Y) < \epsilon$ and $\tilde{h}_Y^{as} = \tilde{h}_X^{as}$.*

Proof. Part (i). Take a partition of X into sets of diameter less than ϵ and observe that (much like topological entropy) the asymptotic entropy on a finite union of compact sets is less than or equal to the maximum of the entropies of these sets. Part (ii) follows from (i) applied to an increasing sequence of R's. □

The following definition is (using his result) adapted from Handel [Ha].

Definition 2.2 *We will say that $f : \mathbb{A} \to \mathbb{A}$ satisfies the pA-hypothesis if for some $n > 0$ there exists an f^n-invariant finite set K and a homeomorphism $g : \mathbb{A} \to \mathbb{A}$ such that :*

(i) *g is pseudo Anosov relative to K;*

(ii) *g and f^n are homotopic rel K;*

(iii) *the rotation set of g is not a point.*

Remark 2.1 *In fact, condition* **(iii)** *is not independent and follows from* **(i)** *and* **(ii)**. *For details the reader can consult D. Fried's paper ([Fd], Theorem H) or, perhaps more in the spirit of our results, see P. Boyland ([Bd], Theorem 11.1))*

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Definition 2.3 *Suppose that we are given a sequence of f-periodic orbits $(P_n)_{n\in\mathbb{N}}$. We call (P_n)* coherent *iff*

(i) *the sequence of compact sets P_n converges to a compact set K in the Hausdorff topology;*

(ii) *the sequence of rotation numbers ρ_{P_n} converges to a value v.*

We will simplify **(i)** *and* **(ii)** *by writing $P_n \to K$ and $\rho_{P_n} \to v$.*

The next result is simply a restatement of a result in [Ha].

Lemma 2.2 (Handel's Lemma) *Suppose that* $(P_n)_{n\in\mathbb{N}}$ *is a coherent sequence of* f*-periodic orbits with* $P_n \to K$ *and* $\rho_{P_n} \to v$*. If* $\rho_K^{meas} = Conv(\rho_K) \neq \{v\}$*, then the pA-hypothesis is satisfied.*

The following lemma implies that the pA-hypothesis is untrue iff rotation numbers vary (uniformly) continuously on periodic points. The advantage is that one need not work with measures at all.

Lemma 2.3 *Let* $cl(\mathcal{P})$ *denote the closure of the set of periodic points of* f*. Then the rotation number map* $x \mapsto \rho(f,x) = \lim_{k\to\infty} \frac{1}{k}\,\phi_F^k(x)$ *is well-defined and continuous on the set* $cl(\mathcal{P})$ *iff whenever* $(P_n)_{n\in\mathbb{N}}$ *is a coherent sequence of* f*-periodic orbits with* $P_n \to K$ *and* $\rho_{P_n} \to v$*, then* $\rho_K^{meas} = Conv(\rho_K) = \{v\}$*.*

Proof:(if part) We will prove that the rotation number is uniformly continuous on the set of periodic points. We will rule out the possibility that $d(x_n, y_n) \to 0$ but $|\rho(f,x_n) - \rho(f,y_n)| > \epsilon$ for some sequences of periodic points $(x_n), (y_n)$. If y_n is in the orbit P_n, we can suppose $P_n \to K$ and $\rho_{P_n} \to \{v\} = \rho_K$, by passing to a subsequence and reindexing. Some subsequence of (x_n) converges to a point of K, so the corresponding orbit Q_n converges to a subset of K, since K is invariant, and $\rho_{Q_n} \to \{v\}$. It follows that the rotation map is well-defined everywhere on the periodic point closure and is the unique (continuous) extension. The "only if" direction is straight forward. □

The following result reveals a pleasant and, perhaps, unexpected difference between asymptotic and topological entropy, and we will take advantage of it.

Lemma 2.4 *Let* X *be a compact subset of* $\mathbb{A}$*. Then the following bound holds:*

$$h_X^{as} \leq \log 2 \cdot diam(\rho(f,X)).$$

Proof: Let $\{F^n(x) = x(n)\}$ denote a positive trajectory of the the lift F, and pr_1 is projection onto the first coordinate. Choose $\rho^+, \rho^- \in \mathbb{R}$ so that $\rho(f,X) \subset (\rho^-, \rho^+)$. Note that we get, for sufficiently high n :

$$(*) \qquad \rho^- \leq pr_1(x(n) - x(0))/n \leq \rho^+.$$

Therefore, there is a constant $C > 0$ such that for all $n \geq 1$

$$(**) \qquad \rho^- - C/n \leq pr_1(x(n) - x(0))/n \leq \rho^+ + C/n.$$

In fact, by the compactness of the fundamental domain and periodicity of $F^n - Id$, C can be chosen independently of $x(0)$.

