Homotopy and Dynamics for Homeomorphisms of Solenoids and Knaster Continua

Jaroslaw Kwapisz[1]

Department of Mathematical Sciences
Montana State University
Bozeman MT 59717-2400
tel: (406) 994 5358
fax: (406) 994 1789
e-mail: jarek@math.montana.edu
web page: http://www.math.montana.edu/~jarek/

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Abstract

We describe homotopy classes of self-homeomorphisms of solenoids and Knaster continua. In particular, we demonstrate that homeomorphisms within one homotopy class have the same (explicitly given) topological entropy and that they are actually semi-conjugated to an algebraic homeomorphism in the case when the entropy is positive.

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1 Introduction

A solenoid goes back to [20, 5] and is an indecomposable continuum that can be visualized as intersection of a nested sequence of progressively thiner solid tori that are each wrapped into the previous one a number of times as suggested by Figure 1.1. Any radial cross-section of a solenoid is a Cantor set each point of which belongs to a densely immersed line, called a *composant*. The wrapping numbers may vary from one torus to another; we shall record their sequence by $\mathcal{P} = \{p_1, p_2, \dots\}$, and we shall refer to the associated solenoid as *the $\mathcal{P}$-adic solenoid*, denoted by $\mathcal{S}_\mathcal{P}$. No generality is lost in assuming that all p_i's are prime.

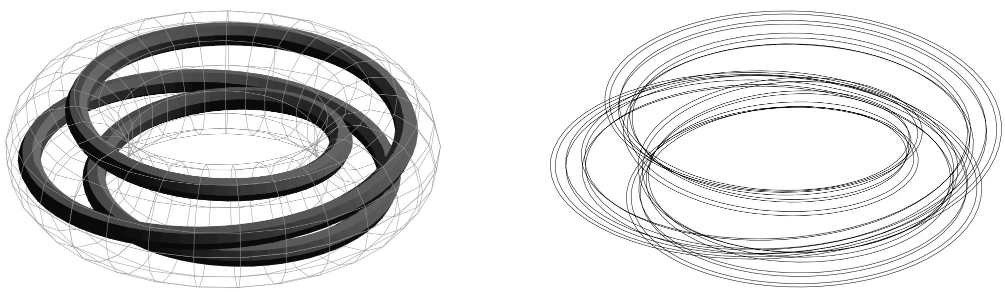

Figure 1.1: Two tori — one wrapped into another 3 times — and an approximation to the $(3, 2, 2, \dots)$-adic solenoid.

For the sequence $\mathcal{P}$, we also have the associated *$\mathcal{P}$-adic Knaster continuum* $\mathcal{K}_\mathcal{P}$ that is an intersection of a nested sequence of disks each traversing the previous one in *a snake-like fashion* a number of times indicated by the corresponding term in $\mathcal{P}$ (see Figure 1.2). $\mathcal{K}_\mathcal{P}$ is related to $\mathcal{S}_\mathcal{P}$ by a 2-1 branched cover $\mathcal{S}_\mathcal{P} \to \mathcal{K}_\mathcal{P}$ (see Section 9 and c.f. [3]).

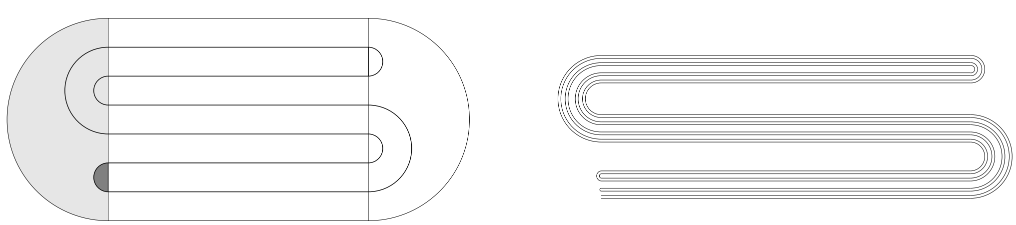

Figure 1.2: Two embedded disks and an approximation to the $(3, 2, 2, \dots)$-adic Knaster continuum.

In dynamics, which serves as our main motivation, one usually encounters $\mathcal{S}_\mathcal{P}$ and $\mathcal{K}_\mathcal{P}$ for periodic sequences $\mathcal{P}$, $\mathcal{P} = \{n, n, \dots\}$. Particularly, for $n = 2$ we get two classical attractors: *the Smale's Solenoid* and *the Smale's Horseshoe* (see e.g. [11]). These are basic examples among a bewildering variety of complicated continua that are observed in chaotic systems. One naturally wonders to what extent the rich structure of such sets determines the underlying dynamics. We solve this problem in the simplest setting of $\mathcal{S}_\mathcal{P}$ and $\mathcal{K}_\mathcal{P}$ by classifying all their homeomorphisms. Let us describe our results for $\mathcal{S}_\mathcal{P}$ now; analogous theorems hold for $\mathcal{K}_\mathcal{P}$ (see Section 9).

$\mathcal{S}_\mathcal{P}$ has a structure of a topological group (see Section 2 or [10]). Its translations roughly turn $\mathcal{S}_\mathcal{P}$ axially and may fix or permute its composants. Besides translations and the involution $r : z \mapsto z^{-1}$, the simplest maps of $\mathcal{S}_\mathcal{P}$ are Frobenius homomorphisms $g_b : \mathcal{S}_\mathcal{P} \to \mathcal{S}_\mathcal{P}$, $z \mapsto z^b$, where $b \in \mathbf{N}$. (The action of g_b roughly wraps $\mathcal{S}_\mathcal{P}$ longitudinally b times onto itself.) We shall prove that g_b is a homeomorphism iff b is $\mathcal{P}$-recurrent (i.e. every prime dividing b repeats infinitely many times in $\mathcal{P}$). In that case, we can also form maps $g_{a/b} := g_b^{-1} \circ g_a$. The compositions $s \circ g_{a/b}$ and $r \circ s \circ g_{a/b}$ where s is a translation and $a, b \in \mathbf{N}$ are co-prime and $\mathcal{P}$-recurrent form a group of *affine homeomorphisms of* $\mathcal{S}_\mathcal{P}$.

THEOREM 1 [1] *Suppose that $\mathcal{P}$ is an infinite sequence of prime numbers and $f : \mathcal{S}_\mathcal{P} \to \mathcal{S}_\mathcal{P}$ is a homeomorphism of the $\mathcal{P}$-adic solenoid. Then there are unique $\mathcal{P}$-recurrent and co-prime natural numbers $a, b \in \mathbf{N}$ such that f is isotopic to the affine map $g = s \circ g_{a/b}$ or $g = r \circ s \circ g_{a/b}$.*

We note that g is determined uniquely up to a translation that fixes the composants of $\mathcal{S}_\mathcal{P}$ so that there are actually uncountably many homotopy classes for homeomorphisms of $\mathcal{S}_\mathcal{P}$ — each class is determined by *the winding ratio* a/b and the composant that contains the image $f(e)$ of the neutral element $e \in \mathcal{S}_\mathcal{P}$.

Our second theorem assures that the topological entropy is constant across any fixed homotopy class.

THEOREM 2 *In the context of Theorem 1, the topological entropies of f and g coincide and are given by*

$$h(f) = h(g) = \log \max\{a, b\}.$$

In particular, if f is homotopic to a translation then the entropy is zero.

Finally, our third theorem shows that if $h(f) > 0$, then f is conjugated to g after perhaps collapsing some arcs in $\mathcal{S}_\mathcal{P}$ to points.

THEOREM 3 *In the context of Theorem 1, if $a/b \neq 1$, then f is semi-conjugated to g; namely, there is a surjective continuous map $h : \mathcal{S}_\mathcal{P} \to \mathcal{S}_\mathcal{P}$ such that $h \circ f = g \circ h$. Moreover, $h^{-1}(z)$ is a point or an arc for any $z \in \mathcal{S}_\mathcal{P}$.*

[1] This result is hardly new; see the comments at the end of this introduction.

It is easy to see that the semi-conjugacy does not in general exist when $a/b = 1$ (c.f. [13]).

Let us mention that the above results show that the set of entropies exhibited by homeomorphisms of $\mathcal{S}_{\mathcal{P}}$ determines all the $\mathcal{P}$-recurrent primes and that this already determines $\mathcal{S}_{\mathcal{P}}$ up to a homeomorphism in the dynamically interesting case when $\mathcal{P}$ is a periodic sequence. Indeed, in [15, 1], it is shown that two solenoids $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\tilde{\mathcal{P}}}$ are homeomorphic iff, after perhaps removing a finite number of terms, $\mathcal{P}$ and $\tilde{\mathcal{P}}$ contain each prime the same number of times. In general, however, non-homeomorphic solenoids may exhibit the same entropies as exemplified by a pair $\mathcal{S}_{(2,3,5,7,\dots)}$ and $\mathcal{S}_{(2,2,3,3,5,5,7,7,\dots)}$ — both solenoids admit only self-homeomorphisms of zero entropy.

The proofs of the three theorems hinge on constructing for an arbitrary homeomorphism of $\mathcal{S}_{\mathcal{P}}$ its lift to $\Lambda_{\mathcal{P}} \times \mathbf{R}$ where $\Lambda_{\mathcal{P}}$ is the cross section of $\mathcal{S}_{\mathcal{P}}$. The lift, although generally no longer a homeomorphism, is a skew product over the base $\Lambda_{\mathcal{P}}$ because it has to permute the composants $\{\omega\} \times \mathbf{R}$. Moreover, the base map on $\Lambda_{\mathcal{P}}$ is purely algebraic and universal across each homotopy class.

To indicate other ingredients making up our arguments, we outline the contents of the sections to follow. In Section 2, we give a formal definition of $\mathcal{S}_{\mathcal{P}}$ and recall the standard identification of the cross section $\Lambda_{\mathcal{P}}$ of $\mathcal{S}_{\mathcal{P}}$ with *an adding machine*. Section 3 collects number-theoretical facts about $\Lambda_{\mathcal{P}}$ used in Section 4 to classify the algebraic homeomorphisms of $\mathcal{S}_{\mathcal{P}}$. Theorem 1 is shown in Section 5 by using "small cross sections" of $\mathcal{S}_{\mathcal{P}}$ to lift an arbitrary homeomorphism of $\mathcal{S}_{\mathcal{P}}$ to $\Lambda_{\mathcal{P}} \times \mathbf{R}$. A short Section 6 combines the results of Sections 4 and 5 to provide an explicit formula for such a lift. Section 7 establishes Theorem 2 by computing the entropy in $\Lambda_{\mathcal{P}} \times \mathbf{R}$ as the sum of the entropies in the fibers and in the base. This requires a suitable version of Bowen's formula from [4], which we prove in the appendix. Section 8 mimics Handel's [9] to implement Katok's idea of global shadowing in order to show Theorem 3. Finally, Section 9 develops the analogues of Theorems 1, 2, and 3 for the Knaster continua by lifting the homeomorphisms from $\mathcal{K}_{\mathcal{P}}$ to $\mathcal{S}_{\mathcal{P}}$.

To end, let us put our results into perspective.[2] Solenoids are old and we inevitably included some known material hoping to make the presentation accessible even to a novice. The author's initial thrust came from conversations with Marcy Barge. In [3], a special case of Theorem 2 was established for $\mathcal{P} = \{n, n, \dots\}$ and f homotopic to a power of the very Frobenius homomorphism g_n. [3] shows also the formula for $h(g_{a/b})$ for all a, b, which however, goes back to [2] and belongs to a long line of works as cited in [14]. Theorem 1, in turn, is not original: it belongs to "folklore" among topologists and can be extracted from any of [15, 18, 12, 16, 7]; in particular, its version for maps fixing the unit element can be found on page 45 in [7]. A quick proof amounts to observing that a homeomorphism of a solenoid must

[2]The author was greatly aided by feedback from the preliminary circulation of the manuscript and is especially grateful to M. Barge, L. Block, J. Keesling, P. Minc, V. Ssembatya, and the anonymous referee.

permute the components according to its cohomological action on the dual group. Nevertheless, to present all three results from a unified point of view, we supply a careful elementary argument. Also, we should add that the analogue of Theorem 1 for $\mathcal{K}_{\mathcal{P}}$ with periodic $\mathcal{P}$ has been attributed in [3] to W. T. Watkins[3] and it follows from Lemma 9.5 in [6]. Theorem 3 seems genuinely new although it relies on tested tools of hyperbolic dynamics and is a natural extension of the classical result in [8, 9]. We mention that our result complements [12], where the group of homeomorphisms of a solenoid was studied as a topological space. Finally, non-locally connected coverings like $\Lambda_{\mathcal{P}} \times \mathbf{R} \to \mathcal{S}_{\mathcal{P}}$ were used in [17]; however, our approach to homeomorphisms of $\mathcal{S}_{\mathcal{P}}$ via their (non-invertible) lifts to $\Lambda_{\mathcal{P}} \times \mathbf{R}$ seems original and is the key idea of this paper.

2 Preliminaries

We fix a sequence $\mathcal{P} = \{p_0, p_1, p_2, \dots\}$ where $p_k > 1$ for $k \geq 1$ are prime numbers, and $p_0 = 1$ is added for convenience. We define *the $\mathcal{P}$-adic solenoid $\mathcal{S}_{\mathcal{P}}$* as the inverse limit space of mappings $z \mapsto z^{p_k}$, $k = 1, 2, \dots$, on the complex unit circle $\mathbf{S} := \{z \in \mathbf{C} : \ |z| = 1\}$; namely,

$$\mathcal{S}_{\mathcal{P}} = \lim_{\longleftarrow}(z \mapsto z^{p_k}) := \{(z_k)_{k=0}^{\infty} : \ z_{k-1} = z_k^{p_k}, \ k \geq 1\}.$$

(This $\mathcal{S}_{\mathcal{P}}$ is homeomorphic to the geometric model discussed in the introduction, which justifies the abuse of notation.) Being a closed subgroup of the product $\prod_{k=0}^{\infty} \mathbf{S}$, $\mathcal{S}_{\mathcal{P}}$ is a compact abelian topological group (c.f. §10 in Chapter II of [10]); we write $z \cdot \tilde{z}$ for the product of its two elements and e for the neutral element $(1, 1, 1, \dots)$. $\mathcal{S}_{\mathcal{P}}$ is also a solenoidal group, meaning that there is a homomorphism $\Gamma : \mathbf{R} \to \mathcal{S}_{\mathcal{P}}$ with dense image, $\overline{\Gamma(\mathbf{R})} = \mathcal{S}_{\mathcal{P}}$. Explicitly,

$$\Gamma(t) := \left(\exp\left(2\pi i t / p_0 \dots p_k\right)\right)_{k=0}^{\infty}, \quad t \in \mathbf{R}.$$

Γ generates a translation flow $T : \mathbf{R} \times \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$, $T^t z := \Gamma(t) \cdot z$, which has an obvious cross section with return time 1 along a subgroup given by

$$\Lambda_{\mathcal{P}} := \{z = (z_k)_{k=0}^{\infty} \in \mathcal{S}_{\mathcal{P}} : \ z_0 = 1\}, \qquad T^1(\Lambda_{\mathcal{P}}) = \Lambda_{\mathcal{P}},$$

(i.e. there is $t \geq 0$ with $T^t z \in \Lambda_{\mathcal{P}}$ for any $z \in \mathcal{S}_{\mathcal{P}}$ and $\{t \in \mathbf{R} : \ T^t z \in \Lambda_{\mathcal{P}}\} = \mathbf{Z}$ for $z \in \Lambda_{\mathcal{P}}$). The return map is the translation by $\gamma := \Gamma(1)$,

$$T := T^1|_{\Lambda_{\mathcal{P}}} : \Lambda_{\mathcal{P}} \to \Lambda_{\mathcal{P}}, \qquad T\omega := \gamma \cdot \omega;$$

and $\mathcal{S}_{\mathcal{P}}$ is homeomorphic with the suspension manifold of T, $\mathcal{S}_{\mathcal{P}} \cong \Lambda_{\mathcal{P}} \times [0,1]/\sim$ where $(\omega, 1) \sim (T\omega, 0)$ for $\omega \in \Lambda_{\mathcal{P}}$. Equivalently, $\mathcal{S}_{\mathcal{P}}$ is the orbit space of $\mathbf{Z}$ acting (discretely) on $\Lambda_{\mathcal{P}} \times \mathbf{R}$ so that $k \in \mathbf{Z}$ is assigned the map

$$D^k : (\omega, x) \mapsto (T^k \omega, x - k).$$

[3] although we could not locate a written account

We denote by π the associated natural projection

$$\pi : \Lambda_{\mathcal{P}} \times \mathbf{R} \to \Lambda_{\mathcal{P}} \times \mathbf{R}/\mathbf{Z} \cong \mathcal{S}_{\mathcal{P}},$$

which is a covering and a homomorphism of topological groups[4] (c.f. [17]). Being a covering, π has *the unique path lifting property* and *the homotopy lifting property* (see sec. 2, chap. 2 in [19]); however, unlike in the usual setting, $\Lambda_{\mathcal{P}} \times \mathbf{R}$ is neither connected nor locally connected so that the standard lifting theorems do not apply. In particular, if $\kappa : \Lambda_{\mathcal{P}} \to \mathbf{Z}$ is any continuous function, then the map

$$D^{\kappa(\cdot)} : \Lambda_{\mathcal{P}} \times \mathbf{R} \to \Lambda_{\mathcal{P}} \times \mathbf{R}, \qquad D^{\kappa(\cdot)} : (\omega, x) \mapsto \left(T^{\kappa(\omega)}\omega, x - \kappa(\omega)\right),$$

is clearly a *covering transformation*, (i.e. $\pi \circ D^{\kappa(\cdot)} = \pi$), yet $D^{\kappa(\cdot)}$ is not invertible for non-constant κ. This (along with other features of T) is made transparent by viewing T as an "adding machine", which we recall below.

