Horseshoes and the Conley Index Spectrum

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Abstract

Given a continuous map on a locally compact metric space and an isolating neighborhood which is decomposed into two disjoint isolating neighborhoods, it is shown that the spectral information of the associated Conley indices is sufficient to conclude the existence of a semi-conjugacy onto the full shift dynamics on two symbols.

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1 Introduction

Algebraic topology has provided significant techniques for the investigation of the dynamics of nonlinear systems. The most celebrated result is probably that of the Lefschetz fixed point theorem which in its simplest form can be stated as follows [3]. Let $f : X \to X$ be a continuous map on a compact manifold and let $f^n : H^n(X) \to H^n(X)$ denote the induced map on n-th cohomology group. If $\sum_{n=0}^{\infty}(-1)^n \mathrm{tr}(f^n) \neq 0$, then f has a fixed point. This has proven to be a powerful technique for determining the existence of fixed points of continuous functions. Through the use of zeta functions the Lefschetz theorem becomes an effective tool for proving the existence of periodic orbits. In the context of gradient flows, Morse theory via the Morse inequalities, has proven to be an extremely important means of relating the local dynamics about hyperbolic fixed points with the global topology of the underlying manifold. Again, these ideas can be extended into systems with periodic orbits, i.e. Morse-Smale diffeomorphisms and flows, and further ([5]). With the discovery of the ubiquity of much more complicated dynamics, often referred to as chaos, and with the realization of the important role it plays in the long-term or asymptotic dynamics of nonlinear systems, attempts to characterize this behavior in terms of algebraic invariants have been made. Perhaps the most notable is M. Shub's entropy conjecture [23] which states that for $f : X \to X$ a C^1 map on a compact manifold a lower bound on the entropy of f is given by the spectral radius of f^* (for a proof when f is a C^∞ map see [27]).

In this paper a new algebraic characterization of chaotic dynamics is presented. In particular, it is shown that the spectrum of the Conley index can be used to guarantee the existence of horseshoes. To be more precise, recall that given a topological space X and a continuous function $f : X \to X$, a set $S \subset X$ is *invariant* if $f(S) = S$ and for $N \subset X$ its *maximal invariant set* is given by

$$\mathrm{Inv}(N, f) := \{x \in N \mid \exists\, \{x_n\}_{n=-\infty}^{\infty} \subset N \text{ such that } x_{n+1} = f(x_n) \text{ and } x_0 = x\}.$$

A compact set $N \subset X$ is an *isolating neighborhood* if

$$\mathrm{Inv}(N, f) \subset \mathrm{int} N$$

and an invariant set S is *isolated* if there exists an isolating neighborhood N such that $S = \mathrm{Inv}(N, f)$. The cohomological Conley index (discussed further below) associates to each isolated invariant set S a graded vector space $CH^*(S)$ and a graded linear automorphism $\chi^*(S) : CH^*(S) \to CH^*(S)$.

At the same time, recall that horseshoes are characterized by shift dynamics on finite sets of symbols. Let

$$\Sigma_2^+ := \{a = (a_n)_{n=0}^{\infty} \mid a_n \in \{0, 1\}\}$$

and let $T : \Sigma_2^+ \to \Sigma_2^+$ be the shift map

$$(T(a))_n = a_{n+1}.$$

Given a linear operator M, let $\sigma(M)$ denote the set of eigenvalues of M where the eigenvalues are repeated according to their multiplicity. Given two sets A and B, their *amalgamation* $A \amalg B$ is obtained by taking the union of A and B but by treating elements common to both A and B as distinct elements in $A \amalg B$. For example, $\{1\} \amalg \{1\} = \{1, 1\}$ where $\{1, 1\}$ is a two element set.

Finally, a set Q is *cyclic* if $Q = \cup_{i=1}^{I} Q_i$ where $Q_i \cap Q_j = \emptyset$ for $i \neq j$ and

$$Q_i = \{z \in \mathbf{C} \mid z^{n_i} = x_i, n_i \in \mathbf{Z}^+, x_i \in \mathbf{Z}\}.$$

The main result of this paper is the following theorem.