Let $R > 0$ be as in the definition of asymptotic entropy.

Define

$$m = R/(\rho^+ - \rho^-).$$

If m is not already an integer, increase ρ^- and decrease ρ^+ so that m is now an integer and $(*)$ is true for $x(m)$. We may have to increase R, but R is independent of $\rho^\pm$ except for having to be sufficiently large. This may also increase C in $(**)$. Now fix those values of m, R, and C.

Let D be an R-box; i.e., $D := pr_1^{-1}(I)$ where I is an arbitrary interval of length R. Let $x(i), y(i)$ be two trajectories in the universal cover both originating in D.

Claim 2.1 *For the above choice of m, the following holds:*

(1) both $x(m)$ and $y(m)$ are in one of the two boxes $D+m\rho^+$ or $D+m\rho^-$;

(2) For $n=0,1,2,...,m$, $d(x(n),y(n)) \leq R+2C+2$.

Proof of Claim. (Part 1) From $(*)$ we get $\inf I + m\rho^- \leq pr_1(x(m)) \leq \sup I + m\rho^+$. That is, $x(m)$ lies in a region swept out as we slide $D+m\rho^-$ to $D+m\rho^+$. But by the choice of m there is no gap between the two. □(Part 2)
We consider the case when $x(m) \in D+m\rho^+$. The other case is similar. Apply $(**)$ to the trajectory $x(i)$ twice — once starting from $x(0)$ and the second time starting from $x(m)$ and running back in time. We get:

$$pr_1(x(n)) \leq pr_1(x(0)) + n\rho^+ + C;$$

$$pr_1(x(n)) \geq pr_1(x(m)) - (m-n)\rho^+ - C.$$

Since $x(0) \in D$ and $x(m) \in D+m\rho^+$ we see that:

$$\inf I + n\rho^+ - C \leq pr_1(x(n)) \leq \sup I + n\rho^+ + C.$$

Of course we have an analogous inequality for $y(n)$. "Subtracting" the two, yields $pr_1(x(n)-y(n)) \leq R+2C$. Our claim follows. □

The claim insures that we can match to each trajectory $x(n)$ a sequence of signs $\sigma_i = \pm$ so that $x(i \cdot m) \in D + m\rho^{\sigma_1} + ... + m\rho^{\sigma_i}$. Also this correspondence is injective on $(R+2C+2)$-separated trajectories by (2). Trivial counting implies that $\tilde{h}(R+2C+2) \leq \log 2/m$. The Lemma follows.

□

Thus, compact sets with positive asymptotic entropy support rotation sets of positive diameter.

Proof of Theorem A(2.1):

We proceed by contradiction. We show that, otherwise, there are invariant sets with small diameters, small rotation sets and yet have full asymptotic entropy. Fixing $\delta > 0$, we may, by the uniform continuity of ρ and compactness of the rotation set, write $\mathrm{cl}\,\mathcal{P}$ as a finite union of compact sets of the form $\mathrm{cl}(\mathcal{P}) \cap \rho^{-1}[r_i - \epsilon, r_i + \epsilon]$ of diameter less than δ. One of those sets must support full asymptotic entropy. This contradicts the preceding bound, if ϵ is chosen so that $2\epsilon \cdot M$ is less than the asymptotic entropy of f restricted to $\mathrm{cl}(\mathcal{P})$. □

2.1 Proof of Theorem B

From now on we will be assuming that f is a diffeomorphism with Hölder continuous derivative, and F is any of its lifts to the universal cover.

Recall that we denote by $\mathcal{M}$ the collection of all invariant Borel probability measures for f. Denote also by $\mathcal{E}$ all those ergodic ones.

Definition 2.4 *Suppose that we are given a probability invariant measure $\mu \in \mathcal{M}$ with ergodic decomposition ([Wa])*

$$\mu = \int_{\mathcal{E}} m \; d\tau(m).$$

Define the Lyapunov exponents

$$\chi(x) = \limsup_n (1/n) \log ||Df^n_x||, \; \text{and } \chi_\mu = \int_X \chi(x) \, d\mu(x).$$

We will say that the measure $\mu \in \mathcal{M}$ is hyperbolic iff for τ-almost all m in $\mathcal{E}$ the Lyapunov exponent χ_m is not zero. We will also denote the metric entropy of μ by h_μ ([Wa]).

Remark 2.2 *The definitions of the Lyapunov exponents and hyperbolic measure given here are for surfaces, the setting of this paper. In higher dimensions, one needs to stipulate that all Lyapunov exponents are nonzero and that there exist exponents of opposite signs.*

We will need the following powerful consequence of Pesin theory (see [Pe]) due to A. Katok [Ka]. Again, we assume the given diffeomorphism is Hölder smooth.