Topologically, $\Lambda_{\mathcal{P}}$ is a $\mathcal{P}$-adic Cantor set: it is homeomorphic to $\mathcal{C}_{\mathcal{P}} := \prod_{k=1}^{\infty}\{0, \ldots, p_k - 1\}$ via

$$\mathcal{C}_{\mathcal{P}} \ni (d_k)_{k=1}^{\infty} \mapsto (z_k)_{k=0}^{\infty} := \exp\left(2\pi i \frac{d_1 + d_2 p_1 + d_3 p_1 p_2 + \cdots + d_k p_1 \cdots p_{k-1}}{p_0 p_1 \cdots p_k}\right).$$

Here, we recognize the sequences $[d_1 \ldots d_k]$ as the $\mathcal{P}$-adic representations of the integers

$$s_k := d_1 + d_2 p_1 + d_3 p_1 p_2 + \cdots + d_k p_1 \cdots p_{k-1}, \tag{2.1}$$

and the group operation on $\Lambda_{\mathcal{P}}$ corresponds to *the addition with carry* performed on $[d_1 \ldots d_k]$'s. More formally, the correspondence $(z_k)_{k=0}^{\infty} \mapsto (s_k)_{k=1}^{\infty}$ is an isomorphism of $\Lambda_{\mathcal{P}}$ as a topological group with the inverse limit of (additive) cyclic groups,

$$\Lambda_{\mathcal{P}} \cong \lim_{\leftarrow} \mathbf{Z}_{p_1 \cdots p_k},$$

where the bonding maps $\mathbf{Z}_{p_1 \cdots p_k} \to \mathbf{Z}_{p_1 \cdots p_{k-1}}$ are given by

$$s \pmod{p_1 \cdots p_k} \mapsto s \pmod{p_1 \cdots p_{k-1}}, \quad k > 1.$$

The action of T corresponds to the addition with carry of $[1000 \ldots\ ]$ and is given on $s \in \lim_{\leftarrow} \mathbf{Z}_{p_1 \cdots p_k}$ simply by

$$T : (s_k) \mapsto (s_k + 1).$$

It follows easily that the orbits of T are dense in $\Lambda_{\mathcal{P}}$; in particular, $\Lambda_{\mathcal{P}} = \overline{\{\gamma^k : \ k \in \mathbf{N}\}}$ and $\mathcal{S}_{\mathcal{P}} = \overline{\{T^t e : \ t \in \mathbf{R}\}}$.

[4] $\mathcal{S}_{\mathcal{P}} = \Lambda_{\mathcal{P}} \times \mathbf{R}/I$ where I is the subgroup $I = \{(\gamma^k, x - k) : \ k \in \mathbf{Z}\}$.

To compute the entropy in Section 7, we shall use the metric $d_{\Lambda_\mathcal{P}}$ on $\Lambda_\mathcal{P}$ given in terms of the $\mathcal{P}$-adic representations by

$$d_{\Lambda_\mathcal{P}}([d_k]_{k=1}^\infty, [\tilde{d}_k]_{k=1}^\infty) := \exp(-\min\{k \geq 1 : d_k \neq \tilde{d}_k;\ \infty\}).$$

Note that $d_{\Lambda_\mathcal{P}}$ is induced by a norm — $\|[d_k]\|_{\Lambda_\mathcal{P}} := \exp(-\min\{k : d_k \neq 0\})$ — and therefore $d_{\Lambda_\mathcal{P}}$ is an invariant metric, i.e. the translations of $\Lambda_\mathcal{P}$ are isometries. On $\Lambda_\mathcal{P} \times \mathbf{R}$ and $\mathcal{S}_\mathcal{P}$, we put the corresponding invariant metrics:

$$d_{\Lambda_\mathcal{P}\times\mathbf{R}}((\omega, x), (\tilde{\omega}, \tilde{x})) := \max\{d_{\Lambda_\mathcal{P}}(\omega, \tilde{\omega}), |x - \tilde{x}|\}$$

and

$$d_{\mathcal{S}_\mathcal{P}}(z, \tilde{z}) := \min\{d_{\Lambda_\mathcal{P}\times\mathbf{R}}((\omega, x), (\tilde{\omega}, \tilde{x})) :\ \pi(\omega, x) = z,\ \pi(\tilde{\omega}, \tilde{x}) = \tilde{z}\}.$$

In this way, the deck map $D : \Lambda_\mathcal{P} \times \mathbf{R} \to \Lambda_\mathcal{P} \times \mathbf{R}$ becomes an isometry, and the covering projection $\pi : \Lambda_\mathcal{P} \times \mathbf{R} \to \mathcal{S}_\mathcal{P}$ becomes a local isometry.

3 Self-similarity of $\Lambda_\mathcal{P}$

Let a sequence of primes $\mathcal{P}$ be fixed as in the previous section. For determination of possible types of homeomorphisms of $\mathcal{S}_\mathcal{P}$ in Section 4, it matters how the number theory of $\mathcal{P}$ correlates with "self similarity of $\Lambda_\mathcal{P}$". Precisely: for $a \in \mathbf{N}$, we have the associated Frobenius endomorphism

$$\phi_a : \Lambda_\mathcal{P} \to \Lambda_\mathcal{P}, \qquad \phi_a : \omega \mapsto \omega^a;$$

and we want to know that ϕ_a is a monomorphism onto a subgroup of index a exactly when a is *$\mathcal{P}$-recurrent* (i.e. any prime that divides a appears in $\mathcal{P}$ infinitely many times). In fact, we shall need a bit more, which necessitates the following definitions.

Let $\mathbf{P}$ be the set of all prime numbers, and let $\alpha_\mathcal{P} : \mathbf{P} \to \{0, 1, 2, \ldots, \infty\}$ be the function that counts primes in $\mathcal{P}$,

$$\alpha_\mathcal{P}(p) := \#\{i \in \mathbf{N} :\ p_i = p\}, \quad p \in \mathbf{P}.$$

For $a \in \mathbf{N}$, we have an analogous count of primes dividing a, $\alpha_a : \mathbf{P} \to \{0, 1, \ldots\}$,

$$\alpha_a(p) := \max\{k \geq 0 :\ p^k \text{ divides } a\}, \quad p \in \mathbf{P}.$$

The product $a = \prod_{p\in\mathbf{P}} p^{\alpha_a(p)}$ can be decomposed according to the number of times p shows in $\mathcal{P}$: the primes p that repeat in $\mathcal{P}$ infinitely many times contribute

$$a_{\mathcal{P},\infty} := \prod_{\alpha_\mathcal{P}(p)=\infty} p^{\alpha_a(p)};$$

the primes p that repeat in $\mathcal{P}$ finitely many times contribute

$$a_{\mathcal{P},+} := \prod_{0\le\alpha_{\mathcal{P}}(p)<\infty} p^{\min\{\alpha_a(p),\alpha_{\mathcal{P}}(p)\}};$$

and there are the primes p that do not occur in $\mathcal{P}$ or occur in $\mathcal{P}$ fewer times than in a,

$$a_{\mathcal{P},-} := \prod_{0\le\alpha_{\mathcal{P}}(p)<\infty} p^{\alpha_a(p)-\min\{\alpha_a(p),\alpha_{\mathcal{P}}(p)\}}.$$

The number

$$a_{\mathcal{P}} := a_{\mathcal{P},\infty}\, a_{\mathcal{P},+}$$

is *the greatest common divisor* of a and the (infinite) product $\prod_{p_i\in\mathcal{P}} p_i$. We have

$$a = a_{\mathcal{P}} a_{\mathcal{P},-} = a_{\mathcal{P},\infty}\, a_{\mathcal{P},+}\, a_{\mathcal{P},-};$$

and $a \in \mathbf{N}$ is $\mathcal{P}$-*recurrent* iff $a = a_{\mathcal{P},\infty}$.

Fact 3.1 *For $a \in \mathbf{N}$, the kernel of ϕ_a has cardinality $|\ker(\phi_a)| = a_{\mathcal{P},+}$. In particular, ϕ_a is a monomorphism iff $a = a_{\mathcal{P},\infty} a_{\mathcal{P},-}$, i.e. every prime in $\mathcal{P}$ that divides a repeats in $\mathcal{P}$ infinitely many times.*

We shall argue for the representation of ϕ_a on $\lim_{\leftarrow} \mathbf{Z}_{p_1\cdots p_k} \equiv \Lambda_{\mathcal{P}}$ (see Section 2) given by

$$\psi_a : \lim_{\leftarrow} \mathbf{Z}_{p_1\cdots p_k} \to \lim_{\leftarrow} \mathbf{Z}_{p_1\cdots p_k}, \qquad \psi_a : s = (s_k)_{k=1}^{\infty} \mapsto as := (as_k)_{k=1}^{\infty}.$$

Proof of Fact 3.1. Let k_0 be large enough so that $a_{\mathcal{P}}$ divides $p_1\cdots p_{k_0}$ and no prime p_k divides $a_{\mathcal{P},+}$ for $k > k_0$. In particular, $a_{\mathcal{P},-}$ and $p_1\cdots p_k/a_{\mathcal{P}}$ are co-prime for $k > k_0$. Consider $s \in \ker(\psi_a)$. We have

$$as_k = a_{\mathcal{P},-}\, a_{\mathcal{P}}\, s_k \equiv 0 \pmod{p_1\cdots p_k}, \quad k \ge 1.$$

Thus, for $k > k_0$, $a_{\mathcal{P},-}\, s_k \equiv 0 \pmod{p_1\cdots p_k/a_{\mathcal{P}}}$ and, by the choice of k_0,

$$s_k \equiv 0 \pmod{p_1\cdots p_k/a_{\mathcal{P}}}. \tag{3.1}$$

Also, for a fixed $k > k_0$ and $l > k$ large enough, $a_{\mathcal{P},\infty}$ divides $p_{k+1}\cdots p_l$ so that $p_1\cdots p_k/a_{\mathcal{P},+}$ divides $p_1\cdots p_l/a_{\mathcal{P}}$. The congruence (3.1) (applied for $k = l$) yields that $s_l \equiv 0 \pmod{p_1\cdots p_k/a_{\mathcal{P},+}}$, and thus $s_k \equiv s_l \equiv 0 \pmod{p_1\cdots p_k/a_{\mathcal{P},+}}$ since $s \in \lim_{\leftarrow} \mathbf{Z}_{p_1\cdots p_k}$. This shows that an arbitrary $s \in \ker(\psi_a)$ is of the form

$$s_k = x_k p_1\cdots p_k/a_{\mathcal{P},+} \in \mathbf{Z}_{p_1\cdots p_k}, \quad k > k_0,$$

where $x_k \in \mathbf{Z}_{a_{\mathcal{P},+}}$. As a result there are at most $a_{\mathcal{P},+} = |\mathbf{Z}_{a_{\mathcal{P},+}}|$ elements in $\ker(\psi_a)$ (because they would all have to differ on some fixed coordinate s_k for sufficiently large k).

To see that $|\ker(\psi_a)| = a_{\mathcal{P},+}$, for any $x_{k_0} \in \mathbf{Z}_{a_{\mathcal{P},+}}$, we exhibit $s \in \ker(\psi_a)$ with $s_{k_0} = x_{k_0} p_1 \cdots p_{k_0} / a_{\mathcal{P},+}$. For $k < k_0$, set $s_k := s_{k_0} \pmod{p_1 \cdots p_k}$. For $k \geq k_0$, since p_{k+1} is co-prime with $a_{\mathcal{P},+}$ by the choice of k_0, we can solve recursively for x_k's the equations

$$x_{k+1} p_{k+1} \equiv x_k \pmod{a_{\mathcal{P},+}}, \; k \geq k_0; \tag{3.2}$$

and we can set $s_k := x_k p_1 \cdots p_k / a_{\mathcal{P},+}$. The recurrence (3.2) guarantees that $s_{k+1} \equiv s_k \pmod{p_1 \cdots p_k}$ for all $k \geq 1$, i.e. that $s \in \lim_{\leftarrow} \mathbf{Z}_{p_1 \cdots p_k}$. It is also immediate that $\psi_a(s) = 0$. □

Fact 3.2 *For $a \in \mathbf{N}$, the image $\phi_a(\Lambda_{\mathcal{P}}) = {\Lambda_{\mathcal{P}}}^a \subset \Lambda_{\mathcal{P}}$ is a subgroup of index*

$$[\Lambda_{\mathcal{P}} : {\Lambda_{\mathcal{P}}}^a] = a_{\mathcal{P}} = a_{\mathcal{P},\infty}\, a_{\mathcal{P},+}.$$

In particular, ϕ_a is an epimorphism iff $a = a_{\mathcal{P},-}$, i.e. no prime in $\mathcal{P}$ divides a.

Proof of Fact 3.2. As before, take k_0 large enough so that $a_{\mathcal{P}}$ divides $p_1 \cdots p_{k_0}$. Note that $a_{\mathcal{P},-}$ and $p_1 \cdots p_k / a_{\mathcal{P}}$ are co-prime for $k > k_0$. Consider a homomorphism $h : \lim_{\leftarrow} \mathbf{Z}_{p_1 \cdots p_k} \to \mathbf{Z}_{a_{\mathcal{P}}}$ given by $h : s \mapsto s_{k_0} \pmod{a_{\mathcal{P}}}$. This is an epimorphism because, given $x \in \mathbf{Z}_{a_{\mathcal{P}}}$, we clearly have $h(s) = x$ for $s = (s_k)_{k=1}^{\infty} := (x \pmod{p_1 \cdots p_k})_{k=1}^{\infty}$. It follows that $[\lim_{\leftarrow} \mathbf{Z}_{p_1 \cdots p_k} : \ker(h)] = a_{\mathcal{P}}$. To finish the proof, we show that $\ker(h) = \psi_a(\lim_{\leftarrow} \mathbf{Z}_{p_1 \cdots p_k})$.