Theorem 1.1 *Let $f : X \to X$ be a continuous function on a locally compact metric space. Assume that $N \subset X$ is an isolating neighborhood which is the disjoint union of compact sets N_0 and N_1, i.e.*

$$N = N_0 \cup N_1.$$

Let $S = \mathrm{Inv}(N, f)$ and $S_i = \mathrm{Inv}(N_i, f)$, $i = 0, 1$. If, for some positive integer n, the component $CH^n(S)$ of $CH^*(S)$ is finite dimensional and

$$\sigma(\chi^n(S_0)) \amalg \sigma(\chi^n(S_1)) \not\subset \sigma(\chi^n(S))$$

or

$$\sigma(\chi^n(S)) \setminus (\sigma(\chi^n(S_0)) \amalg \sigma(\chi^n(S_1)))$$

is not cyclic, then there exist a positive integer d and a continuous surjection $\rho : S \to \Sigma_2^+$ such that the diagram

$$\begin{array}{ccc} \mathrm{Inv}(N, f) & \xrightarrow{f^d} & \mathrm{Inv}(N, f) \\ \Big\downarrow \rho & & \Big\downarrow \rho \\ \Sigma^+ & \xrightarrow{T} & \Sigma^+ \end{array}$$

commutes.

Because the amalgamation operation, rather than union, is used on the spectra of the Conley indices, the hypothesis of this theorem involves multiplicity of eigenvalues of the index isomorphisms.

To begin to put Theorem 1.1 in perspective, let us begin by contrasting it with the entropy conjecture. First, it is well known that the conjecture is false for arbitrary continuous functions [18], and hence, is restricted in its applicability to smooth manifolds. Theorem 1.1 applies to rather general topological spaces and does not require any smoothness. Second, except in dimensions less than or equal to two and under additional smoothness assumptions [11] there are no general results of the nature that positive entropy necessarily implies the existence of horseshoes. However, the existence of the semi-conjugacy onto the full shift of Theorem 1.1 implies positive entropy. Third, the entropy conjecture is necessarily global, i.e. it can only be applied to an entire compact invariant manifold. One of the strengths of Theorem 1.1 is that it can be applied "locally", i.e. N may be a small subset of X. Conversely, however, this theorem cannot be applied to an entire manifold if it is connected, which may in some cases be a severe handicap. Finally, to apply the entropy conjecture only one quantity, $f^* : H^*(X) \to H^*(X)$ needs to be calculated. To apply Theorem 1.1 one must be able to determine the sets N, N_i, and show that they are isolating neighborhoods. Furthermore, one must be able to compute the spectra (not just the radii) of the associated indices[1].

While the difficulty of finding isolating neighborhoods and computing the Conley index must not be dismissed it should also not be exaggerated. The index (and hence the spectrum) is invariant under continuation [1, 25]. Thus, as with degree theory, the actual computations may be performed in the setting of a simple system for which a homotopy to the problem of interest exists. Alternately, recent results indicate that it is possible to use the computer to rigorously compute isolating neighborhoods and their corresponding Conley indices [4, 13, 14, 15].

Theorem 1.1 is sharp. In [2] it is shown if $\sigma(\chi^n(S_0)) \amalg \sigma(\chi^n(S_1)) \subset \sigma(\chi^n(S))$ and $\sigma(\chi^n(S)) \setminus (\sigma(\chi^n(S_0)) \amalg \sigma(\chi^n(S_1)))$ is cyclic, there exists a dynamical system $f : X \to X$ with an isolating neighborhood N satisfying the hypothesis of Theorem 1.1 such that the set of trajectories in S which are not entirely contained in N_0 and N_1 is finite. In particular, if $\sigma(\chi^n(S)) \setminus (\sigma(\chi^n(S_0)) \amalg \sigma(\chi^n(S_1))) = \cup_{i=1}^{I} Q_i$, where Q_i is defined as above, then one can construct examples such that $S \setminus (S_0 \cup S_1)$ consists of distinct periodic orbits with periods n_i, $i = 1, \cdots, I$.