Theorem 2.3 *The support of a hyperbolic invariant measure is contained in the closure of the periodic points $cl(\mathcal{P})$ of f; in fact, it is contained in the closure of the subset of periodic points with associated transverse homoclinic intersections.*

Remark 2.3 *In [Ka] (Theorem 4.1) hyperbolic measures are required to be ergodic, but our definition coupled with the preceding theorem leads to the same conclusion, for if the supports of almost all the ergodic components of μ are in the periodic closure, then so too is the support of μ.*

Recall also the Margulis-Pesin-Ruelle inequality:

$$h_\mu \leq \chi_\mu \tag{1}$$

valid for invariant Borel probability measures $\mu \in \mathcal{M}$. In particular, any ergodic measure with positive metric entropy is hyperbolic.

Taking into account Theorem 2.3, it is easy to see that Theorem B(2.2) is the consequence of the following proposition, and the remainder of the paper will be devoted to its proof:

Proposition 2.1 (Main Proposition) *Suppose that $R > 0$ is given. If $\tilde{h}(R) > 0$ then there exists an f-invariant hyperbolic measure $\nu \in \mathcal{M}$ such that*

$$\tilde{h}_{supp(\nu)}(R/2) \geq \tilde{h}(R).$$

Anti-Renormalization.

Topological entropy is immune to changes of scale (in the lift), which is not the case for rotation numbers or, as a consequence, asymptotic entropy. We want to take advantage of this lack of scale invariance which we call "anti-renormalization". Let us first prove that, in the context of the preceding proposition, we can always assume that R satisfies the estimate

$$(2) \qquad 1 > 2(\sup \|\phi_F\| + R).$$

Otherwise we pick $M \in \mathbb{N}$ such that $M > 2(\sup \|\phi_F\| + R)$ and consider an M-cyclic cover of f i.e. the factorization of $F : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ by the action of the subgroup $M \cdot \mathbb{Z} \subset \mathbb{Z}$. Up on rescaling the metric by the factor M^{-1}, we can consider this map as acting on $\mathbb{A}$. Call it g. Hypothesis 2 holds for g if R is replaced by $R' := M^{-1}R$.

Moreover the Proposition 2.1 for g and R' implies 2.1 for f and R, as well. In fact, if we find a hyperbolic g-invariant measure μ with $\tilde{h}_{supp(\mu)}(g, M^{-1}R/2) \geq \tilde{h}(g, M^{-1}R)$ we can push μ forward by the covering map to an f-invariant ν. Pushing forward by a local diffeomorphism preserves ergodicity and Lyapunov exponents. Thus ν is hyperbolic. Also, since at the level of the universal cover f and g differ only by rescaling the metric by the factor M, it follows that $\tilde{h}(f, R) = \tilde{h}(g, M^{-1}R)$ and $\tilde{h}_{supp(\nu)}(f, R/2) = \tilde{h}_{supp(\mu)}(g, M^{-1}R/2)$. Thus ν is the measure we were looking for.

The advantage of working under assumption (2) is that we can deal with $h_X(R)$ which is defined on the compact $\mathbb{A}$ unlike $\tilde{h}_X(R)$ (defined on $\widetilde{\mathbb{A}}$). Indeed, we have the following useful fact:

Fact 2.2 (the reason for anti-renormalization) *If 2 holds; that is,*

$$1 > 2(\sup \|\phi_F\| + R),$$

then

$$\tilde{h}_X(R) = h_X(R).$$

Proof. Due to Lemma 2.1 we can assume that $diam(X) < R$. There is also no loss of generality in assuming that $diam(\pi^{-1}(X) \cap [0,1]^2) < R$ (otherwise move the fundamental domain $[0,1]^2$ suitably). Our claim about entropies follows as soon as we prove that if $\tilde{x}, \tilde{y} \in \pi^{-1}(X) \cap [0,1]^2$ is a pair of points with $\tilde{d}_n(\tilde{x}, \tilde{y}) \geq R$, then also $d_n(x, y) \geq R$ for $x := \pi(\tilde{x}), y := \pi(\tilde{y})$. To see that the distances really behave as stated, we can check what happens otherwise. We would have $d(f^j(x), f^j(y)) < R, j = 0, ..., n-1$, but $\tilde{d}(F^i(\tilde{x}), F^i(\tilde{y})) \geq R$ for some $i \in \{0, ..., n-1\}$. Assume that i is minimal with respect to this property. Observe that $i \geq 1$, because $diam(\pi^{-1}(X) \cap [0,1]^2) < R$. Since $d(f^i(x), f^i(y)) < R$, one can find $k \in \mathbb{Z}$ so that $\tilde{d}(F^i(\tilde{x}), F^i(\tilde{y}) + k) < R$. Obviously $k \neq 0$. We have the following inequalities