If $s \in \psi_a(\lim_{\leftarrow} \mathbf{Z}_{p_1 \cdots p_k})$, then $s \in \ker(h)$ because $s_{k_0} \equiv 0 \pmod{a}$. For the other inclusion, observe that if $h(s) = 0$, then $s_{k_0} \equiv 0 \pmod{a_{\mathcal{P}}}$; and $s_k \equiv 0 \pmod{a_{\mathcal{P}}}$ for $k \geq k_0$ because $s_k \equiv s_{k_0} \pmod{p_1 \cdots p_{k_0}}$ and $a_{\mathcal{P}}$ divides $p_1 \cdots p_{k_0}$. Hence, $s_k = x_k a_{\mathcal{P}} \pmod{p_1 \cdots p_k}$ for some $x_k \in \mathbf{Z}_{p_1 \cdots p_k / a_{\mathcal{P}}}$, $k \geq k_0$. Because $a_{\mathcal{P},-}$ and $p_1 \cdots p_k / a_{\mathcal{P}}$ are co-prime for $k > k_0$, we can solve the equations $a_{\mathcal{P},-}\, y_k \equiv x_k \pmod{p_1 \cdots p_k / a_{\mathcal{P}}}$ for $y_k \in \mathbf{Z}_{p_1 \cdots p_k / a_{\mathcal{P}}}$, $k \geq k_0$. It follows that $a_{\mathcal{P},-}\, a_{\mathcal{P}}\, y_k \equiv a_{\mathcal{P}}\, x_k \equiv s_k \pmod{p_1 \cdots p_k}$ so that $\psi_a(y) = s$, i.e. $s \in \psi_a(\lim_{\leftarrow} \mathbf{Z}_{p_1 \cdots p_k})$. □

4 Algebraic Homeomorphisms

We list in this section the self-homeomorphisms of solenoids that are to serve as the representatives of the homotopy classes. (We shall proceed in a completely elementary fashion that avoids Pontriagin's duality.)

The Inverse: Taking inverse with respect to the group operation in $\mathcal{S}_{\mathcal{P}}$ yields a homeomorphism

$$r : \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}, \qquad r : z \mapsto z^{-1}.$$

The corresponding map on the cover,

$$R : \Lambda_{\mathcal{P}} \times \mathbf{R} \to \Lambda_{\mathcal{P}} \times \mathbf{R}, \qquad R : (\omega, x) \mapsto (\omega^{-1}, -x),$$

is a lift of r.

Translations: For $w \in \mathcal{S}_\mathcal{P}$, the translation by w yields a homeomorphism

$$s_w : \mathcal{S}_\mathcal{P} \to \mathcal{S}_\mathcal{P}, \qquad z \mapsto w \cdot z.$$

If $(\sigma, t) \in \Lambda_\mathcal{P} \times \mathbf{R}$ is a lift of w, $\pi(\sigma, t) = w$, then

$$S_{(\sigma,t)} : \Lambda_\mathcal{P} \times \mathbf{R} \to \Lambda_\mathcal{P} \times \mathbf{R}, \qquad S_{(\sigma,t)} : (\omega, x) \mapsto (\omega \cdot \sigma, x + t)$$

is a lift of s_w.

Frobenius Automorphisms: For $a \in \mathbf{N}$, we have the Frobenius endomorphism

$$g_a : \mathcal{S}_\mathcal{P} \to \mathcal{S}_\mathcal{P}, \qquad g_a : z \mapsto z^a,$$

which lifts to the corresponding Frobenius endomorphism on $\Lambda_\mathcal{P} \times \mathbf{R}$,

$$G_a : \Lambda_\mathcal{P} \times \mathbf{R} \to \Lambda_\mathcal{P} \times \mathbf{R}, \qquad G_a : (\omega, x) \mapsto (\phi_a(\omega), ax).$$

($\phi_a : \Lambda_\mathcal{P} \to \Lambda_\mathcal{P},\ \phi_a(\omega) = \omega^a$ was discussed in Section 3.)

Lemma 4.1 *The Frobenius endomorphism g_a is a homeomorphism of $\mathcal{S}_\mathcal{P}$ iff a is $\mathcal{P}$-recurrent, i.e. $a = a_{\mathcal{P},\infty}$.*

The idea is that g_a is a homeomorphism exactly when ϕ_a is 1-1 and $[\Lambda_\mathcal{P} : \phi_a(\Lambda_\mathcal{P})] = a$, which will translate to $a = a_{\mathcal{P},\infty}$ by the results of the previous section. The first condition is clear because $\phi_a = g_a|_{\Lambda_\mathcal{P}}$, and the second condition is to prevent the fundamental domain $\Lambda_\mathcal{P} \times [0,1]$ from overlapping in the image of G_a.

Proof of Lemma 4.1. Observe that g_a is surjective for any $a \in \mathbf{N}$ because $g_a(\Gamma(t)) = \Gamma(at)$ for $t \in \mathbf{R}$ and $\overline{\Gamma(a\mathbf{R})} = \mathcal{S}_\mathcal{P}$.

Suppose that $a = a_{\mathcal{P},\infty}$. We shall first prove that g_a is a homeomorphism by deriving an explicit formula for its (lifted) inverse. (This may not be the shortest route but the formula will be crucial in all subsequent sections.) From Fact 3.2, $\Lambda_\mathcal{P}$ decomposes into a disjoint union of a clopen cosets

$$\Lambda_\mathcal{P} = {\Lambda_\mathcal{P}}^a \cup \gamma \cdot {\Lambda_\mathcal{P}}^a \cup \ldots \cup \gamma^{a-1} \cdot {\Lambda_\mathcal{P}}^a. \tag{4.1}$$

Let $I_a : \Lambda_\mathcal{P} \to \mathbf{Z}_a \simeq \Lambda_\mathcal{P}/{\Lambda_\mathcal{P}}^a$ be the natural projection so that $\omega \in \gamma^{I_a(\omega)}{\Lambda_\mathcal{P}}^a$ for $\omega \in \Lambda_\mathcal{P}$. Define $G_{1/a} : \Lambda_\mathcal{P} \times \mathbf{R} \to \Lambda_\mathcal{P} \times \mathbf{R}$ by the formula

$$G_{1/a} : (\omega, x) \mapsto \left(\phi_a^{-1}(\gamma^{-I_a(\omega)} \cdot \omega), \frac{x + I_a(\omega)}{a} \right). \tag{4.2}$$

$G_{1/a}$ is manifestly continuous; and we claim that $G_{1/a}$ descends to a continuous map $g_{1/a} : \mathcal{S}_\mathcal{P} \to \mathcal{S}_\mathcal{P}$. It suffices to see that, given $(\omega, x) \in \Lambda_\mathcal{P} \times \mathbf{R}$, $G_{1/a} \circ D(\omega, x) =$

$D^k \circ G_{1/a}(\omega, x)$ for some $k \in \mathbf{N}$:
If $I_a(\omega) < a-1$, so that $I_a(\gamma \cdot \omega) = I_a(\omega)+1$, then

$$\begin{aligned} G_{1/a} \circ D(\omega, x) = G_{1/a}(\gamma \cdot \omega, x-1) &= \left(\phi_a^{-1}\left(\gamma^{-I_a(\gamma\cdot\omega)} \cdot \gamma \cdot \omega\right), \frac{x-1+I_a(\gamma\cdot\omega)}{a} \right) \\ &= \left(\phi_a^{-1}\left(\gamma^{-I_a(\omega)} \cdot \omega\right), \frac{x+I_a(\omega)}{a} \right) = G_{1/a}(\omega, x). \end{aligned}$$

If $I_a(\omega) = a-1$, so that $\gamma \cdot \omega \in \Lambda_{\mathcal{P}}$, then

$$G_{1/a} \circ D(\omega, x) = G_{1/a}(\gamma \cdot \omega, x-1) = \left(\phi_a^{-1}(\gamma \cdot \omega), \frac{x-1}{a} \right)$$

coincides with

$$D \circ G_{1/a}(\omega, x) = \left(\gamma \cdot \phi_a^{-1}(\gamma^{-a+1} \cdot \omega), \frac{x+a-1}{a} - 1 \right) = \left(\gamma \cdot \phi_a^{-1}(\gamma^{-a+1} \cdot \omega), \frac{x-1}{a} \right)$$

because $\gamma \cdot \phi_a^{-1}(\gamma^{-a+1} \cdot \omega) = \phi_a^{-1}(\gamma \cdot \omega)$, which is easily verified by applying ϕ_a to both sides and simplifying. Thus, indeed, $g_{1/a}$ is well defined.

Note that $G_{1/a}$ is a left inverse of G_a, $G_{1/a} \circ G_a = \mathbf{Id}_{\Lambda_{\mathcal{P}} \times \mathbf{R}}$. Therefore, since g_a is surjective (unlike G_a), $g_{1/a}$ is the (two-sided) inverse of g_a, which makes g_a a homeomorphism.

To prove the other implication, suppose that g_a is a homeomorphism. If $a = q_1 \cdots q_m$ is the prime decomposition of a, then $g_a = g_{q_1} \circ \cdots \circ g_{q_m}$ where the g_{q_i}'s are homeomorphisms as well. Let us consider then a single homeomorphism g_q for a prime $q = q_i$. If q appears in $\mathcal{P}$, then q must be $\mathcal{P}$-recurrent since otherwise $\phi_q = g_q|_{\Lambda_{\mathcal{P}}}$ is not 1-1 by Fact 3.1. If q does not appear in $\mathcal{P}$, then Facts 3.2 and 3.1 imply that $\phi_q : \Lambda_{\mathcal{P}} \to \Lambda_{\mathcal{P}}$ is a homeomorphism and so are the restrictions

$$g_q|_{\pi(\Lambda_{\mathcal{P}} \times \{k/q\})} : \pi(\Lambda_{\mathcal{P}} \times \{k/q\}) \to \Lambda_{\mathcal{P}}, \qquad k = 0, \ldots, q-1.$$

Therefore, each point of $\Lambda_{\mathcal{P}}$ has (exactly) q preimages, which contradicts g_q being a homeomorphism. This shows that any prime factor q_i, and thus a, is $\mathcal{P}$-recurrent. □

By combining the Frobenius automorphisms, their inverses, translations, and possibly the inverse involution we arrive with affine homeomorphisms of $\mathcal{S}_{\mathcal{P}}$ — among which those that map $\Lambda_{\mathcal{P}}$ to itself we shall call normalized affine homeomorphisms.

Definition 4.1 (affine homeomorphisms) *Suppose that $\sigma \in \Lambda_{\mathcal{P}}$ and $a, b \in \mathbf{N}$ are co-prime and $\mathcal{P}$-recurrent. We define a homeomorphism*

$$g_{a/b,\sigma} : \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$$

to be the factor of $G_{a/b,\sigma} : \Lambda_{\mathcal{P}} \times \mathbf{R} \to \Lambda_{\mathcal{P}} \times \mathbf{R}$ *given by*

$$G_{a/b,\sigma} : (\omega, x) \mapsto \left(\sigma \cdot \phi_{a/b}(\omega), \frac{a(x + I_b(\omega))}{b}\right) \tag{4.3}$$

where

$$\phi_{a/b}(\omega) := \phi_a \circ \phi_b^{-1}(\gamma^{-I_b(\omega)} \cdot \omega) \tag{4.4}$$

and $I_b : \Lambda_{\mathcal{P}} \to \mathbf{Z}_b$ *is determined by* $\gamma^{-I_b(\omega)} \cdot \omega \in {\Lambda_{\mathcal{P}}}^b$.
A map $g : \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$ *is called* a normalized affine homeomorphism *iff* $g = g_{a/b,\sigma}$ *or* $g = r \circ g_{a/b,\sigma}$ *for some* a, b, σ *as above.*
A map $g : \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$ *is called* an affine homeomorphism *iff* $g = T^t \circ g_{a/b,\sigma}$ *or* $g = r \circ T^t \circ g_{a/b,\sigma}$ *for some* a, b, σ *as above and* $t \in \mathbf{R}$.

Note that g_a and g_b commute so that $g_{a/b,\sigma} = s_\sigma \circ g_a \circ g_b^{-1} = s_\sigma \circ g_b^{-1} \circ g_a$. Also, if $\tilde{a}$ and $\tilde{b}$ fail to be co-prime, one may define $g_{\tilde{a}/\tilde{b},\sigma} := g_{a/b,\sigma}$ where $a/b = \tilde{a}/\tilde{b}$ is the reduced fraction — which allows one to write[5]

$$g_{a_1/b_1,\sigma_1} \circ g_{a_2/b_2,\sigma_2} = g_{a_1/b_1 \cdot a_2/b_2, \sigma_1 \cdot \phi_{a_1/b_1}(\sigma_2)} \qquad \text{and} \qquad g^{-1}_{a/b,\sigma} = g_{b/a,\phi_{b/a}(\sigma)}. \tag{4.5}$$

At the same time, keep in mind that if $b \neq 1$, then $G_{a/b,\sigma} = S_\sigma \circ G_a \circ G_{1/b} \neq S_\sigma \circ G_{1/b} \circ G_a$, and $G_{a/b,\sigma}$ is neither 1-1 nor surjective.

Remark 4.1 *The formula (4.4) takes on a more friendly appearance when interpreted on* $\lim_{\leftarrow} \mathbf{Z}_{p_1 \cdots p_k} \cong \Lambda_{\mathcal{P}}$. *Namely, if* $w \in \lim_{\leftarrow} \mathbf{Z}_{p_1 \cdots p_k}$ *corresponds to* $\sigma \in \Lambda_{\mathcal{P}}$, $a, b \in \mathbf{N}$ *are co-prime and* b *is* $\mathcal{P}$*-recurrent, then* $\sigma \cdot \phi_{a/b} : \Lambda_{\mathcal{P}} \to \Lambda_{\mathcal{P}}$ *is conjugated to* $w + \psi_{a/b} : \lim_{\leftarrow} \mathbf{Z}_{p_1 \cdots p_k} \to \lim_{\leftarrow} \mathbf{Z}_{p_1 \cdots p_k}$ *given by*

$$(w + \psi_{a/b})(s)_k = w_k + a(s_k \div b) \pmod{p_1 \cdots p_k}, \quad s \in \lim_{\leftarrow} \mathbf{Z}_{p_1 \cdots p_k},$$

valid for k *large enough to guarantee that* b *divides* $p_1 \cdots p_k$. *(Here the symbol* $\div$ *stands for the integer division.)*

We finish by classifying the affine homeomorphisms with respect to homotopy.

Lemma 4.2 *Two normalized affine homeomorphisms* $g_{a/b,\sigma}$ *and* $g_{a_1/b_1,\sigma_1}$ *are homotopic iff* $a = a_1$, $b = b_1$, *and* $T^n \sigma = \sigma_1$ *for some* $n \in \mathbf{Z}$.

Proof. To prove necessity, it suffices to argue for $a_1 = b_1 = 1$ and $\sigma_1 = e$ (by (4.5)). Suppose that $f_t : \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$, $t \in [0,1]$, is a homotopy such that $f_0 = g_{a/b,\sigma}$ and $f_1 = \mathbf{Id}$. As a covering, π has a homotopy lifting property so that there is a homotopy $F_t : \Lambda_{\mathcal{P}} \times \mathbf{R} \to \Lambda_{\mathcal{P}} \times \mathbf{R}$, $t \in [0,1]$, covering f_t and such that $F_0 = G_{a/b,\sigma}$.

[5] That is the affine homeomorphisms are a skew product of the group of $\mathcal{P}$-adic fractions and $\mathcal{S}_{\mathcal{P}}$.