Finally, the Conley index is a topological generalization of the Morse index. In particular, if S is a hyperbolic fixed point, then from the Conley index of S one can determine the Morse index of S. The Conley index, of course, is defined for more general invariant sets and, in particular, does not require hyperbolicity. As was mentioned above, the power of Morse theory arises from the Morse inequalities which relate local and global information (see [5, Chapter 6]). Conley's connection matrices represent a generalization of this ([6, 7, 8, 9]). The key to all these relations is the existence of a Lyapunov function and hence the non-existence of recurrent dynamics between the different local objects. Theorem 1.1 shows that the Conley index extends the Morse index in yet another direction. One of the key steps in the proof of this result is Proposition 3.1. If one interprets this as a statement that the hypothesis of the theorem prevents the existence of a polynomial relation between the spectra of $\sigma(\chi^n(S_i))$, $i = 0, 1$,

[1] The algorithm described in the appendix of [10] remedies this last shortcoming

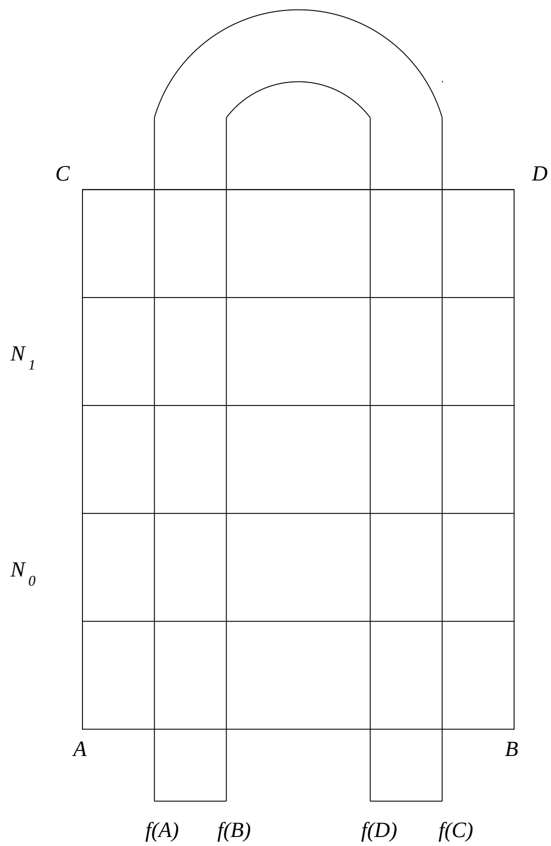

Figure 1

and $\sigma(\chi^n(S))$, then Theorem 1.1 can be thought of as showing that the lack of such a polynomial implies the existence of full shift dynamics, a considerably stronger statement than just the existence of recurrent dynamics.

On a more concrete level, consider the quintessential example of the Smale horseshoe [21, 11, 19, 12]. Let $f : S^2 \to S^2$ and let $N = N_0 \cup N_1 \subset S^2$ be as indicated in Figure 1.

Since f is a diffeomorphism, f^* is the identity map. Thus, the entropy conjecture provides no information concerning the existence of the complicated dynamics. On the other hand, $\sigma(\chi^1(S_0)) = \{1\}$, $\sigma(\chi^1(S_1)) = \{-1\}$, and $\sigma(\chi^1(S)) = \emptyset$, and hence, Theorem 1.1 applies. In this case of course one may use analysis, in the form of the uniform expansion and contraction of f on N, to conclude that d (the number of iterations of f needed to construct the semi-conjugacy) can be chosen equal to one and ρ can be chosen to be a homeomorphism. These latter conclusions cannot be obtained with Theorem 1.1.

Even the following, much weaker, question cannot be answered positively under the above mentioned hypothesis. Given a periodic orbit $\gamma \subset \Sigma_2^+$ does $\rho^{-1}(\gamma)$ contain a periodic orbit? To see this consider $F : S^2 \times S^1 \to S^2 \times S^1$ defined by

$$F(x, e^{2\pi i\theta}) = (f(x), e^{2\pi i\theta+1}),$$

i.e., the horseshoe crossed with irrational rotation on the circle. The map F clearly has no periodic orbits though the theorem still applies. There are, however, reasonable additional conditions on the space X and the map f which can be imposed which force the existence of periodic orbits. The following proposition, which can be proven as [24, Theorem 2.3], is a case in point.

Proposition 1.2 *Assume the hypothesis of Theorem 1.1 are satisfied, X is an ENR space and*

$$\sigma(\chi^q(S_1)) = \sigma(\chi^q(S_2)) = \emptyset \tag{1}$$

for all $q \in \mathbf{Z}$ and $q \neq n$. Then, each periodic point of T is an image of a periodic point of f^d, with the same principal period.