$$\begin{aligned} 1 \leq \tilde{d}(F^i(\tilde{y}), F^i(\tilde{y}) + k) &\leq \tilde{d}(F^i(\tilde{y}), F^{i-1}(\tilde{y})) + \tilde{d}(F^{i-1}(\tilde{y}), F^{i-1}(\tilde{x})) + \\ &\quad \tilde{d}(F^{i-1}(\tilde{x}), F^i(\tilde{x})) + \tilde{d}(F^i(\tilde{x}), F^i(\tilde{y}) + k) \leq \\ &\quad \sup \|\phi_F\| + R + \sup \|\phi_F\| + R; \end{aligned}$$

which contradicts $(*)$. $\square$

Continuing the Proof of the Main Proposition 2.1.

Since $h(R) > 0$ we have a sequence (n_k) tending to infinity and (d_{n_k}, R)-separated subsets S_{n_k} of $\mathbb{A}$ with

$$\lim_{k\to\infty} \frac{1}{n_k} \log(\mathrm{card} S_{n_k}) = h(R).$$

Set $E_{n_k} := S_{n_k} \cup ... \cup f^{n_k - 1}(S_{n_k})$ and denote by $\delta_{E_{n_k}}$ the uniformly distributed probability measures carried by the sets E_{n_k}. Passing perhaps to a subsequence of the sequence (n_k), we can assume that the measures $\delta_{E_{n_k}}$ converge in the *weak**-topology as $k \to \infty$. Call the limit measure μ.

From Misiurewicz's proof of the Variational Principle; i.e. $h^{top} = \sup_\mu h_\mu$ (see e.g., pp. 189-190 in [Wa]), we know that $h_\mu > 0$. However μ need not be hyperbolic. Also it is not clear to the authors whether one can expect that $h_{supp(\mu)}(R) = h(R)$. What we are going to prove instead is a stronger version of inequality $h_{supp(\mu)}(R/2) \geq h(R)$ as in the next lemma.

Lemma 2.5 (First Main Lemma.) *For any* $0 < \eta < h(R)$ *there exists* $\tau > 0$ *such that the following property holds:*

Hypothesis 2.1 *If the compact set* Λ *satisfies* $\mu(\Lambda) > 1 - \tau$, *then*

$$\frac{1}{q} \log(s_\Lambda(d_q, R/2)) \geq h(R) - \eta$$

for all $q \in \mathbb{N}$.

Corollary 2.1 *We have :*

$$h_{supp(\mu)}(R/2) \geq h(R).$$

Let us postpone the proof of the First Main Lemma and finish the argument proving the Main Proposition 2.1. If μ is hyperbolic, there is nothing left to do except for setting $\nu := \mu$ which ends the proof. Thus, from now on we will assume that this is not the case. This means that as we look at the ergodic decomposition of μ

$$\mu := \int_{\mathcal{E}} m \; d\tau(m),$$

for a positive τ-measure set of ergodic measures m we may have $\chi_m = 0$ which forces $h_m = 0$. However, the τ-measure of this set cannot be 1, since (as in [Wa])

$$\int_{\mathcal{E}} h_m \; d\tau(m) = h_\mu > 0.$$

Thus if we set $\mathcal{H} := \{m \in \mathcal{E} : \chi_m > 0\}$ the following formulas define invariant probability measures:

$$\nu := \frac{1}{\tau(\mathcal{H})} \cdot \int_{\mathcal{H}} m \; d\tau(m);$$

$$\nu_0 := \frac{1}{\tau(\mathcal{E}-\mathcal{H})} \cdot \int_{\mathcal{E}-\mathcal{H}} m \; d\tau(m).$$

Clearly, μ is a nontrivial convex combination of ν and ν_0. Moreover, ν is hyperbolic and ν_0 has zero metric entropy ($h_{\nu_0} = 0$).

Vanishing of the metric entropy for a measure does not imply that there is no topological entropy carried on its support. However, the following lemma holds.