Being a lift of $f_1 = \mathbf{Id}$, F_1 must be a covering transformation, i.e. $F_1 = D^{\kappa(\cdot)}$ for some continuous function $\kappa : \Lambda_{\mathcal{P}} \to \mathbf{Z}$ (c.f. Section 2). Now, if $\mathrm{pr}_{\mathbf{R}} : \Lambda_{\mathcal{P}} \times \mathbf{R} \to \mathbf{R}$ is the projection on the second factor, then the compactness of $\Lambda_{\mathcal{P}}$ implies that f_t has a bounded displacement; namely,

$$|\mathrm{pr}_{\mathbf{R}}(F_0(\omega, x)) - \mathrm{pr}_{\mathbf{R}}(F_1(\omega, x))| = |a(x + I_b(\omega))/b - (x - \kappa(\omega))| < C$$

for some $C > 0$ independent on $x \in \mathbf{R}$ and $\omega \in \Lambda_{\mathcal{P}}$. It follows (by considering $x \to \infty$) that $a/b = 1$, i.e. $a = 1$ and $b = 1$. Moreover — $\Lambda_{\mathcal{P}}$ being totally disconnected — F_t must permute the leaves $\omega \times \mathbf{R}$ in $\Lambda_{\mathcal{P}} \times \mathbf{R}$, which is to say that the projection $\mathrm{pr}_{\Lambda_{\mathcal{P}}}(F_t(\omega, x))$ onto $\Lambda_{\mathcal{P}}$ is constant over the range $(t, x) \in [0, 1] \times \mathbf{R}$, i.e. $\sigma \cdot \omega = \gamma^{\kappa(\omega)} \cdot \omega$ for all $\omega \in \Lambda_{\mathcal{P}}$. In this way, $\kappa(\cdot)$ must be constant and $\sigma = \gamma^{\kappa}$ — which finishes the proof of necessity.

For sufficiency, observe that the flow T^t provides isotopies from $g_{a/b,\sigma}$ to $g_{a/b,\gamma^k \cdot \sigma}$ and back. □

5 Homotopy Classes (Proof of Theorem 1)

In this section, we fix a homeomorphism $f : \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$ and construct an isotopy of f to an affine homeomorphism g of $\mathcal{S}_{\mathcal{P}}$ in order to prove Theorem 1.

We record first the simple fact that f is an orbit equivalence between the flow T^t and itself or T^t and its inverse, T^{-t}. (This suggests that the homotopy type of f is solely determined by the way f permutes the orbits of T^t.)

Fact 5.3 *For every $z \in \mathcal{S}_{\mathcal{P}}$, there is a homeomorphism $\tau : \mathbf{R} \to \mathbf{R}$ such that*

$$f \circ T^t z = T^{\tau(t)} \circ f(z), \quad t \in \mathbf{R}.$$

Depending on whether $\tau(t)$ is increasing or decreasing we shall call f *orientation preserving* or *orientation reversing* — (which z is used is irrelevant because $\{T^t(z) : t \in \mathbf{R}\}$ winds densely in $\mathcal{S}_{\mathcal{P}}$.) At the expense of replacing f with $r \circ f$, we may assume that f is orientation preserving.

Proof of Fact 5.3. Fix $z \in \mathcal{S}_{\mathcal{P}}$. By path lifting, we have $\alpha : \mathbf{R} \to \Lambda_{\mathcal{P}} \times \mathbf{R}$ such that $\pi \circ \alpha(t) = f \circ T^t z$ for $t \in \mathbf{R}$. Because $\Lambda_{\mathcal{P}}$ is totally disconnected, $\mathrm{pr}_{\Lambda_{\mathcal{P}}} \circ \alpha$ is constant and so $\alpha(t) = (\omega, \tau(t))$ for some $\tau : \mathbf{R} \to \mathbf{R}$ and some $\omega \in \Lambda_{\mathcal{P}}$. τ is 1-1 because f is 1-1. Also, $\tau(\mathbf{R}) = \mathbf{R}$ because otherwise $\pi(\{(\omega, \tau(t)) : \pm t > 0\}) = \{f \circ T^t z : \pm t > 0\}$ would not be dense in $\mathcal{S}_{\mathcal{P}}$. □

In order to lift f to a map of $\Lambda_{\mathcal{P}} \times \mathbf{R}$, we shall use "small cross-sections of T^t". For $N \in \mathbf{N}$, let $c := p_1 \cdots p_N$. Clearly, $c = c_{\mathcal{P}}$ so that $[\Lambda_{\mathcal{P}} : \Lambda_{\mathcal{P}}{}^c] = c$ by Fact 3.2. Hence, $\Lambda_{\mathcal{P}}{}^c$ is a cross-section of the flow T^t with constant return time c. Moreover, since $\Lambda_{\mathcal{P}}{}^c = \{\omega \in \Lambda_{\mathcal{P}} : \ \omega_i = 1, \ \forall\ 0 \le i \le N\}$, we see that

$\lim_{N\to\infty} \mathrm{diam}(\Lambda_{\mathcal{P}}{}^c) = 0$ (with respect to any compatible metric). In this way, we can find $N \in \mathbf{N}$, $\epsilon, \delta \in (0, 1/4)$, and $x_0 \in (0, 1)$ such that

$$f \circ \pi(\Lambda_{\mathcal{P}}{}^c \times (-\delta, \delta)) \subset \pi(\Lambda_{\mathcal{P}} \times (x_0 - \epsilon, x_0 + \epsilon)), \tag{5.1}$$

$$\mathrm{diam}\left(\mathrm{pr}_0 \circ f^{-1} \circ \pi(\omega \times (x_0 - \epsilon, x_0 + \epsilon))\right) < 1/2, \ \omega \in \Lambda_{\mathcal{P}}, \tag{5.2}$$

$$\mathrm{dist}\left(\mathrm{pr}_0 \circ f(z_1), \mathrm{pr}_0 \circ f(z_2)\right) < 1/4 \text{ for } z_1 \cdot z_2^{-1} \in \Lambda_{\mathcal{P}}{}^c, \ z_1, z_2 \in \mathcal{S}_{\mathcal{P}}, \tag{5.3}$$

where $\mathrm{pr}_0 : \mathcal{S}_{\mathcal{P}} \to \mathbf{S}$ is the projection on the zeroth coordinate, $\mathrm{pr}_0((z_i)_{i=0}^{\infty}) = z_0$, and the metric on $\mathbf{S}$ is induced from $\mathbf{R}$ via the exponential $t \mapsto \exp(2\pi i t)$.

In particular, $\Lambda_{\mathcal{P}}{}^c$ is chosen to be small enough cross-section so that a localized isotopy will easily "straighten" $f(\Lambda_{\mathcal{P}}{}^c)$ and put it into a fiber of $\mathrm{pr}_0 : \mathcal{S}_{\mathcal{P}} \to \mathbf{S}$, which then can be further isotoped into $\Lambda_{\mathcal{P}}$ via T^t. This is formalized in the first step below.

Step 1: Construct an isotopy of f to $\overline{f}$ such that $\overline{f}(\Lambda_{\mathcal{P}}{}^c) \subset \Lambda_{\mathcal{P}}$.

Construction. From (5.1), $f(\Lambda_{\mathcal{P}}{}^c)$ is *evenly covered* by π; in particular, we have a well defined and continuous $\xi := \mathrm{pr}_{\Lambda_{\mathcal{P}}} \circ \tilde{\pi}^{-1} \circ f|_{\Lambda_{\mathcal{P}}{}^c} : \Lambda_{\mathcal{P}}{}^c \to \Lambda_{\mathcal{P}}$ and $\theta = \mathrm{pr}_{\mathbf{R}} \circ \tilde{\pi}^{-1} \circ f|_{\Lambda_{\mathcal{P}}{}^c} : \Lambda_{\mathcal{P}}{}^c \to \mathbf{R}$ where $\tilde{\pi} := \pi|_{\Lambda_{\mathcal{P}} \times (x_0 - \epsilon, x_0 + \epsilon)}$. Clearly, $f|_{\Lambda_{\mathcal{P}}{}^c}(\omega) = \pi(\xi(\omega), \theta(\omega))$. Observe that ξ is 1-1. Indeed, if $\omega = \xi(\omega_1) = \xi(\omega_2)$ for $\omega_1, \omega_2 \in \Lambda_{\mathcal{P}}{}^c$, then (5.1) implies that both ω_1 and ω_2 are contained in $f^{-1} \circ \pi(\omega \times (x_0 - \epsilon, x_0 + \epsilon))$, which intersects $\Lambda_{\mathcal{P}}$ only once by (5.2) — hence, $\omega_1 = \omega_2$.

Thus we can invert ξ, and $\theta \circ \xi^{-1}|_{\xi(\Lambda_{\mathcal{P}}{}^c)}$ is a continuous function. Let $\hat{\theta} : \Lambda_{\mathcal{P}} \to \mathbf{R}$ be its continuous extension to $\Lambda_{\mathcal{P}}$. To construct a homotopy $h_\lambda : \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$, $\lambda \in [0, 1]$, supported on $\pi(\Lambda_{\mathcal{P}} \times [x_0 - 2\epsilon, x_0 + 2\epsilon])$, fix a vector field X supported on $[x_0 - 2\epsilon, x_0 + 2\epsilon]$ and such that $X = 1$ on $[x_0 - \epsilon, x_0 + \epsilon]$. Denote by X^t the flow of X. For $z \in \pi(\Lambda_{\mathcal{P}} \times (x_0 - 2\epsilon, x_0 + 2\epsilon))$, set

$$h_\lambda(z) = \pi\left(\omega, X^{\lambda[x_0 - \hat{\theta}(\omega)]} x\right)$$

where $(\omega, x) \in \Lambda_{\mathcal{P}} \times (x_0 - 2\epsilon, x_0 + 2\epsilon)$ is the (unique) lift of z. Extend h_λ by identity to the whole $\mathcal{S}_{\mathcal{P}}$. This definition assures that $h_0 = \mathbf{Id}$ and $h_1 \circ f(\Lambda_{\mathcal{P}}{}^c) = \pi \circ (\xi(\Lambda_{\mathcal{P}}{}^c) \times x_0)$. Moreover, h_λ is a homeomorphism for each $\lambda \in [0, 1]$: it is manifestly surjective, and it is 1-1 because it preserves the composants $\pi(\omega \times \mathbf{R})$ and solves an ODE on each of them. In this way, the isotopy given by $T^{-\lambda x_0} \circ h_\lambda \circ f$, $\lambda \in [0, 1]$, deforms f to $\overline{f} := T^{-x_0} \circ h_1 \circ f$ such that $\overline{f}(\Lambda_{\mathcal{P}}{}^c) \subset \Lambda_{\mathcal{P}}$ — as required. □

Step 2: Adjust $\overline{f}$ by translation so that the neutral element e is a fixed point; namely, define

$$\hat{f} := s_{\overline{f}(e)^{-1}} \circ \overline{f}$$

so that $\hat{f}(e) = e$.

Fact 5.4 *$\hat{f}(\Lambda_{\mathcal{P}}{}^c)$ is a cross-section of T^t with some constant return time $d \in \mathbf{N}$.*

Proof. Fix $\omega_0 \in \Lambda_{\mathcal{P}}{}^c$ and set $\alpha(t) := \hat{f} \circ T^t(\omega_0)$ for $t \in \mathbf{R}$. By Fact 5.3, $\alpha(t) = T^{\tau(t)}\omega$ for some $\omega \in \Lambda_{\mathcal{P}}$ and $\tau : \mathbf{R} \to \mathbf{R}$ an increasing homeomorphism (since we assumed that f preserved orientation). Because $\alpha(0), \alpha(c) \in \hat{f}(\Lambda_{\mathcal{P}}{}^c) \subset \Lambda_{\mathcal{P}}$, the winding number of the curve $\mathrm{pr}_0 \circ \alpha|_{[0,c]}$ into the zeroth circle is well defined — denote it by $d \in \mathbf{N}$. Since c is the first return time of ω_0 to $\Lambda_{\mathcal{P}}{}^c$ and $\hat{f}$ is an orbit equivalence, d is the first return time for $\hat{f}(\omega_0)$ to $\hat{f}(\Lambda_{\mathcal{P}}{}^c)$. The condition (5.3) assures that d does not depend on ω_0 because, as we vary $\omega_0 \in \mathcal{S}_{\mathcal{P}}{}^c$, the different curves $\mathrm{pr}_0 \circ \alpha(t)$ never depart from each other further than by $1/4$, i.e. $\mathrm{dist}\,(\mathrm{pr}_0 \circ \alpha(t), \mathrm{pr}_0 \circ \tilde{\alpha}(t)) \leq 1/4$ if $\tilde{\alpha}(t) := \hat{f} \circ T^t(\tilde{\omega}_0)$ for some $\tilde{\omega}_0 \in \Lambda_{\mathcal{P}}{}^c$. □

Denote $\phi := \hat{f}|_{\Lambda_{\mathcal{P}}{}^c}$. Observe that $\phi(\gamma^{cn}) = \gamma^{dn}$ for $n \in \mathbf{Z}$ because the first return to $\Lambda_{\mathcal{P}}{}^c$ corresponds via $\hat{f}$ to the first return to $\hat{f}(\Lambda_{\mathcal{P}})$ after time d (see Fact 5.4). Since $\overline{\{\gamma^n : \ n \in \mathbf{N}\}} = \Lambda_{\mathcal{P}}$, we conclude that

$$\phi \circ \phi_c = \phi_d. \tag{5.4}$$

(Here $\phi_k(\omega) = \omega^k$, see Section 3.) In particular, $\hat{f}(\Lambda_{\mathcal{P}}{}^c) = \Lambda_{\mathcal{P}}{}^d$.

Fact 5.5 *The return times c and d are products of primes in $\mathcal{P}$, i.e. $c = c_{\mathcal{P}}$ and $d = d_{\mathcal{P}}$. Moreover, $c/d = c_{\mathcal{P},\infty}/d_{\mathcal{P},\infty}$.*

Proof. From Fact 3.2, the return time to $\Lambda_{\mathcal{P}}{}^d$ equals $[\Lambda_{\mathcal{P}} : \Lambda_{\mathcal{P}}{}^d] = d_{\mathcal{P}}$ which shows that $d = d_{\mathcal{P}}$. That $c = c_{\mathcal{P}}$ follows from the definition of c. For the "moreover part", note that ϕ is 1-1 so that $|\ker(\phi_c)| = |\ker(\phi_d)|$ by (5.4), which translates to $c_{\mathcal{P},+} = d_{\mathcal{P},+}$ via Fact 3.1. Hence, $c/d = c_{\mathcal{P}}/d_{\mathcal{P}} = (c_{\mathcal{P},+}c_{\mathcal{P},\infty})/(d_{\mathcal{P},+}d_{\mathcal{P},\infty}) = c_{\mathcal{P},\infty}/d_{\mathcal{P},\infty}$. □

Step 3: There is an isotopy of $\hat{f}$ to the affine homomorphism $g_{c_{\mathcal{P},\infty}/d_{\mathcal{P},\infty}} := g_{c_{\mathcal{P},\infty}} \circ g^{-1}_{d_{\mathcal{P},\infty}}$.

The idea is that (5.4) forces $\hat{f}$ to agree with $g_{c/d}$ on $\Lambda_{\mathcal{P}}{}^c$, and this agreement can be easily spread onto the whole $\mathcal{S}_{\mathcal{P}}$ by "ironing out" the nonlinearities along the fibers $\omega \times \mathbf{R}$.