Remark 1.3 In essence the proof of Proposition 1.2 is an application of the Lefschetz fixed point theorem. The constraint (1) implies that the trace condition only needs to be checked in the n^{th} cohomology level, the same level at which the spectral information for Theorem 1.1 is assumed. It appears that obtaining general nontrivial conditions on the necessary relationships of the spectra at different levels is a difficult problem.

To motivate the details of the next two sections and to indicate the difficulties of demonstrating Theorem 1.1 a sketch of the proof is presented here. The first step is to define the semi-conjugacy ρ using the neighborhoods N_0 and N_1 as the basis for the symbolic dynamics. With this in mind define $\Theta : S \to \{0, 1\}$ by

$$\Theta(x) = \begin{cases} 0 & \text{if } x \in N_0, \\ 1 & \text{if } x \in N_1. \end{cases}$$

Since N_0 and N_1 are disjoint, Θ is continuous. At this point the standard construction of the relationship between the dynamics of f^d and the symbolic dynamics is obtained by defining $\rho : S \to \Sigma_2^+$ by

$$\rho(x) := (\Theta(x), \Theta(f^d(x)), \Theta(f^{2d}(x)), \Theta(f^{3d}(x)), \ldots).$$

Clearly, ρ is continuous and $T \circ \rho = \rho \circ f^d$. However, it remains to show that ρ is surjective. This is where the algebraic topology of the Conley index is required. In particular, as will be explained in detail later, if the Conley index associated to a particular sequence of symbols is nontrivial, then the sequence lies in the image of ρ. The difficult point is to determine a priori from the indices of S and S_i which symbol sequences have non-trivial indices. This is the content of Section 3 (c.f. also [10]).

However, one must first deal with another issue: how to compare the Conley indices of different isolated invariant sets. The variant of the Conley index used in the hypothesis of Theorem 1.1 is due to M. Mrozek [16] and is defined as follows. An *index pair* for an isolated invariant set S with respect to the map f consists of a pair of compact set (N, L) satisfying:

1. $S = \mathrm{Inv}(\mathrm{cl}(N \setminus L), f) \subset \mathrm{int}(N \setminus L)$,
2. $f(L) \cap N \subset L$,
3. $f(N \setminus L) \subset N$.

Observe that $f : (N, L) \to (N \cup f(L), L \cup f(L))$ and that excision gives rise to an isomorphism $e^* : H^*(N, L) \to H^*(N \cup f(L), L \cup f(L))$. Thus, one can define $f^*_{N,L} := f^* \circ e^* : H^*(N, L) \to H^*(N, L)$. Let

$$\mathrm{gker} f^*_{N,L} := \{v \in H^*(N, L) \mid (f^*_{N,L})^k(v) = 0 \text{ for some } k \in \mathbf{Z}^+\}.$$

Mrozek showed that $CH^*(S) := H^*(N, L)/\mathrm{gker} f^*_{N,L}$ and the induced automorphism $\chi^*(S) : CH^*(S) \to CH^*(S)$ are invariants of S, i.e. they are independent of the index pair (N, L) of S. The problem with this construction is that the index is given in terms of equivalence classes of $CH^*(N, L)$. More precisely, given $CH^*(S)$ and $\chi^*(S)$ there is no natural means by which one can reconstruct $f^*_{N,L} : H^*(N, L) \to H^*(N, L)$. This is not a purely academic problem. The first result in the spirit of Theorem 1.1 can be found in [12]. As the reader is invited to check, much of the effort of that paper is dedicated to overcoming this difficulty.

In the next section we shall briefly describe a more general version of the Conley index due to A. Szymczak [25]. The important point of Szymczak's result is that we are able to use the map $f^*_{N,L} : H^*(N, L) \to H^*(N, L)$ on a representative of the index. Because $f^*_{N,L}$ is defined directly from the topology, it is straightforward to identify algebraic generators of $H^*(N, L)$ and their images under $f^*_{N,L}$ with subsets of N and their images under f. Thus while the assumptions of the theorem are made in terms of the spectral information on the Conley index (information which is preserved by both Mrozek and Szymczak's versions of the index) the final step of the proof implicitly uses the choice of representative of the index $f^*_{N,L} : H^*(N, L) \to H^*(N, L)$ which only makes sense in Szymczak's formulation of the index.