Lemma 2.6 (Second Main Lemma.) *Suppose that an invariant probability measure $\nu_0 \in \mathcal{M}$ satisfies $h_{\nu_0} = 0$. Then for any $r, \kappa > 0$ there exists a compact set L with $\mu(L) \geq 1 - \kappa$ such that there are arbitrarily large values of q for which the set L can be covered by $\exp(\kappa q)$ sets of d_q-diameter less than r.*

Before proving this lemma, we will complete, with its aid, the proof of the Main Proposition:

Conclusion of the proof of the main proposition

Take an arbitrary $\eta > 0$ with $\eta < h(R)$. Since the measures ν and ν_0 are mutually singular, we have two disjoint Borel sets $\Sigma \subset supp(\nu)$ and $\Sigma_0 \subset supp(\nu_0)$ such that $\nu(\Sigma) = \nu_0(\Sigma_0) = 1$ and $\nu(\Sigma_0) = \nu_0(\Sigma) = 0$. Let $\tau > 0$ be as in the First Main Lemma. Set $\kappa := \min\{\tau/2, \eta, (h(R) - \eta)/2\}$ and $r := R/2$ and let L be as in Second Main Lemma. Choose a compact set $M_0 \subset L \cap \Sigma_0$ such that $\nu_0(M_0) > 1 - \tau$, and a compact set $M \subset \Sigma$ with $\nu(M) > 1 - \tau$. Clearly $\mu(M \cup M_0) > 1 - \tau$. Thus, by the First Main Lemma we have for any $q \in \mathbb{N}$ a $(d_q, R/2)$-separated set in $\Lambda := M \cup M_0$ with at least $\exp((h(R) - \eta)q)$ points. However according to the Second Main Lemma, for certain arbitrarily large q only $\exp(\kappa q)$ of these points may belong to M_0 (for no two can sit in the same element of the covering). This means that there exists a sequence (q_k), with $\lim_k q_k = \infty$, such that $M \subset supp(\nu)$ contains a $(d_{q_k}, R/2)$-separated set consisting of at least $\exp((h(R) - \eta)q_k) - \exp(\kappa q_k)$ points. Taking logarithms, dividing by q_k and letting k tend to infinity, we finally conclude that

$$h_{supp(\nu)}(R/2) \geq h(R) - \eta.$$

Since η was arbitrary, Proposition 2.1 is proved. □

At this point to complete the proof of Theorem B(2.2) we have to establish the First Main Lemma and the Second Main Lemma.

The proof of the Second Main Lemma relies on the following fact.

Lemma 2.7 *Suppose that the invariant probability measure $\nu_0 \in \mathcal{M}$ is such that $h_{\nu_0} = 0$. Let ξ be a finite measurable partition of $\mathbb{A}$. There exists a sequence (δ_n), $0 < \delta_n \to 0$ as $n \to \infty$ and compact sets L_n , $\nu_0(L_n) > 1 - 2\sqrt{\delta_n}$ such that for every $n \in N$ the set L_n can be covered by $\exp(\sqrt{\delta_n} \cdot n)$ sets of d_n-diameter less than $diam(\xi)$.*

Proof:(of 2.7) Fix n. We will write ξ^n for the partition $\xi \vee f^{-1}\xi \vee ... \vee f^{-n+1}\xi$. By $\xi^n(x)$ we mean the element of ξ^n containing x. Define:

$$\delta_n := \frac{1}{n} H(\xi^n) = \frac{1}{n} \int -\log(\nu_0(\xi^n(x))d\nu_0(x).$$

Recall that by definition $\frac{1}{n}H(\xi^n) \to h_{\nu_0}(\xi) = 0$ as $n \to \infty$. By Markov's inequality we have

$$\nu_0\{x : -\log \nu_0(\xi^n(x)) \geq n\sqrt{\delta_n}\} \leq \frac{\int -\log(\nu_0(\xi^n(x))\, d\nu_0(x)}{n\sqrt{\delta_n}} = \sqrt{\delta_n}.$$

Thus there exists a compact subset L_n of $\{x : \nu_0(\xi^n(x)) > \exp(-n\sqrt{\delta_n})\}$ that satisfies $\nu_0(L_n) \geq 1 - 2\sqrt{\delta_n}$. By the definition all elements of ξ^n intersecting L_n have measure at least $\exp(-n\sqrt{\delta_n})$ therefore there can be at most $\exp(n\sqrt{\delta_n})$ of them. They cover L_n and of course they have d_n-diameter not exceeding $diam(\xi)$. □

Proof of Second Main Lemma.

We are given $r, \kappa > 0$ arbitrary. Choose for ξ any finite partition of $\mathbb{A}$ with $diam(\xi) < r$. Let (δ_n) and L_n be as in Lemma 2.7 for $n \in \mathbb{N}$. We can take a sequence (n_k) so that $\sum_k 2\sqrt{\delta_{n_k}} < \kappa$ and $\max\{\sqrt{\delta_{n_k}}\} < \kappa$. It is easy to see that the set L defined as $L := \bigcap_k L_{n_k}$ satisfies the required property.(Actually any n_k can be taken for q.) □

Proof of the First Main Lemma.