Construction. We only produce a homotopy and leave derivation of formulas for an isotopy as an exercise. Because $\hat{f}(\Lambda_{\mathcal{P}}{}^c) = \Lambda_{\mathcal{P}}{}^d$ we can *cut* $\mathcal{S}_{\mathcal{P}}$ along the cross sections $\Lambda_{\mathcal{P}}{}^c$ and $\Lambda_{\mathcal{P}}{}^d$ and lift g to $\tilde{g}$ as in the diagram below

$$\begin{array}{ccc} \Lambda_{\mathcal{P}}{}^c \times [0,c] & \xrightarrow{\tilde{g}} & \Lambda_{\mathcal{P}}{}^d \times [0,d] \\ \downarrow & & \downarrow \\ \mathcal{S}_{\mathcal{P}} & \xrightarrow{g} & \mathcal{S}_{\mathcal{P}} \end{array}$$

where the vertical arrows realize the boundary identifications $(\omega, c) \sim (T^c\omega, 0)$ and $(\omega, d) \sim (T^d\omega, 0)$.

Since $\hat{f}$ preserves the flow lines of T, (5.4) implies that $\tilde{g}(\omega^c, x) := (\omega^d, h(\omega^c, x))$, $(\omega^c, x) \in {\Lambda_{\mathcal{P}}}^c \times [0, c]$, for some $h : {\Lambda_{\mathcal{P}}}^c \times [0, c] \to [0, d]$ that maps ${\Lambda_{\mathcal{P}}}^c \times \{0, c\}$ to $\{0, d\}$. (Note that ω^d is well in terms of ω^c defined because $\ker(\phi_d) = \ker(\phi_c)$ by (5.4).) We write a homotopy $\tilde{g}_\lambda : {\Lambda_{\mathcal{P}}}^c \times [0, c] \to {\Lambda_{\mathcal{P}}}^d \times [0, d]$, $\lambda \in [0, 1]$,

$$\tilde{g}_\lambda(\omega^c, x) := \left(\omega^d,\, d/c \cdot x + (1 - \lambda) \cdot \left(h(\omega^c, x) - d/c \cdot x\right)\right).$$

Observe that $\tilde{g}$ respects the identifications on ${\Lambda_{\mathcal{P}}}^c \times \{0, c\}$ and ${\Lambda_{\mathcal{P}}}^d \times \{0, d\}$ so that it descends to a homotopy $\hat{g}_\lambda : \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$. Clearly, $\hat{g}_0 = \hat{f}$ and we claim that $\hat{g}_1 = g_{c_{\mathcal{P},\infty}/d_{\mathcal{P},\infty}}$. Indeed, given $z \in \mathcal{S}_{\mathcal{P}}$, we take its lift $(\omega^c, x) \in {\Lambda_{\mathcal{P}}}^c \times [0, c]$ and see that $c/d = c_{\mathcal{P},\infty}/d_{\mathcal{P},\infty}$ implies that

$$\tilde{g}_1\left(\omega^{c\, c_{\mathcal{P},\infty}}, c_{\mathcal{P},\infty}\, x\right) = \left(\omega^{d\, c_{\mathcal{P},\infty}}, d/c \cdot c_{\mathcal{P},\infty}\, x\right) = \left(\omega^{c\, d_{\mathcal{P},\infty}}, d_{\mathcal{P},\infty}\, x\right),$$

which shows that $\hat{g}_1(z^{c_{\mathcal{P},\infty}}) = z^{d_{\mathcal{P},\infty}}$ for all $z \in \mathcal{S}_{\mathcal{P}}$, i.e. $\hat{g}_1 = g_{c_{\mathcal{P},\infty}/d_{\mathcal{P},\infty}}$. □

Conclusion of the proof of Theorem 1. The progression of the three steps above shows that if $f : \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$ is a homeomorphism, then $g_{c_{\mathcal{P},\infty}/d_{\mathcal{P},\infty}}$ is isotopic to $s_{\omega^{-1}} \circ f$ in the orientation preserving case or to $s_{\omega^{-1}} \circ r \circ f$ in the orientation reversing case. (Here $\omega := \overline{f}(e)$.) Thus f is isotopic to $g_{c_{\mathcal{P},\infty}/d_{\mathcal{P},\infty},\omega}$ or to the $r \circ g_{c_{\mathcal{P},\infty}/d_{\mathcal{P},\infty},\omega}$. □

6 General Form of Homeomorphism

The results of the two previous sections yield a very explicit description of homeomorphisms of $\mathcal{S}_{\mathcal{P}}$:

Corollary 6.1 *If $f : \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$ is an orientation preserving homeomorphism, then f has a lift $F : \Lambda_{\mathcal{P}} \times \mathbf{R} \to \Lambda_{\mathcal{P}} \times \mathbf{R}$ of the form*

$$F : (\omega, x) \mapsto \left(\sigma \cdot \phi_{a/b}(\omega), \frac{a(x + I_b(\omega))}{b} + \delta(\omega, x)\right) \tag{6.1}$$

where a and b are co-prime $\mathcal{P}$-recurrent, $\sigma \in \Lambda_{\mathcal{P}}$, and $\delta : \Lambda_{\mathcal{P}} \times \mathbf{R} \to \mathbf{R}$ is a bounded continuous function invariant under the deck map: $\delta \circ D = \delta$, i.e. $\delta(\gamma \cdot \omega, x - 1) = \delta(\omega, x)$ for all $(\omega, x) \in \Lambda_{\mathcal{P}} \times \mathbf{R}$. If $f : \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$ is orientation reversing, then it has a lift of the form $R \circ F$ (where, recall, $R(\omega, x) = (\omega^{-1}, -x)$). Moreover, the numbers a and b are uniquely determined by the homotopy type of f.

Observe that $F \circ D = D^{\kappa(\cdot)} \circ F$ where $\kappa(\omega, x) = 0$ if $I_b(\omega) < b - 1$ and $\kappa(\omega, x) = a$ if $I_b(\omega) = b - 1$ (c.f. the proof of Lemma 4.1).

Proof. Theorem 1 supplies a homotopy $f_\lambda : \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$, $\lambda \in [0, 1]$, that connects $f = f_1$ to an affine homeomorphism $g = f_0$. Let $F_\lambda : \Lambda_{\mathcal{P}} \times \mathbf{R} \to \Lambda_{\mathcal{P}} \times \mathbf{R}$, $\lambda \in [0, 1]$, be a lift of this homotopy such that F_0 is the lift $G_{a/b,\sigma}$ of the affine homeomorphism (see Definition 4.1). Recall that, since π is a covering, $F_\lambda(\omega, x)$ is obtained by unique path lifting of the path $[0,1] \ni \lambda \mapsto f_\lambda(z)$, $z = \pi(\omega, x)$, starting at a point $(\mu, y) := G_{a/b,\sigma}(\omega, x)$. Total disconnectedness of $\Lambda_\mathcal{P}$ forces the lifted path to have a constant projection onto $\Lambda_\mathcal{P}$, i.e. to be of the form $(\mu, y + \tilde{\delta}(\omega, x, \lambda))$ for some continuous $\tilde{\delta} : \Lambda_\mathcal{P} \times \mathbf{R} \times [0,1] \to \mathbf{R}$. Thus the corollary holds with $\delta(\omega, x) := \tilde{\delta}(\omega, x, 1)$. The equivariance of δ is immediate from the uniqueness of path lifting. The uniqueness of a and b is a consequence of Lemma 4.2. □

7 Entropy (Proof of Theorem 2)

To determine the entropy of an arbitrary homeomorphism $f : \mathcal{S}_\mathcal{P} \to \mathcal{S}_\mathcal{P}$ and prove Theorem 2, we shall pass to the lift F of f given by (6.1), compute the entropies generated by F in the base $\Lambda_\mathcal{P}$ and in the individual fibers $\omega \times \mathbf{R}$, and apply a version of Bowen's formula for the entropy of skew-products (as shown in the appendix). The references [4] and [21] may be consulted for the definition of topological entropy. All the entropies are computed with respect to the metrics on $\Lambda_\mathcal{P}$, $\mathcal{S}_\mathcal{P}$, and $\Lambda_\mathcal{P} \times \mathbf{R}$ constructed in Section 2.

Lemma 7.3 *For a map F as in Corollary 6.1, a compact segment $I \subset \mathbf{R}$, and $\omega \in \Lambda_\mathcal{P}$, the entropy of a compact set $\omega \times I$ with respect to F and the metric $d_{\Lambda_\mathcal{P} \times \mathbf{R}}$ satisfies*

$$h(F, \omega \times I) \leq \max\{0, \log(a/b)\}.$$

The lemma hinges on the well known basic fact that, given two partitions of a line segment: one into p subsegments $\{I_i\}$ and another into q subsegments $\{J_j\}$, their *refinement* $\{I_i\} \vee \{J_j\} := \{I_i \cap J_j\}_{i,j}$ can have at most $p+q-1$ non-empty elements. This indicates that the entropy is produced solely by exponential stretching, which is controlled by a/b. (Thus, in fact, $h_{d_{\Lambda_\mathcal{P} \times \mathbf{R}}}(F, \omega \times \mathbf{R}) = \max\{0, \log(a/b)\}$.)

Proof of Lemma 7.3. Fix arbitrary $\epsilon, \lambda > 0$ such that $\lambda > \max\{1, |a/b|\}$. Set $\omega_k := \mathrm{pr}_{\Lambda_\mathcal{P}} \circ F^k(\omega \times I)$ and $I_k := \mathrm{pr}_\mathbf{R} \circ F^k(\omega \times I)$, $k \geq 0$. It is easy to see from (6.1) that there is $C > 0$ such that $|I_k| \leq C\lambda^k$ for $k \geq 0$. Let $\mathcal{A}_k$ be a partition of $\omega_k \times I_k$ into closed segments of equal length $\rho \in [\epsilon/2, \epsilon]$ so that the cardinality $\#\mathcal{A}_k \leq 2C\lambda^k/\epsilon$. The partition $\mathcal{A}^n := \mathcal{A}_0 \vee F^{-1}\mathcal{A}_1 \vee \ldots \vee F^{-n+1}\mathcal{A}_{n-1}$ has cardinality $\#\mathcal{A}^n \leq 2C/\epsilon \cdot (1 + \ldots + \lambda^n + n)$. Since any selector set of $\mathcal{A}^n$ is an (ϵ, n)-spanning set in $\omega \times I$, we obtain $h_{d_{\Lambda_\mathcal{P} \times \mathbf{R}}}(F, \omega \times I) \leq \log\lambda$ after passing to the limit with $n \to \infty$ and then with $\epsilon \to \infty$. We are done by the arbitrariness of λ. □

Lemma 7.4 *For $a, b \in \mathbf{N}$ co-prime with b $\mathcal{P}$-recurrent, the topological entropy of the map $\sigma \cdot \phi_{a/b} : \Lambda_\mathcal{P} \to \Lambda_\mathcal{P}$ is*

$$h(\sigma \cdot \phi_{a/b}) = \log b.$$

Proof of Lemma 7.4. It is more convenient to argue for $\psi := w + \psi_{a/b} : \lim_{\leftarrow} \mathbf{Z}_{p_1 \cdots p_k} \to \lim_{\leftarrow} \mathbf{Z}_{p_1 \cdots p_k}$ that is conjugated to $\sigma \cdot \phi_{a/b}$ (as explained in Remark 4.1). Let $(k_n)_{n=1}^{\infty}$ be a non-decreasing sequence such that $p_1 \cdots p_{k_n}$ is divisible by b^n, $n \in \mathbf{N}$; and set $\epsilon := \exp(-k_1)$. We claim that if s and $\tilde{s} \in \lim_{\leftarrow} \mathbf{Z}_{p_1 \cdots p_k}$ are such that $s_k \not\equiv \tilde{s}_k \pmod{b^n}$ for some $k \geq k_n$, then s and $\tilde{s}$ are (n, ϵ)-separated. This would yield $h(\psi) \geq \log b$ because, by using the natural embedding $\kappa : \mathbf{N} \to \lim_{\leftarrow} \mathbf{Z}_{p_1 \cdots p_k}$ such that $(\kappa(n)_k)_{k=1}^{\infty} := (n \pmod{p_1 \cdots p_k})_{k=1}^{\infty}$, we exhibit an (n, ϵ)-separated set $S_n := \kappa(\{0, ..., b^n - 1\}) \subset \lim_{\leftarrow} \mathbf{Z}_{p_1 \cdots p_k}$ and $\# S_n = b^n$.

We verify the claim by induction with respect to n. For $n = 1$, the $\mathcal{P}$-adic expansions $[d_k]$ and $[\tilde{d}_k]$ of s and $\tilde{s}$ — see (2.1) — must differ on some digit $d_{k_*} \neq \tilde{d}_{k_*}$ for $k_* \leq k_1$ since otherwise $\tilde{s}_k - s_k$ would be divisible by b. Thus $\mathrm{d}_{\Lambda_{\mathcal{P}}}(s, \tilde{s}) \geq \epsilon$ by the definitions of ϵ and the distance $\mathrm{d}_{\Lambda_{\mathcal{P}}}$ (Section 2). Now, for the induction step, assume that the claim holds for $n - 1$ and that $s_k \not\equiv \tilde{s}_k \pmod{b^n}$. Unless $\mathrm{d}_{\Lambda_{\mathcal{P}}}(s, \tilde{s}) \geq \epsilon$, and there is nothing to show, we have $s_k \equiv \tilde{s}_k \pmod{b}$ for $k \geq k_1$ by what we have already shown. Hence, $(s_k \div b) \not\equiv (\tilde{s}_k \div b) \pmod{b^{n-1}}$. Since a is co-prime with b, it follows that

$$\psi(s)_k = w_k + a(s_k \div b) \not\equiv w_k + a(\tilde{s}_k \div b) = \psi(\tilde{s})_k \pmod{b^{n-1}},$$

and we conclude that $\psi(s)$ and $\psi(s)$ are $(n - 1, \epsilon)$-separated by the induction hypothesis. This makes s and $\tilde{s}$ (n, ϵ)-separated.

To prove $h(\psi) \leq \log b$, it suffices to show that if $s, \tilde{s} \in \lim_{\leftarrow} \mathbf{Z}_{p_1 \cdots p_k}$ are such that $s_k \equiv \tilde{s}_k \pmod{p_1 \cdots p_{k_1} b^n}$ for some $k > k_1$ large enough so that $p_1 \cdots p_{k_1} b^n$ divides $p_1 \cdots p_k$, then s and $\tilde{s}$ are (n, ϵ)-close. Indeed, $\tilde{S}_n := \kappa\left(\{0, \ldots, p_1 \cdots p_{k_1} b^n - 1\}\right)$ is then an (n, ϵ)-spanning set with cardinality $p_1 \cdots p_{k_1} b^n$ — and ϵ can be made arbitrarily small by increasing k_1. Thus, suppose that $s_k \equiv \tilde{s}_k \pmod{p_1 \cdots p_{k_1} b^n}$. Then $s_k \equiv \tilde{s}_k \pmod{p_1 \cdots p_{k_1}}$ and so $\mathrm{d}_{\Lambda_{\mathcal{P}}}(s, \tilde{s}) \leq \epsilon$. Also, $s_k \div b \equiv \tilde{s}_k \div b \pmod{p_1 \cdots p_{k_1} b^{n-1}}$ so that

$$\psi(s)_k = w_k + a(s_k \div b) \equiv w_k + a(\tilde{s}_k \div b) = \psi(\tilde{s})_k \pmod{p_1 \cdots p_{k_1} b^{n-1}},$$

which shows that $\mathrm{d}_{\Lambda_{\mathcal{P}}}(\psi(s), \psi(\tilde{s})) \leq \epsilon$. By repeating the argument, $\psi^i(s)_k \equiv \psi^i(\tilde{s})_k \pmod{p_1 \cdots p_{k_1} b^{n-i}}$ so that $\mathrm{d}_{\Lambda_{\mathcal{P}}}(\psi^i(s), \psi^i(\tilde{s})) \leq \epsilon$ for $i = 0, \ldots, n-1$. □

Our Theorem 2 amounts to the following proposition.