2 The Index à la Szymczak

As was mentioned in the introduction the spectral information of the indices of S and S_i, $i = 0, 1$, is related using the extension of the Conley index developed by Szymczak. To describe his construction one begins by recalling the *category of objects with a morphism*. The reader is referred to [25, 26] for further details.

Given a category $\mathcal{K}$, the associated category of objects with a morphism, $\mathcal{K}_m$ is defined as follows. The objects of $\mathcal{K}_m$ are given by

$$\mathrm{Ob}(\mathcal{K}_m) := \{(V, f) \mid V \in \mathrm{Ob}(\mathcal{K}) \text{ and } f \in \mathrm{Mor}_{\mathcal{K}}(V, V)\}$$

and the morphisms between objects in $\mathcal{K}_m$ are the equivalence classes

$$\mathrm{Mor}_{\mathcal{K}_m}((V, f), (V', f')) := \left(\{\psi \in \mathrm{Mor}_{\mathcal{K}}(V, V') \mid \psi \circ f = f' \circ \psi\} \times \mathbf{Z}^+\right) / \sim$$

where

$$(\psi_1, n_1) \sim (\psi_2, n_2) \quad \Longleftrightarrow \quad \exists\, k \in \mathbf{Z}^+,\ \psi_2 \circ f^{n_1+k} = \psi_1 \circ f^{n_2+k}.$$

The morphisms in $\mathcal{K}_m$ will be denoted by $[\psi, n]$. The composition of two morphisms is given by

$$[\psi_1, n_1] \circ [\psi_2, n_2] = [\psi_1 \circ \psi_2, n_1 + n_2].$$

With this setting, $\mathcal{K}_m$ is a well defined category with the identity morphism of any object (V, f) given by $[\mathrm{Id}_V, 0]$.

This category is used to extend the Conley index as follows. Given an index pair (N, L), f induces a continuous map on the pointed quotient spaces

$$f_{N,L} : (N/L, [L]) \to (N/L, [L]).$$

The index pair (N, L) is called *regular* if $f_{N,L}^{-1}([L])$ is a neighborhood of $[L]$.

Let S be an isolated invariant set with respect to a continuous map $f : X \to X$. The *Conley index* of S is the class of all objects isomorphic to $((N/L, [L]), f_{N,L})$ in the category $\mathcal{H}_m$ where (N, L) is an index pair for S and $\mathcal{H}$ is the category of pointed topological spaces with homotopy classes of basepoint preserving maps serving as morphisms. In [25] it is shown that this definition is independent of the index pair chosen and hence the index can be written as

$$h(S, f) := [(N/L, [L]), f_{N,L}].$$

Using the Alexander-Spanier cohomology functor with field coefficients one obtains

$$h^*(S, f) := \left[H^*(N/L, [L]), f^*_{N,L}\right]$$

where the equivalence class consists of all isomorphic objects in the category of graded vector spaces with graded linear maps, $\mathcal{V}_m$. In what follows $h^*(S, f) = 0$ will denote the equivalence class containing the trivial graded vector space with the trivial graded map.

The following theorem, while trivial, is fundamental.

Theorem 2.1 *If $h^*(S, f) \neq 0$, then $S \neq \emptyset$.*

The notation that will be introduced next is used to relate the indices of S with those of isolated invariant subsets of S. Given $\omega = (\omega_0, \omega_1, \dots, \omega_k) \in \{0,1\}^{k+1}$, define

$$N_\omega := \bigcap_{j=0}^{k} f^{-j}(N_{\omega_j})$$

and

$$S_\omega := \mathrm{Inv}(N_\omega, f^{k+1}),$$

where N_ω is an isolating neighborhood for f^{k+1} with $S_\omega \subset S$ by [24, Proposition 2.1]. By [17, Corollary 2] there exists a regular index pair (K, L) for S such that $K \subset N$. Let $\Pi : K \to K/L$ be the quotient map and given $x \in K$ set $[x] := \Pi(x)$. To simplify the notation set $(Y, \bar{y}) := (K/L, [L])$ and $F := f_{K,L} : (Y, \bar{y}) \to (Y, \bar{y})$. Define