Lemma 2.8 (Technical version of First Main Lemma.) *For any* $0 < \eta < h(R)$ *there exists* $\tau > 0$ *such that the following property holds for all* $q \in N$*:*

Hypothesis 2.2 *If* U *is open with* $\mu(\partial U) = 0$ *and* $\mu(U) > 1-\tau$*, then any open covering of* U *with sets of* d_q*-diameter not exceeding* R *has at least* $\exp((h(R)-\eta)q)$ *elements.*

Proof of First Main Lemma from its technical version.

Set $U_r := B_r(\Lambda)$, $r > 0$. Clearly $\mu(U_r) > 1 - \tau$ for all $r > 0$. Moreover there exists a sequence $r_n \to 0$ such that $\mu(\partial U_{r_n}) = 0$. Every set U_{r_n} satisfies Hypothesis 2.2. For every $n \in \mathbb{N}$ choose $Q_n \subset U_{r_n}$ a maximal $(d_q, R/2)$-separated subset of U_{r_n}. By maximality, open d_q-balls of radius $R/2$ with centers in the points of Q_n form a cover of U_{r_n} with d_q-diameter R. Thus we have an estimate

$$\text{card } Q_n \geq \exp((h(R)-\eta)q).$$

Consider a set Q which is a Hausdorff limit of a subsequence of compact sets Q_n. It is easy to see that Q is a $(d_q, R/2)$-separated subset of Λ and $\text{card}\, Q \geq \exp((h(R)-\eta)q)$. □

Here are some preliminaries for the proof of Lemma 2.8. Define $\phi : (0,1) \to \mathbb{R}$ by the formula:

$$\phi(x) := -x \cdot \log x - (1-x)\cdot \log(1-x), x \in (0,1).$$

The function ϕ extends continuously to $[0,1]$ so that $\phi(0) = \phi(1) = 0$. The following elementary estimate is a consequence of Sterling's formula for factorials :

$$\sum_{k=0}^{\lceil \alpha n \rceil} \binom{n}{k} \leq n \cdot \exp(n\phi(\alpha)), \tag{3}$$

for n sufficiently large and $0 < \alpha < 1$.

Fix $\mathcal{C}$ any finite open covering of $\mathbb{A}$ with sets of diameter not exceeding R. We will use in our argument the fact that for $l \in \mathbb{N}$ the covering $\mathcal{C}^l$ consists of at most $(\mathrm{card}\,\mathcal{C})^l$ open sets all of d_l-diameter not exceeding R.

Proof of the technical version of the First Main Lemma

We are given an arbitrary $0 < \eta < h(R)$. Choose $\tau > 0$ small enough to satisfy :

$$\tau < \eta/(8\log(\mathrm{card}\,\mathcal{C})); \tag{4}$$

$$\phi(2\tau) < \eta/4. \tag{5}$$

(Definition of ϕ and other preliminaries immediately precedes this proof.) Let U be a set satisfying the hypothesis of 2.2 i.e. $\mu(\delta U) = 0$ and $\mu(U) > 1 - \tau$. Fix $q \in \mathbb{N}$ arbitrarily. Assume also that $\mathcal{B}$ is an open covering of U with d_q diameter not exceeding R. Our goal is to estimate the cardinality of $\mathcal{B}$.

By the definition of μ as a weak* accumulation point of probability measures $\delta_{E_{n_k}}$ (see the paragraph preceding the first main lemma) and assumptions on U we have

$$\Delta_{n_k} := \mathrm{card}\,(U^c \cap E_{n_k})/\mathrm{card}\,E_{n_k} \to \mu(U^c) < \tau.$$

We can fix $n := n_k$ large enough so that the inequality (3) holds and :

$$\Delta_n \le \tau; \tag{6}$$

$$\frac{1}{n}\log\mathrm{card}\,S_n \ge h(R) - \eta/4; \tag{7}$$

$$\frac{1}{n}\log n < \eta/4. \tag{8}$$

For any $x \in S_n$ define the set of 'good times' I_x as follows

$$I_x := \{k : f^k(x) \in U, 0 \le k \le n-1\}.$$

Claim 2.2 *A substantial number of points have mostly good times:*

$$card\{x \in S_n : card\, I_x \ge (1 - 2\Delta_n)n\} \ge \frac{1}{2} card\, S_n.$$

Proof: This is a manifestation of Markov's inequality. Indeed we have

$$\Delta_n = \frac{1}{\mathrm{card}\,E_n}\sum_{x\in S_n}\mathrm{card}\,I_x^c \ge$$

$$\frac{1}{\mathrm{card}\,E_n}\mathrm{card}\,\{x \in S_n : \mathrm{card}\,I_x^c \ge 2\Delta_n n\}2\Delta_n n.$$