Proposition 7.1 *The topological entropy $h(f)$ of a homeomorphism $f : \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$ is given by*

$$h(f) = \max\{0, \log(a/b)\} + \log b = \log \max\{a, b\}$$

where a and b are as in Corollary 6.1.

Proof. It suffices to argue for f that is orientation preserving (otherwise, consider f^2). First, note that $h(f) = h(F, K)$ for $K := \Lambda_{\mathcal{P}} \times [-M, M]$ where $M \geq 1$ (c.f. Theorem 8.12 in [21]). To see that, fix $1/4 > r > 0$ so that $\pi|_{B(p,r)} : B(p, r) \to B(\pi(p), r)$ is a surjective isometry for all $p \in \Lambda_{\mathcal{P}} \times \mathbf{R}$

(where $B(p,r) = \{z : d(z,p) < r\}$). Consider any $\epsilon \in (0,r)$ small enough that $d(F(p),F(q)) < r$ if $d(p,q) < \epsilon$. One can verify that if E is (n,ϵ)-spanning for $\mathcal{S}_{\mathcal{P}}$, then $\tilde{E} := \pi^{-1}(E) \cap \{p \in K : \ d(p,K) \le \epsilon\}$ is (n,ϵ)-spanning for K; and $\#\tilde{E} \le (2M+1)\#E$, which implies $h(F,K) \le h(f)$. Also, if $\tilde{E}$ is (n,ϵ)-spanning for K, then $E := \pi(\tilde{E})$ is (n,ϵ)-spanning for $\mathcal{S}_{\mathcal{P}}$, which implies $h(f) \le h(F,K)$.

Since passing to f^{-1} interchanges the roles of a and b (c.f. (4.5)) and $h(f) = h(f^{-1})$, we may assume that $a/b \le 1$. Suppose first that $a/b < 1$. For some large $M > 1$, $K = \Lambda_{\mathcal{P}} \times [-M,M]$ is invariant under F, $F(K) \subset K$. The projection $K \to \Lambda_{\mathcal{P}}$ factors $F|_K$ to $\sigma \cdot \phi_{a/b} : \Lambda_{\mathcal{P}} \to \Lambda_{\mathcal{P}}$. Since entropy cannot increase under factoring, $h(F|_K) \ge h(\sigma \cdot \phi_{a/b}) = \log b$, where we also used Lemma 7.4. Bowen's theorem (Theorem 17 in [4], c.f. Theorem 5), combined with Lemma 7.3, yields

$$h(F|_K) \le h(\sigma \cdot \phi_{a/b}) + \sup_{\omega \in \Lambda_{\mathcal{P}}} h(F, \omega \times [-M,M]) = \log b + 0.$$

Thus $h(f) = h(F,K) = h(F|_K) = \log b$, and we are done for $a/b \ne 1$.

Suppose now $a/b = 1$, that is $a = b = 1$. One easily checks that Theorem 5 (formulated and shown in the appendix) can be applied to $X := \Lambda_{\mathcal{P}} \times \mathbf{R}$ with $G := \mathbf{Z}$ acting as described in Section 2 and $K := \Lambda_{\mathcal{P}} \times [-1,1]$, $Y := \Lambda_{\mathcal{P}}$, $T := F$, and $S := \sigma \cdot \phi_1$. Therefore, we obtain

$$h(f) = h(F,K) \le h(\sigma \cdot \phi_1) + \sup_{\omega \in \Lambda_{\mathcal{P}}} h(F, \omega \times [-1,1]) = \log 1 + 0 = 0.$$

□

8 Global Shadowing (Proof of Theorem 3)

We shall prove in this section Theorem 3, that is we fix a homeomorphism $f : \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$ with the winding ratio $a/b \ne 1$ and show that f is semi-conjugated to the affine homeomorphism $g : \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$ isotopic to f provided by Theorem 1, (i.e. we construct a continuous surjective map $h : \mathcal{S}_{\mathcal{P}} \to G$ such that $h \circ f = g \circ h$). As before, we shall argue mostly at the level of lifts F and $G : \Lambda_{\mathcal{P}} \times \mathbf{R} \to \Lambda_{\mathcal{P}} \times \mathbf{R}$ of f and g. By Corollary 6.1, these exist as a pair of equivariantly homotopic maps of the form

$$F(\omega,x) = (\phi(\omega), \lambda x + \delta(\omega,x)) \quad \text{and} \quad G(\omega,x) = (\phi(\omega), \lambda x + \Delta(\omega)),$$

where $\lambda = \pm a/b$ according to whether f is orientation preserving or not, and $\Delta(\cdot) = a/b \cdot I_b(\cdot)$. Note that we may assume that $|\lambda| > 1$ at the expense of passing to f^{-1} if necessary. We shall also fix $\mu \in (1,|\lambda|)$ and set $C := (\sup|\delta| + \sup|\Delta|)/(|\lambda| - \mu) < +\infty$.

Let us fix (for a moment) points $(\omega,x) \in \Lambda_{\mathcal{P}} \times \mathbf{R}$ and $(\omega,y) \in \Lambda_{\mathcal{P}} \times \mathbf{R}$ together with the sequences

$$x_n := \mathrm{pr}_{\mathbf{R}}(F^n(\omega,x)) \quad \text{and} \quad y_n := \mathrm{pr}_{\mathbf{R}}(G^n(\omega,y)),\ n \ge 0.$$

In adopting Katok's notion of global shadowing and by emulating [9], we say that (ω, x) is C-*shadowed* by (ω, y), and write $(\omega, x) \rightsquigarrow (\omega, y)$, iff $|x_n - y_n| \leq C$ for all $n \geq 0$. The following proposition shows that $H : \Lambda_{\mathcal{P}} \times \mathbf{R} \to \Lambda_{\mathcal{P}} \times \mathbf{R}$ that associates to (ω, x) its C-shadow (ω, y) is well defined and factors to the sought after semi-conjugacy h.

Proposition 8.2 *(i) For any $(\omega, x) \in \Lambda_{\mathcal{P}} \times \mathbf{R}$, there is a unique (ω, y) that C-shadows (ω, x), i.e. $(\omega, x) \rightsquigarrow (\omega, y)$.*
(ii) $H : \Lambda_{\mathcal{P}} \times \mathbf{R} \to \Lambda_{\mathcal{P}} \times \mathbf{R}$, $(\omega, x) \rightsquigarrow (\omega, y)$, is continuous.
(iii) $D \circ H = H \circ D$ so that H factors to a map $h : \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$, $\pi \circ H = h \circ \pi$.
(iv) $H \circ F = G \circ H$ and $h \circ f = g \circ h$.
(v) h is surjective.

Proof of Proposition 8.2. (i) Fix $(\omega, x) \in \Lambda_{\mathcal{P}} \times \mathbf{R}$. Let $\rho = 1$ if f is orientation preserving or $\rho = -1$ otherwise (so that $\lambda = \rho \cdot a/b$). Consider the open sets

$$E^- := \{y \in \mathbf{R} : \ \rho^n y_n < \rho^n x_n - C \text{ for some } n \geq 0\}$$

and

$$E^+ := \{y \in \mathbf{R} : \ \rho^n y_n > \rho^n x_n + C \text{ for some } n \geq 0\}.$$

We have to show that $\mathbf{R} \setminus (E^- \cup E^+)$ is a single point as this is the set of y's for which $(\omega, x) \rightsquigarrow (\omega, y)$. The choice of the constant C guarantees that, given $k \geq 0$,

$$\rho^k y_k > \rho^k x_k + C \quad \text{implies} \quad \rho^n y_n > \rho^n x_n + \mu^{n-k} C \text{ for } n \geq k$$

and

$$\rho^k y_k < \rho^k x_k - C \quad \text{implies} \quad \rho^n y_n < \rho^n x_n - \mu^{n-k} C \text{ for } n \geq k.$$

Hence E^- and E^+ are disjoint. The monotonicity of the maps $x \to \lambda x + \delta(\omega, x)$ and $x \to \lambda x + \Delta(\omega)$ implies that E^- contains with each y all points in $(-\infty, y]$, i.e. E^- is an infinite segment $E^- = (-\infty, y^-)$ for some $y^- \in \mathbf{R}$. Similarly, $E^+ = (y^+, \infty)$ for some $y^+ \in \mathbf{R}$. Since $y^-, y^+ \notin E^- \cup E^+$, we have $(\omega, x) \rightsquigarrow (\omega, y^-)$ and $(\omega, x) \rightsquigarrow (\omega, y^+)$; however, $y_n^+ - y_n^- = \lambda^n (y^+ - y^-)$ and thus necessarily $y^+ = y^-$, which ends the proof of (i).

(ii) Fix an arbitrary $\epsilon > 0$. Pick $n \in \mathbf{N}$ so that $3C/|\lambda|^n \leq \epsilon$. Consider (ω, x) and $(\tilde{\omega}, \tilde{x}) \in \Lambda_{\mathcal{P}} \times \mathbf{R}$. Take $\delta > 0$ such that $d_{\Lambda_{\mathcal{P}} \times \mathbf{R}}((\omega, x), (\tilde{\omega}, \tilde{x})) < \delta$ forces $|\tilde{x}_k - x_k| \leq C$ and $\Delta(\phi^k(\omega)) = \Delta(\phi^k(\tilde{\omega}))$ for $k = 0, ..., n$ — note that Δ is locally constant. For $(\omega, y) := H(\omega, x)$ and $(\tilde{\omega}, \tilde{y}) := H(\tilde{\omega}, \tilde{x})$, we see then that

$$|\tilde{y}_n - y_n| \leq |\tilde{y}_n - \tilde{x}_n| + |\tilde{x}_n - x_n| + |y_n - x_n| \leq 3C$$

and

$$|\tilde{y}_n - y_n| = |\lambda|^n |\tilde{y} - y|,$$

which combine to yield $|\tilde{y} - y| \leq 3C/|\lambda|^n \leq \epsilon$. This establishes continuity of H.

(iii) Since F is a lift, we have for all $(\omega, x) \in \Lambda_{\mathcal{P}} \times \mathbf{R}$ that $F \circ D(\omega, x) = D^{k(\omega)} \circ F$, where $k: \Lambda_{\mathcal{P}} \to \mathbf{Z}$. Because G is equivariantly homotopic to F, unique path lifting forces that $G \circ D(\omega, x) = D^{k(\omega)} \circ G$ with the same k. It follows that if $(\omega, x) \rightsquigarrow (\omega, y)$ then $D(\omega, x) \rightsquigarrow D(\omega, y)$. This establishes equivariance of H and secures the existence of factor h.

(iv) is a simple consequence of the definition of C-shadowing.

(v) Fix $(\omega, y) \in \Lambda_{\mathcal{P}} \times \mathbf{R}$. We can reverse the roles of f and g in the proof of part (i) and conclude that the orbit of (ω, y) under G is C-shadowed by an orbit under F of some $(\omega, x) \in \Lambda_{\mathcal{P}} \times \mathbf{R}$. (Uniqueness of (ω, x) may be lost due to non-linearities in F, c.f. Remark 8.2 below.) □

Remark 8.2 *The fibers of the semi-conjugacy h are arcs of the form $h^{-1}(z) = \pi(\omega \times (x^-, x^+))$. In particular, h is guaranteed to be 1-1, and thus a conjugacy if "f is C^1-close enough to g" so that F is expanding the fibers. This happens for example when $x \mapsto \lambda x + \delta(\omega, x)$ has Lipschitz inverse with a constant $L < 1$ that is uniform in ω.*

Remark 8.3 *Because $h^{-1}(z)$'s are arcs, the fiber entropy is zero in Bowen's Theorem 17 in [4]; and we conclude $h(f) = h(g)$. This leads to an alternative proof of Theorem 2 for f with the winding ratio $a/b \neq 1$.*

9 Knaster continua

Knaster continua can be realized by identifying each point of a solenoid with its inverse, i.e. we have $\mathbf{Z}_2$ acting via the involution $r: \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$, $r: z \mapsto z^{-1}$, and the $\mathcal{P}$-adic Knaster continuum is

$$\mathcal{K}_{\mathcal{P}} := \mathcal{S}_{\mathcal{P}}/\mathbf{Z}_2.$$

We shall denote by $\mu: \mathcal{S}_{\mathcal{P}} \to \mathcal{K}_{\mathcal{P}}$ the natural projection. Observe that the restriction to $\mathcal{S}_{\mathcal{P}}$ of the product of *the circle flattening maps*, $\prod_{k=0}^{\infty} \mathbf{S} \to \prod_{k=0}^{\infty} [-1, 1]$ where $(e^{i\theta_k})_{k=0}^{\infty} \mapsto (\cos \theta_k)_{k=0}^{\infty}$, establishes a homeomorphism of $\mathcal{K}_{\mathcal{P}}$ with the inverse limit of interval maps

$$\mathcal{K}_{\mathcal{P}} \cong \lim_{\leftarrow} (P_{p_k}: \ [-1, 1] \to [-1, 1])$$

where $P_p(\cos \theta) = \cos^p \theta$ is the Chebyshev polynomial and has exactly p monotonic laps each of which is surjective. One can now verify that thus defined $\mathcal{K}_{\mathcal{P}}$ is homeomorphic to the Knaster continuum pictured in the introduction.

The projection μ fails to be a cover at the fixed points of r, and it is easy to verify that we have two cases:

Even Case: *$\mathcal{P}$ contains infinitely many 2's* — then the neutral element e is the only fixed point of r;

Odd Case: $\mathcal{P}$ *contains only finitely many* 2*'s* — then, besides e, there is exactly one more fixed point $e^* = (1, \dots 1, -1, -1, -1, \dots)$ where the number of the leading 1's equals the maximal k for which $p_k = 2$ (or there are no 1's if all p_k's are odd). In the "odd case", $e^* \cdot e^* = e$; and the two points e and e^* are interchanged by the translation $s_{e^*} : z \mapsto e^* \cdot z$, which factors to an involution $\hat{s} : \mathcal{K}_\mathcal{P} \to \mathcal{K}_\mathcal{P}$.

We distinguish a class of *standard homeomorphisms of* $\mathcal{K}_\mathcal{P}$ (c.f. [3]) that are the factors (through μ) of the affine homeomorphisms of $\mathcal{S}_\mathcal{P}$. Since $g_{a/b,\sigma} \circ r = r \circ g_{a/b,r(\sigma)}$, an affine homeomorphism $g_{a/b,\sigma}$ commutes with r iff $r(\sigma) = \sigma$ i.e. $\sigma \in \{e, e^*\}$. The standard homeomorphisms come then in two kinds: those fixing $\mu(e)$,

$$\hat{g}_{a/b,+} : \mathcal{K}_\mathcal{P} \to \mathcal{K}_\mathcal{P}, \qquad \mu \circ g_{a/b,e} = \hat{g}_{a/b,+} \circ \mu,$$

and those interchanging e and e^*,

$$\hat{g}_{a/b,-} : \mathcal{K}_\mathcal{P} \to \mathcal{K}_\mathcal{P}, \qquad \mu \circ g_{a/b,e^*} = \hat{g}_{a/b,-} \circ \mu$$

— the latter are well defined only in the "odd case". As before, a, b run over pairs of $\mathcal{P}$-recurrent co-prime natural numbers. Clearly, $\hat{g}_{a/b,-} = \hat{s} \circ \hat{g}_{a/b,+}$.