$$Y_i := \Pi(K \cap N_i) \cup \{\bar{y}\}, \quad i = 0, 1,$$

and continuous maps $r_i, F_i : (Y, \bar{y}) \to (Y, \bar{y})$ given by

$$r_i([x]) := \begin{cases} x & \text{if } x \in Y_i, \\ \bar{y} & \text{otherwise} \end{cases}$$

and

$$F_i := F \circ r_i.$$

Given $\omega \in \{0,1\}^{k+1}$ set

$$F_\omega := F_{\omega_k} \circ \dots \circ F_{\omega_1} \circ F_{\omega_0}.$$

The following proposition indicates that the Conley indices of isolated invariant subsets of S can be viewed in terms of different graded linear maps defined on the same graded vector space.

Proposition 2.2 [24, Lemma 3.1] *Given* $\omega \in \{0,1\}^{k+1}$

$$h^*(S_\omega, f^{k+1}) = [H^*(Y, \bar{y}), F_\omega^*].$$

The final point which needs to be checked is that the spectral properties of F_i^* agree with the spectral information assumed in the statement of Theorem 1.1. First observe that given any linear map $M : V \to V$, there exists an integer N such that for any $n \geq N$,

$$\mathrm{rank}(M^N) = \mathrm{rank}(M^n) \in \mathbf{Z}^+ \cup \{\infty\}.$$

In this case the *asymptotic rank* of M is given by

$$\mathrm{arank}(M) := \mathrm{rank}(M^N).$$

Proposition 2.3 *The spectral decomposition of isomorphic objects in $\mathcal{V}_m$ differs only in the space corresponding to* 0 *eigenvalue.*

PROOF. Notice that given $n \in \mathbf{Z}^+$ if (V^*, φ^*) is an object in $\mathcal{V}_m$, then it is isomorphic to $(V^*/\mathrm{gker}(\varphi^*)^n, \tilde{\varphi}^*)$, where $\tilde{\varphi}^*$ is induced by φ^*.

■

3 Existence of Generators

As was indicated in the introduction, the spectral assumptions of Theorem 1.1 must be used to obtain two distinct generators of a free semi-group. This will be done using an argument based on the trace formula. As will become clear the arguments of this section are independent of the index theory and work for arbitrary linear maps. In fact, they uncover an aspect of a broader problem concerning random sequences drawn from a finite (say with l elements) pool of linear maps. Selecting only the sequences with the composition of positive rank gives rise to a subshift of $\{1, \ldots, l\}^{\mathbf{N}}$; and, from this perspective, what follows (Proposition 3.4) amounts to deciding when a power of this subshift factors onto the full shift $\{0, 1\}^{\mathbf{N}}$. As shown in [10], existence of a factor (or even embedded) horseshoe is equivalent to non-vanishing of topological entropy, which in turn is always detected by a certain spectral decomposition and can be tested for by a finite algorithm. A general theory of these so called *cocyclic subshifts* offers a large and yet unexplored spectrum of problems on the boundary of topological dynamics, ergodic theory, combinatorics and algebra — the interested reader may want to consult [10] for an introduction.

The following notation will be used to generate and discuss products of matrices. Let $\omega = (\omega_0, \ldots, \omega_k) \in \{0, 1, \ldots, l\}^{k+1}$. Its *length* is denoted by $p(\omega) = k + 1$. Given also $\tilde{\omega} \in \{0, 1, \ldots, l\}^{\tilde{k}+1}$, we have the concatenation $\omega\,\tilde{\omega} \in \{0, 1, \ldots, l\}^{k+\tilde{k}+2}$, $\omega\,\tilde{\omega} = (\omega_0, \ldots, \omega_k, \tilde{\omega}_0, \ldots, \tilde{\omega}_k)$. The sequence obtained by repeating ω i-times is therefore denoted by $\omega^i \in \{0, \ldots, l\}^{(k+1)i}$. The sequence ω is *minimal* if there does not exist $\alpha \in \{0, 1, \ldots, l\}^j$ such that $\omega = \alpha^i$ for some i.