Our claim follows immediately. (Just remember that $\mathrm{card}\,E_n = n \cdot \mathrm{card}\,S_n$.) □

Claim 2.3 *There exists $I \subset \{0,1,...,n-1\}$ with at least $(1-2\Delta_n)n$ elements such that*

$$card\{x \in S_n : I_x = I\} \geq \frac{1}{2} card\, S_n/(n \exp n\phi(2\Delta_n)).$$

Proof: Consider the mapping $x \mapsto I_x$ restricted to the set $\{x \in S_n : \text{card}\, I_x \geq (1 - 2\delta_n)n\}$. In view of Claim 2.1, we only need to prove that the range of this restricted map has cardinality less than $n \exp n\phi(2\Delta_n)$. This in turn is a piece of standard combinatorics supplemented by the inequality (3). □

Fix I as in Claim 2.3 and denote by S_n' the set $\{x \in S_n : I_x = I\}$. Define inductively an increasing finite sequence $(j_0, ..., j_r)$ as follows: $j_0 := \min I, j_1 := \min(I - \{0, ..., j_0 + q - 1\}), j_2 := \min(I - \{0, ..., j_1 + q - 1\}), j_3 := \min(I - \{0, ..., j_2 + q - 1\})$, etc ... (as long as the min is taken over nonempty set). Using this sequence we define a family of open sets $\mathcal{A}$ by the formula

$$\mathcal{A} := \mathcal{C}^{j_0} \vee f^{-j_0}\mathcal{B} \vee f^{-(j_0+q)}\mathcal{C}^{j_1-j_0-q} \vee f^{-j_1}\mathcal{B} \vee f^{-(j_1+q)}\mathcal{C}^{j_2-j_1-q} \vee \ldots.$$

Claim 2.4 $card\mathcal{A} \leq ((card\mathcal{C})^{2\Delta_n n}) \cdot ((card\mathcal{B})^{\lceil n/q \rceil})$.

This is a consequence of the following inequality:

$$\text{card}\,\mathcal{A} \leq (\text{card}\,\mathcal{C})^{j_0} \cdot \text{card}\,\mathcal{B} \cdot (\text{card}\,\mathcal{C})^{j_1-j_0-q} \cdot \text{card}\,\mathcal{B} \cdot (\text{card}\,\mathcal{C})^{j_2-j_1-q} \cdot \ldots.$$

Indeed, from the definition of the numbers j_p, the difference between two consecutive ones is at least q so there can be at most $\lceil n/q \rceil$ factors card $\mathcal{B}$. On the other hand, we see that the intervals $\{j_p + q, ..., j_{p+1}\}$ are disjoint from I. Consequently, due to the choice of I (so that card $I^c \leq 2\Delta_n n$) the cumulative exponent over card $\mathcal{C}$ can not exceed $2\Delta_n n$. □

Now notice that $\mathcal{A}$ covers S_n'. Also, its d_n diameter is not exceeding R so no two distinct points of S_n' are in the same element of $\mathcal{A}$. Hence, we have the estimate card $S_n' \leq$ card $\mathcal{A}$. Combining this with Claim 2.3 and the bound on card S_n' provided by Claim 2.2, we write:

$$\frac{1}{2}\text{card}\, S_n/(n \exp n\phi(2\Delta_n) \leq (\text{card}\,\mathcal{C})^{2\Delta_n n}(\text{card}\,\mathcal{B})^{\lceil n/q \rceil}.$$

Taking logarithms and dividing by q we arrive at

$$\frac{1}{q}\log \text{card}\,\mathcal{B} \geq \frac{1}{n}\log \text{card}\, S_n + \frac{1}{n}\log(1/2) - \frac{1}{n}\log n - \phi(2\Delta_n) - 2\Delta_n \log \text{card}\,\mathcal{C} \geq$$

$$h(R) - \eta/4 - \eta/4 - \phi(2\tau) - 2\tau\log(\text{card}\,\mathcal{C}) \geq$$

$$h(R) - \eta/2 - \eta/4 - \eta/4 \geq h(R) - \eta.$$

We used our settings for τ and n from the beginning of the proof to obtain the last two inequalities. (See (7), (8) and (6) for the first one and (5), (4) for the other.) This ends the proof of the technical version of First Main Lemma, and we are, therefore, completely done with the proof of Theorem 1.1. □

Remark 2.4 *It is natural to wonder if one can improve this theorem to obtain a "rotary" transverse homoclinic point simply from the fact of nonvanishing asymptotic entropy. A rotary homoclinic point is one for which the the union of the stable and unstable manifolds together with the periodic point form a homologically nontrivial set in the annulus. One would have to impose a very strong somewhat unnatural hypothesis on the measure μ, associated with the fixed scale R in the Main Proposition.*

3 Annulus Homeomorphisms: Asymptotic entropy implies periodic orbits

In previous work ([KS]) the authors proved that positive asymptotic entropy implies the existence of a chain transitive set having a nontrivial rotation set. This, in turn, by a result of J. Franks [F], implies the existence of an interval of rotation numbers and infinitely many periodic orbits in the annulus. This correlates well with the Hölder smooth case and suggests that one might be able to remove the smoothness requirement in Theorem A and derive the pA hypothesis directly from nonvanishing asymptotic entropy. We summarize what is known of the C^0 case in what follows.