THEOREM 4 *If* $\hat{f} : \mathcal{K}_\mathcal{P} \to \mathcal{K}_\mathcal{P}$ *is a homeomorphism, then*
(i) there are $\mathcal{P}$*-recurrent and co-prime* $a, b \in \mathbf{N}$ *such that* $\hat{f}$ *is isotopic to a standard homeomorphism* g*,* $g = \hat{g}_{a/b,-}$ *or* $g = \hat{g}_{a/b,+}$*;*
(ii) $\hat{f}$ *is semi-conjugated to* g *provided* $a/b \neq 1$*;*
(iii) $\hat{f}$ *has topological entropy* $h(\hat{f}) = h(g) = \log \max\{a, b\}$.

We shall reduce the theorem to our results for $\mathcal{S}_\mathcal{P}$ via the following proposition.

Proposition 9.3 *For any homeomorphism* $\hat{f} : \mathcal{K}_\mathcal{P} \to \mathcal{K}_\mathcal{P}$*, there exists a unique orientation preserving homeomorphism* $f : \mathcal{S}_\mathcal{P} \to \mathcal{S}_\mathcal{P}$ *that is a lift of* $\hat{f}$*,* $\mu \circ f = \hat{f} \circ \mu$.

It follows that $r \circ f$ is the only other lift of $\hat{f}$. Before starting the proof, we put a metric $d_{\mathcal{K}_\mathcal{P}}$ on $\mathcal{K}_\mathcal{P}$ given on $u, v \in \mathcal{K}_\mathcal{P}$ by $d_{\mathcal{K}_\mathcal{P}}(u, v) := \min\{d_{\mathcal{S}_\mathcal{P}}(w, z) : \ \mu(w) = u, \ \mu(z) = v, \ w, z \in \mathcal{S}_\mathcal{P}\}$. We mention that, by using the $\mathcal{P}$-adic representation of $\Lambda_\mathcal{P}$, it is easy to verify that $\omega \mapsto \omega^{-1}$ is an isometry of $\Lambda_\mathcal{P}$ an that $\mathbf{Z}_2$ acts on $\mathcal{S}_\mathcal{P}$ isometrically so that μ is a local isometry at all points beside e and e^*. We will use the following lemma which expresses the idea that the points z and z^{-1} in $\mathcal{S}_\mathcal{P}$ can be distinguished by looking at the "orientation" of their composants projected to $\mathcal{K}_\mathcal{P}$.

Lemma 9.5 *There exists* $\rho > 0$ *such that if* $\epsilon < \rho$*, and* $\tau : [0, \infty) \to [0, \infty)$ *is a homeomorphism, and*

$$d_{\mathcal{K}_\mathcal{P}}(\mu(T^t z), \mu(T^{-\tau(t)} z)) < \epsilon \quad \textit{for all } t \geq 0,$$

then

$$d_{\mathcal{S}_\mathcal{P}}(z, z^{-1}) < 2\epsilon.$$

Proof: Since $\pi : \Lambda_{\mathcal{P}} \times \mathbf{R} \to \mathcal{S}_{\mathcal{P}}$ is a local isometry, there is $\rho > 0$ such that $d_{\mathcal{S}_{\mathcal{P}}}(T^t z, T^{-\tau(t)} z) = t + \tau(t)$ for $t, \tau(t) \in [0, \rho]$. Our hypothesis, $d_{\mathcal{K}_{\mathcal{P}}}(\mu(T^t z), \mu(T^{-\tau(t)} z)) = \min\left\{d_{\mathcal{S}_{\mathcal{P}}}\left(T^t z, T^{-\tau(t)} z\right), d_{\mathcal{S}_{\mathcal{P}}}\left(T^t z, (T^{-\tau(t)} z)^{-1}\right)\right\} < \epsilon < \rho$, applied at the instance when $t + \tau(t) = \epsilon$, yields that $d_{\mathcal{K}_{\mathcal{P}}}(\mu(T^t z), \mu(T^{-\tau(t)} z)) = d_{\mathcal{S}_{\mathcal{P}}}\left(T^t z, (T^{-\tau(t)} z)^{-1}\right) < \epsilon$. By the triangle inequality, $d_{\mathcal{S}_{\mathcal{P}}}(z, z^{-1}) \le d_{\mathcal{S}_{\mathcal{P}}}(z, T^t z) + d_{\mathcal{S}_{\mathcal{P}}}\left(T^t z, T^{\tau(t)}(z^{-1})\right) + d_{\mathcal{S}_{\mathcal{P}}}\left(T^{\tau(t)}(z^{-1}), z^{-1}\right) < t + \epsilon + \tau(t) = 2\epsilon$. □

Under the projection $\eta := \mu \circ \pi : \Lambda_{\mathcal{P}} \times \mathbf{R} \to \mathcal{K}_{\mathcal{P}}$, the lines $\omega \times \mathbf{R}$ map onto the composants of $\mathcal{K}_{\mathcal{P}}$, of which all are immersed $\mathbf{R}$'s except for the composants of $\mu(e)$ and $\mu(e^*)$ which are immersed half-lines $[0, \infty)$. We shall use the immersion $i_{\mathcal{S}_{\mathcal{P}}} : \mathbf{R} \to \mathcal{S}_{\mathcal{P}}$, $i_{\mathcal{S}_{\mathcal{P}}}(t) = \pi(e, t)$, and the immersion $i_{\mathcal{K}_{\mathcal{P}}} : [0, \infty) \to \mathcal{K}_{\mathcal{P}}$ $i_{\mathcal{K}_{\mathcal{P}}} = \mu \circ i_{\mathcal{S}_{\mathcal{P}}}|_{[0,\infty)}$.

Proof of Proposition 9.3: Fix a homeomorphism $\hat{f} : \mathcal{K}_{\mathcal{P}} \to \mathcal{K}_{\mathcal{P}}$. We may assume that $\hat{f}$ fixes $\mu(e)$; otherwise $\hat{s} \circ \hat{f}$ fixes $\mu(e)$ — and if f is its lift then $s_{e^*} \circ f$ is the lift of $\hat{f}$. We have a homeomorphism $\nu : [0, \infty) \to [0, \infty)$ given by $\nu := {i_{\mathcal{K}_{\mathcal{P}}}}^{-1} \circ \hat{f} \circ i_{\mathcal{K}_{\mathcal{P}}}$, which we use to define on a dense subset of $\mathcal{S}_{\mathcal{P}}$ a map $f_0 : i_{\mathcal{S}_{\mathcal{P}}}([0, \infty)) \to i_{\mathcal{S}_{\mathcal{P}}}([0, \infty))$ by $f_0(i_{\mathcal{S}_{\mathcal{P}}}(t)) = i_{\mathcal{S}_{\mathcal{P}}} \circ \nu(t)$, $t \ge 0$. We claim that f_0 is uniformly continuous so that it uniquely extends to a continuous map $f_1 : \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$. Note that then f_1 is a lift of $\hat{f}$ and that any such lift that preserves the orientation of the composant through e must coincide with f_1. Also f_1 must be a homeomorphism: an analogous reasoning applied to f^{-1} yields the inverse of f_1.

To prove the claim, fix $\epsilon > 0$ such that $\epsilon/2 < \rho$ where ρ is an in Lemma 9.5. There is $\delta > 0$ such that $d_{\mathcal{K}_{\mathcal{P}}}(\hat{f}(u), \hat{f}(v)) < \epsilon/8$ when $d_{\mathcal{K}_{\mathcal{P}}}(u, v) < \delta$, $u, v \in \mathcal{K}_{\mathcal{P}}$. Suppose that $d_{\mathcal{S}_{\mathcal{P}}}(i_{\mathcal{S}_{\mathcal{P}}}(x_1), i_{\mathcal{S}_{\mathcal{P}}}(x_2)) < \delta$. We shall show that $d_{\mathcal{S}_{\mathcal{P}}}(f_0 \circ i_{\mathcal{S}_{\mathcal{P}}}(x_1), f_0 \circ i_{\mathcal{S}_{\mathcal{P}}}(x_2)) < \epsilon$. Because translations are isometries of $\mathcal{S}_{\mathcal{P}}$, $d_{\mathcal{S}_{\mathcal{P}}}(i_{\mathcal{S}_{\mathcal{P}}}(x_1 + t), i_{\mathcal{S}_{\mathcal{P}}}(x_2 + t)) < \delta$; and thus also $d_{\mathcal{K}_{\mathcal{P}}}(i_{\mathcal{K}_{\mathcal{P}}}(x_1 + t), i_{\mathcal{K}_{\mathcal{P}}}(x_2 + t)) < \delta$ for all $t \ge 0$. Since we assumed that $\hat{f}(\mu(e)) = \mu(e)$, the composant of e is mapped onto itself, so (c.f. Fact 5.3)

$$\hat{f} \circ i_{\mathcal{K}_{\mathcal{P}}}(x_i + t) = i_{\mathcal{K}_{\mathcal{P}}}(y_i + \tau_i(t))$$

where $y_i = \nu(x_i)$ and $\tau_i : [0, \infty) \to [0, \infty)$ is a homeomorphism, $i = 1, 2$. By the choice of δ,

$$d_{\mathcal{K}_{\mathcal{P}}}(i_{\mathcal{K}_{\mathcal{P}}}(y_1 + \tau_1(t)), i_{\mathcal{K}_{\mathcal{P}}}(y_2 + \tau_2(t))) < \epsilon/8,\ t \ge 0. \tag{9.1}$$

If $d_{\mathcal{S}_{\mathcal{P}}}(i_{\mathcal{S}_{\mathcal{P}}}(y_1), i_{\mathcal{S}_{\mathcal{P}}}(y_2)) < \epsilon/8$, we are done. Otherwise, $d_{\mathcal{S}_{\mathcal{P}}}(i_{\mathcal{S}_{\mathcal{P}}}(y_1), i_{\mathcal{S}_{\mathcal{P}}}(-y_2)) < \epsilon/8$ from the definition of $d_{\mathcal{K}_{\mathcal{P}}}$ and (9.1) at $t = 0$. Therefore, $d_{\mathcal{S}_{\mathcal{P}}}(i_{\mathcal{S}_{\mathcal{P}}}(y_1 + t), i_{\mathcal{S}_{\mathcal{P}}}(-y_2 + t)) < \epsilon/8$ and so $d_{\mathcal{K}_{\mathcal{P}}}(\mu \circ i_{\mathcal{S}_{\mathcal{P}}}(y_1 + t), \mu \circ i_{\mathcal{S}_{\mathcal{P}}}(-y_2 + t)) < \epsilon/8$, for all $t \ge 0$. Together with (9.1), this yields that

$$d_{\mathcal{K}_{\mathcal{P}}}(\mu \circ i_{\mathcal{S}_{\mathcal{P}}}(y_2 + \tau(t)), \mu \circ i_{\mathcal{S}_{\mathcal{P}}}(-y_2 + t)) < \epsilon/8 + \epsilon/8 = \epsilon/4$$

where $\tau := \tau_2 \circ \tau_1^{-1}$ and $t \ge 0$. From Lemma 9.5, $d_{\mathcal{S}_{\mathcal{P}}}(i_{\mathcal{S}_{\mathcal{P}}}(y_2), i_{\mathcal{S}_{\mathcal{P}}}(-y_2)) < 2\epsilon/4 = \epsilon/2$, and so $d_{\mathcal{S}_{\mathcal{P}}}(i_{\mathcal{S}_{\mathcal{P}}}(y_1), i_{\mathcal{S}_{\mathcal{P}}}(y_2)) < 2\epsilon/2 = \epsilon$. □

Proof of Theorem 4.

(i) By Proposition 9.3, $\hat{f}$ lifts to a homeomorphism $f : \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$. As before, we may suppose that $\hat{f}(\mu(e)) = \mu(e)$ and $f(e) = e$, as otherwise one can consider $\hat{s} \circ \hat{f}$. Corollary 6.1 supplies a lift $F : \Lambda_{\mathcal{P}} \times \mathbf{R} \to \Lambda_{\mathcal{P}} \times \mathbf{R}$ of f, and a suitable adjustment by a deck transformation assures that $F(e, 0) = (e, 0)$, i.e. $\sigma = e$ in (6.1). We write a homotopy $F_t : \Lambda_{\mathcal{P}} \times \mathbf{R} \to \Lambda_{\mathcal{P}} \times \mathbf{R}$, $t \in [0, 1]$, given by

$$F_t(\omega, x) = \left(\phi_{a/b}(\omega), \frac{a(x + I_b(\omega))}{b} + t \cdot \delta(\omega, x) \right)$$

where a, b are co-prime $\mathcal{P}$-recurrent and δ is continuous and equivariant, $\delta \circ D = \delta$. We claim that F_t factors through $\mu \circ \pi$ to a homotopy $\hat{f}_t : \mathcal{K}_{\mathcal{P}} \to \mathcal{K}_{\mathcal{P}}$, which connects $\hat{f}$ to the standard map $\hat{g}_{a/b,+}$. To prove the claim, it is to show that there is $\kappa : \Lambda_{\mathcal{P}} \to \mathbf{Z}$ such that

$$F_t \circ R = D^{\kappa(\cdot)} \circ R \circ F_t \tag{9.2}$$

for all $t \in [0, 1]$. The above equation, explicitly written, amounts to the two following identities imposed for all $(\omega, t) \in \Lambda_{\mathcal{P}} \times \mathbf{R}$:

$$\phi_{a/b}(\omega^{-1}) = \gamma^{k(\omega)} \cdot \left(\phi_{a/b}(\omega)\right)^{-1} \tag{9.3}$$

and

$$a/b \cdot I_b(\omega^{-1}, -x) + a/b \cdot I_b(\omega, x) + k(\omega) = -t \cdot \left(\delta(\omega^{-1}, -x) + \delta(\omega, x)\right). \tag{9.4}$$

Both F_0 and F_1 factor to a map of $\mathcal{K}_{\mathcal{P}}$: F_0 covers $\hat{g}_{a/b,+}$ and F_1 covers $\hat{f}$. Thus (9.2) is satisfied for $t = 0, 1$ — and this is with the same κ determined uniquely by (9.3). Moreover, inspection of (9.4) at $t = 0$ and $t = 1$ yields $\delta \circ R = -\delta$. It follows that (9.4), and thus (9.2), holds for all $t \in [0, 1]$.
(ii) Let F_0 and F_1 be as in the proof of (i) above. Suppose that $H : \Lambda_{\mathcal{P}} \times \mathbf{R} \to \Lambda_{\mathcal{P}} \times \mathbf{R}$ is the semi-conjugacy, $H \circ F_1 = H \circ F_0$ between F_1 and F_0 obtained via global shadowing as in Proposition 8.2. From (9.2), if (ω, y) shadows (ω, x) then $R(\omega, y)$ shadows $R(\omega, x)$, i.e. $H \circ R = R \circ H$. It follows that H descends to a semi-conjugacy $h : \mathcal{K}_{\mathcal{P}} \to \mathcal{K}_{\mathcal{P}}$ such that $h \circ \hat{f}_1 = h \circ \hat{f}_0$.
(iii) Because μ is finite-to-one, the entropy of $\hat{f}$ coincides with that of its lift $f : \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}}$ by Theorem 17 in [4] (c.f. the appendix). Theorem 2 completes the proof. □

10 Appendix: Bowen's Theorem

The following result generalizes Theorem 17 in [4], which can be obtained as the special case when $X = K$ is compact and G is a trivial group.

THEOREM 5 *Suppose the following hypotheses are satisfied:*

(i) (X, d) and (Y, e) are metric spaces, Y is compact, and $\pi : X \to Y$ is continuous and surjective;

(ii) a group G acts on X and Y by isometries (on the left), and $\forall_{g\in G}\ g\circ\pi = \pi\circ g$;

(iii) $K\subset X$ is a compact set intersecting every orbit of G and there are $\eta, b > 0$ with $\#\{g\in G:\ g(A)\cap K\neq\emptyset\}\leq b$ for all $A\subset X$ with $\mathrm{diam}(A)\leq\eta$;

(iv) $T: X\to X$ and $S: Y\to Y$ are continuous and $\pi\circ T = S\circ\pi$;

(v) $\forall_{g\in G}\exists_{h\in G}\ T\circ g = h\circ T$.