3.1 Asymptotic entropy and chain transitive components

There are many good references for chain recurrence, and we refer the reader to any of those (e.g. [Co]) for detailed background information.

Definition 3.1 *An f-periodic ϵ-chain is a sequence $\{x_0, \dots, x_n = x_0\}$ with $d(f(x_i), x_{i+1}) < \epsilon$ for $0 \leq i < n$ and $n > 2$. The point x is chain recurrent if for every $\epsilon > 0$, there exists a periodic-ϵ chain containing x. Denote the set of chain recurrent points as $\mathcal{CR}$. Two points x and y lie in the same chain transitive component $\mathcal{C}$, if for all $\epsilon > 0$, there exists a periodic ϵ chain containing x and y.*

Recall that the nonwandering set $\Omega(f)$ is a subset of $\mathcal{CR}$.

Proposition 3.1 *If the asymptotic entropy h^{as} is positive, then the (restricted) asymptotic entropy $h^{as}_{\mathcal{CR}} > 0$*

Proof: By the corollary to Proposition 2.1 from the preceding section, the asymptotic entropy is nonzero on the closure of the recurrent points (= the Birkhoff center). That is so, because we identified invariant measures whose supports displayed positive R-asymptotic entropy $\tilde{h}(f, R)$ for each $R > 0$ (Proposition 2.1), and the support of an invariant measure is contained in the closure of the recurrent points of f by the Poincaré recurrence theorem (e.g. [Wa]). Finally, recurrent points are chain recurrent, and the chain recurrent set is closed. □

However, the following question, which we pose as a conjecture, remains open:

Conjecture 3.1 *Let f denote an annulus homeomorphism isotopic to the identity with chain recurrent set $\mathcal{CR}$. If $h^{as}_{\mathcal{CR}} > 0$, then there exists a chain transitive component $\mathcal{C}$ of $\mathcal{CR}$ such that $h^{as}_{\mathcal{C}} \geq (1/2)h^{as}_{\mathcal{CR}}$.*

(The factor "1/2" arises for technical reasons from the way in which asymptotic entropy is estimated on subsets.) Such a result for topological entropy is not all that difficult (but without the factor 1/2), but scaling differences between topological and asymptotic entropies seem to make this conjecture difficult to prove. The result is true, however, whenever the "$\limsup_{R\to\infty}$" in the definition of asymptotic entropy can be replaced by a limit.

We can prove that for arbitrary homeomorphisms isotopic to the identity, positive asymptotic entropy forces infinitely many periodic orbits. To do this, we will need the following lemma, due to J. Franks [Fr]:

Lemma 3.1 *If a chain transitive set contains points x and y with rotation numbers $\rho(f,x) = r$ and $\rho(f,y) = s$, $r < s$, then*

a) the full rotation set of f contains the interval $[r,s]$, and

b) for each reduced rational $p/q \in [r,s]$ there exists a periodic point of period q and rotation number p/q.

We need two additional lemmas:

Lemma 3.2 *Let X denote a chain transitive subset of $f : \mathbb{A} \to \mathbb{A}$ such that $\rho(f,X) \subset (a,b)$. then there exists a number $\delta > 0$ such that if $d(x,X) < \delta$, then $\rho(f,x) \subset (a,b)$.*

Lemma 3.3 *If every chain transitive set admits at most a single rotation number, then the rotation number mapping is continuous on the set of chain recurrent points.*

For a proof of the first lemma, see [Sw], where a slightly stronger result is proved. The second lemma is clearly an immediate corollary of the first.

Theorem 3.1 *If the asymptotic entropy is nonzero, then there are infinitely many periodic orbits corresponding to a nontrivial rotation interval.*

Proof: If some chain component admits more than one rotation number then the theorem follows from the result of Franks above. Otherwise, the rotation number mapping is continuous on the chain recurrent set by the last lemma. From Lemma 2.1 in the last section, asymptotic entropy can always be supported on sets having small diameters and, using continuity, small rotation set diameters. The bound in Lemma 2.4 now implies that $h^{as} = 0$. □

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