Then we have the following inequality for topological entropies

$$h(T, K)\leq h(S) + \sup_{y\in Y} h(T, \pi^{-1}(y)\cap K). \tag{10.1}$$

In applications, the natural projection $\pi_G : X\to X/G$ is typically a covering onto a compact Hausdorff space, and then (iii) is satisfied by any compact $K\subset X$ with $\pi_G(K) = X/G$. For $t : X/G\to X/G$ obtained as the quotient of T by G, (10.1) yields then

$$h(t)\leq h(S) + \sup_{y\in Y} h(T, \pi^{-1}(y)) \tag{10.2}$$

because $h(T, \pi^{-1}(y)\cap K)\leq h(T, \pi^{-1}(y))$ and $h(t)\leq h(T, K)$. (In fact, $h(t) = h(T, K)$.)

Example: Fix $\omega, \gamma\in\mathbf{R}$ with γ irrational and let ϕ be a $\mathbf{Z}^2$-periodic function on $\mathbf{R}^2$, e.g. $\phi(x_1, x_2) = \frac{1}{4}(\sin(2\pi x_1) + \sin(2\pi x_2) + 5)$. Let $X = \mathbf{R}/\mathbf{Z}\times\mathbf{R}$ and $T(x_1, x_2) := (x_1, x_2) + (\omega, 0) + \phi(x_1, x_2)(1, \gamma)$ where x_1 is taken mod 1. Set $Y = \mathbf{R}/\mathbf{Z}$ and $S(x_1) = x_1 + \omega$ so that $\pi\circ T = S\circ\pi$ for $\pi(x_1, x_2) = x_1 - x_2/\gamma$. Finally, let the action of $G = \mathbf{Z}$ on X be generated by $(x_1, x_2)\mapsto(x_1, x_2 + 1)$ and take $K = \mathbf{R}/\mathbf{Z}\times[0, 1]$. Taking the quotient of T by G yields $t : \mathbf{R}^2/\mathbf{Z}^2\to\mathbf{R}^2/\mathbf{Z}^2$ that is a homotopic to the identity torus map that permutes leaves of a dense foliation. If t is a homeomorphism, then $h(T, \pi^{-1}(y)) = 0$ (c.f. Lemma 7.3), and (10.2) implies that $h(t) = 0$.

Question: Is the rotation set of t a single point in the case when t is a homeomorphism and ω, γ, and 1 are rationally independent?

In the rationally dependent case, the answer can be extracted from [13].

Proof of Theorem 5. The argument is an extension of that of Bowen and we use many definitions and notations from [4]. In particular, $e_n(x, y) := \max_{i=0}^{n-1} e(S^i(x), S^i(y))$ and $B_n(z, r, T) := \{x\in X : d(T^i(x), T^i(z)) < r,\ 0\leq i < n\}$.

Set $a := \sup_{y\in Y} h(T, \pi^{-1}(y)\cap K)$. Let $\alpha, M > 0$ be arbitrary and let $\epsilon > 0$ be arbitrary yet small enough so that $d(T(x), T(\tilde{x}))\leq\eta/2$ if $d(x, \tilde{x}) < 2\epsilon$. For each $y\in Y$, pick a $(m(y), \epsilon)$-spanning set $E_y\subset X$ for $\pi^{-1}(y)\cap K$ so that

$$m(y)\geq M \qquad\text{and}\qquad a + \alpha\geq h(T, \pi^{-1}(y)\cap K)\geq\frac{1}{m(y)}\ln\#E_y. \tag{10.3}$$

Set $U_y := \bigcup_{z\in E_y} B_{m(y)}(z, 2\epsilon, T)$. Observe that $\pi^{-1}(y) \cap K \subset U_y$; and compactness of K assures that

$$\exists_{\gamma=\gamma(y)>0}\ \pi^{-1}(B(y,\gamma)) \cap K \subset U_y, \tag{10.4}$$

where $B(y,\gamma)$ is the ball of radius γ about y. Fix $y_1, \dots, y_r$ so that $\{B(y_i, \gamma(y_i))\}_{i=1}^r$ is a covering of Y, and let $\delta > 0$ be the Lebesgue number of this covering.

Consider an arbitrary $n \in \mathbf{N}$, and let E_n be an (n, δ)-spanning set for Y. Fix $x \in E_n$. Consider also an arbitrary $(n, 4\epsilon)$-separated subset $F \subset X \cap K$. We shall estimate the cardinality of the set $F_x := \{p \in F : \ e_n(\pi(p), x) \le \delta\}$. To this end, we shall assign to each $p \in F_x$ a sequence $(\mathbf{g}, \mathbf{z}) = (g_0, \dots, g_q; z_0, \dots, z_q)$ where $q = q(p)$ depends on p, $g_j \in G$, and $z_{i_j} \in E_{y_{i_j}}$ for some $i_j \in \{1, \dots, r\}$, $j = 0, \dots, q$. We shall refer to $(\mathbf{g}, \mathbf{z})$ as *the code of* p. Fix $p \in F_x$. We proceed recursively.

Step 0: Define $t_0 := 0$, $p_0 := p$, and $x_0 := x$. Set $g_0 = \mathbf{e}$ where $\mathbf{e}$ is the neutral element of G, and select i_0 so that $\overline{B}(x, \delta) \subset B(y_{i_0}, \gamma(y_{i_0}))$. By (10.4), we can pick $z_0 \in E_{y_{i_0}}$ with

$$d_{m(y_{i_0})}(z_0, p_0) < 2\epsilon. \tag{10.5}$$

Step s+1: Suppose that $(g_0, \dots, g_s; z_0, \dots, z_s)$ are already defined, and so are $(i_0, \dots, i_s)$. If

$$t_{s+1} := t_s + m(y_{i_s}) = m(y_{i_0}) + \ldots + m(y_{i_s}) \ge n, \tag{10.6}$$

we stop, set $q = s$, and $(\mathbf{g}, \mathbf{z}) = (g_0, \dots, g_q; z_0, \dots, z_q)$ is the code of p. Otherwise, when $t_{s+1} < n$, we define $p_{s+1} = T^{m(y_{i_s})}(p_s) = T^{t_{s+1}}(p)$, $x_{s+1} = S^{m(y_{i_s})}(x_s) = S^{t_{s+1}}(x)$, and proceed as follows. Pick $g_{s+1} \in G$ so that $g_{s+1}(p_{s+1}) \in K$. Select i_{s+1} so that $\overline{B}(g_{s+1}(x_{s+1}), \delta) \subset B(y_{i_{s+1}}, \gamma(y_{i_{s+1}}))$.

From $e_n(\pi(p), x) \le \delta$, $e(S^{t_{s+1}} \circ \pi(p), S^{t_{s+1}}(x)) \le \delta$; and $e(S^{t_{s+1}} \circ \pi(p), S^{t_{s+1}}(x)) = e(g_{s+1} \circ S^{t_{s+1}} \circ \pi(p), g_{s+1} \circ S^{t_{s+1}}(x)) = e(g_{s+1} \circ \pi \circ T^{t_{s+1}}(p), g_{s+1}(x_{s+1})) = e(\pi \circ g_{s+1}(p_{s+1}), g_{s+1}(x_{s+1}))$ by the hypotheses (ii) and (iv). Therefore,

$$e(\pi \circ g_{s+1}(p_{s+1}), g_{s+1}(x_{s+1})) \le \delta \tag{10.7}$$

By (10.7), the choice of i_{s+1}, and (10.4), we can pick $z_{s+1} \in E_{y_{i_{s+1}}}$ with

$$d_{m(y_{i_{s+1}})}(z_{s+1}, g_{s+1}(p_{s+1})) < 2\epsilon. \tag{10.8}$$

This ends the iteration step.

Claim 1 *If $p, \tilde{p} \in F_x$ have the same code $(\mathbf{g}, \mathbf{z}) = (g_0, \dots, g_q; z_0, \dots, z_q)$, then $p = \tilde{p}$.*

Indeed, for $s = 0, \ldots, q$, we have $d_{m(y_{i_s})}(z_s, g_s(p_s)) < 2\epsilon$ — c.f. (10.5) and (10.8) — that is $d(T^k(z_s), T^k \circ g_s(p_s)) < 2\epsilon$ for $k = 0, \ldots, m(y_{i_s}) - 1$. Hypothesis (v), supplies $h_{s,k} \in G$, $0 \leq s \leq q$, $0 \leq k < m(y_{i_s})$, with $T^k \circ g_s = h_{s,k} \circ T^k$; and we obtain

$$d(h_{s,k}^{-1} \circ T^k(z_s), T^k(p_s)) = d(T^k(z_s), h_{s,k} \circ T^k(p_s)) < 2\epsilon, \tag{10.9}$$

since G acts by isometries (hypothesis (ii)). From (10.9) and its analogue for $\tilde{p}$, $d(T^i(p), T^i(\tilde{p})) < 4\epsilon$ for $i = 0, \ldots, t_{q+1} - 1$. Since $t_{q+1} - 1 \geq n - 1$ by (10.6), and $p, \tilde{p} \in F$, we must have $p = \tilde{p}$, which proves the claim.

Set $\mu := \max_{i=1}^r m(y_i)$.

Claim 2 $\#\{(\mathbf{g}, \mathbf{z}) : \ (\mathbf{g}, \mathbf{z})$ *is a code of some* $p \in F_x\} \leq b^{n/M} e^{(a+\alpha)(n+\mu)}$.

Note that the two claims put together imply $\#F \leq \sum_{x \in E_n} \#F_x \leq \#E_n \cdot b^{n/M} e^{(a+\alpha)(n+\mu)}$. Since n, ϵ, F and E_n were arbitrary, we conclude that $h(T, K) \leq h(S) + \ln b/M + (a + \alpha)$ so that (10.1) follows from arbitrariness of $M, \alpha > 0$.

It remains to demonstrate Claim 2. To ease the exposition, we build a weighted graph as follows. For vertices we take all $(\mathbf{g}; \mathbf{z}) = (g_0, \ldots, g_s; z_0, \ldots, z_s)$ that are initial segments of a code, i.e. $(g_0, \ldots, g_q; z_0, \ldots, z_q)$ with $q \geq s$ is a code of some $p \in F_x$ for certain $g_{s+1}, \ldots, g_q$ and $z_{s+1}, \ldots, z_q$. We also attach to each such vertex a weight equal to $m(y_{i_s})$ and call s the level of the vertex. Then we place a directed edge from $(\mathbf{g}, \mathbf{z}) = (g_0, \ldots, g_s; z_0, \ldots, z_s)$ to $(\tilde{\mathbf{g}}, \tilde{\mathbf{z}}) = (\tilde{g}_0, \ldots, \tilde{g}_{s+1}; \tilde{z}_0, \ldots, \tilde{z}_{s+1})$ iff $g_i = \tilde{g}_i$ and $z_i = \tilde{z}_i$ for $0 \leq i \leq s$; and we say that $(\tilde{\mathbf{g}}, \tilde{\mathbf{z}})$ follows $(\mathbf{g}, \mathbf{z})$. The resulting graph $\mathcal{G}$ is clearly a collection of directed trees.

Claim 3 *Fix a vertex* $(\mathbf{g}; \mathbf{z}) = (g_0, \ldots, g_s; z_0, \ldots, z_s)$ *and consider the set* V *of all vertices* $(\tilde{\mathbf{g}}; \tilde{\mathbf{z}}) = (g_0, \ldots, g_{s+1}; z_0, \ldots, z_{s+1})$ *following* $(\mathbf{g}; \mathbf{z})$. *There are at most* b *different* $g_{s+1} \in G$ *that appear in* V. *Moreover, for* $g \in G$, *all vertices in* $V_g := \{(\tilde{\mathbf{g}}; \tilde{\mathbf{z}}) \in V : \ g_{s+1} = g\}$ *have the same weight equal to some* m *and* $\#V_g \leq e^{(a+\alpha)m}$.

For a proof, let B collect all $p \in F_x$ with an initial segment of their code coinciding with $(\mathbf{g}; \mathbf{z})$. Setting $k = m(y_{i_s}) - 1$ in (10.9) yields $d(h_{s,k}^{-1} \circ T^k(z_s), T^{t_{s+1}-1}(p)) < 2\epsilon$ for all $p \in B$. By the choice of ϵ, $d(T \circ h_{s,k}^{-1} \circ T^k(z_s), T^{t_{s+1}}(p)) < \eta/2$ for $p \in B$ so that $\mathrm{diam}(A) \leq \eta$ for $A := T^{t_{s+1}}(B)$. Any g_{s+1} appearing in V has $p \in B$ with $g_{s+1}(p_{s+1}) \in K$ so that $g_{s+1}(A) \cap K \neq \emptyset$. Hypothesis (iii) allows then for at most b possibilities for g_{s+1}. Once g_{s+1} is fixed so is i_{s+1} and there are at most $\#E_{y_{i_{s+1}}} \leq e^{(a+\alpha)m(y_{i_{s+1}})}$ possibilities for $z_{s+1} \in E_{y_{i_{s+1}}}$. Also, all the vertices in $V_{g_{s+1}}$ have the same weight $m(y_{i_{s+1}})$. This ends the proof of Claim 3.

Each code $(\mathbf{g}; \mathbf{z}) = (g_0, \ldots, g_q; z_0, \ldots, z_q)$ of $p \in F_x$ determines a unique maximal path in $\mathcal{G}$ starting at some root (a level-0 vertex). The weight of such path is $t_{q+1} = m(y_{i_0}) + \ldots + m(y_{i_q}) = t_q + m(y_{i_q}) < n + \mu$. Claim 2 is then a consequence of Claim 3 and the following general lemma.

Lemma 10.6 *Suppose that* $\mathcal{G}$ *is a collection of trees with weighted vertices and there are* $b, M, A > 0$ *so that* *(a) no vertex is lighter than M;*

(b) there is m_0 so that there are at most e^{Am_0} roots and each root is heavier than m_0;

(c) for any vertex, all vertices following it can be grouped into sets $U_1, \ldots, U_{\tilde{b}}$, $\tilde{b} \leq b$, so that $\#U_i \leq e^{Am_i}$ and each vertex in U_i is heavier than m_i for some $m_i \geq 0$, $i = 1, \ldots, \tilde{b}$.

If no path in $\mathcal{G}$ is heavier than N, then the number of different maximal paths (starting at a root) does not exceed $b^{N/M}e^{AN}$.

Proof. We proceed by induction on the height h defined as the maximal level of a vertex in $\mathcal{G}$. If $h = 0$, then $\mathcal{G}$ has only roots; their number does not exceed e^{Am_0} where $N \geq m_0$ by (b). If $h > 0$, for any fixed root R that is followed by some level-1 vertices, we group those vertices into $U_1, \ldots, U_{\tilde{b}}$, $\tilde{b} \leq b$, as stipulated by hypothesis (c). For every maximal path P in $\mathcal{G}$ starting at R, there is a unique $i \in \{1, \ldots, \tilde{b}\}$ so that P passes through some $W \in U_i$; let Q be the maximal subpath of P that starts at W. There are at most $b^{(N-M)/M}e^{A(N-m_0)}$ possibilities for Q for any fixed i since, by (c), the induction hypothesis applies to the collection $\mathcal{G}_i$ of the maximal subtrees of $\mathcal{G}$ that are rooted at the vertices in U_i. By summing over all roots and all i, we see that the number of possibilities for P cannot exceed

$$e^{Am_0} \cdot b \cdot b^{N/(N-M)}e^{A(n-m_0)} = b^{n/M}e^{An}.$$

□

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