Now let $\mathcal{A} := (A_i : V \to V \mid i = 0, 1, \ldots, l)$ be a a finite sequence of linear maps defined on a finite dimensional vector space V. Define

$$A := \sum_{i=0}^{l} A_i$$

and, given $\omega \in \{0, 1, \ldots, l\}^{k+1}$, let

$$A(\omega) := A_{\omega_0} \circ A_{\omega_1} \circ \ldots \circ A_{\omega_k}.$$

Finally, set

$$\mathcal{G} = \mathcal{G}_{\mathcal{A}} := \{\omega \in \{0, 1, \ldots, l\}^{n+1} \mid n \in \mathbf{N},\ \omega \text{ is minimal},\ A(\omega) \text{ is not nilpotent}\}.$$

The symbol $\#(\mathcal{G})$ will be used to denote the cardinality of the set $\mathcal{G}$.

Recall the following trace formula, [5, Lemma 5.2]. For any matrix M and complex number z,

$$\frac{1}{\det(I - zM)} = \exp\left(\sum_{m=1}^{\infty} \frac{\operatorname{tr}(M^m)}{m}\, z^m\right). \tag{2}$$

The following equality is at the heart of the computations of this section (c.f. Theorem 9.1 in [10]).

Proposition 3.1 *If $\#(\mathcal{G}) < \infty$, then for all $z \in \mathbf{C}$,*

$$\det(I - zA) = \prod_{\omega\in\mathcal{G}} \det\left(I - z^{p(\omega)} A(\omega)\right)^{1/p(\omega)}. \tag{3}$$

Proof. Using the fact that the cardinality of $\mathcal{G}$ is finite to reverse the order of summation and (2) one has that,

$$\begin{aligned}
-\ln(\det(I - zA)) &= -\ln\left(\det\left(I - z\sum_{i=0}^{l} A_i\right)\right)\\
&= \sum_{m=1}^{\infty} \frac{\operatorname{tr}\left((\sum_{i=0}^{l} A_i)^m\right)}{m}\, z^m\\
&= \sum_{m=1}^{\infty} \sum_{\omega\in\{0,\dots,l\}^m} \frac{\operatorname{tr}(A(\omega))}{m}\, z^m\\
&= \sum_{\omega\in\mathcal{G}} \sum_{n=1}^{\infty} \frac{\operatorname{tr}(A(\omega)^n)}{np(\omega)}\, z^{np(\omega)}\\
&= \sum_{\omega\in\mathcal{G}} \frac{1}{p(\omega)} \sum_{n=1}^{\infty} \frac{\operatorname{tr}(A(\omega)^n)}{n}\, (z^{p(\omega)})^n\\
&= \sum_{\omega\in\mathcal{G}} -\frac{1}{p(\omega)} \ln\left(\det(I - z^{p(\omega)} A(\omega))\right)\\
&= -\ln\left(\prod_{\omega\in\mathcal{G}} \det\left(I - z^{p(\omega)} A(\omega)\right)^{1/p(\omega)}\right).
\end{aligned}$$

Corollary 3.2 *If $\#(\mathcal{G}) < \infty$, then $\#(\mathcal{G}) \le \operatorname{rank}(A)$.*

From now on we restrict our attention to $\mathcal{A} = \{A_0, A_1\}$ and hence $A = A_0 + A_1$.

Proposition 3.3 *If $\#(\mathcal{G}_\mathcal{A}) = \infty$, then there are $k_1, \ldots, k_m \in \mathbf{N}$ and $\omega_1, \ldots, \omega_m \in \mathcal{G}_\mathcal{A}$ not all equal ($m \geq 2$) such that*

$$\mathrm{arank}(A(\omega_i)) = \mathrm{arank}\left(A(\omega_1^{k_1})A(\omega_2^{k_2}) \cdots A(\omega_m^{k_m})\right) > 0$$

for $i = 1, \ldots, m$.

Proof. Let $\mathcal{G}_\mathcal{A}^r := \{\omega \in \mathcal{G}_\mathcal{A} \mid \mathrm{arank}(A(\omega)) = r\}$, $r \in \mathbf{N}$. Since $\#(\mathcal{G}_\mathcal{A}) = \infty$, there is $q \in \mathbf{N}$ with $\#(\mathcal{G}_\mathcal{A}^q) = \infty$ and $\#(\mathcal{G}_\mathcal{A}^r) < \infty$ for $r < q$. Set $p_* := \prod \{p(\omega) \mid \omega \in \mathcal{G}_\mathcal{A}^r, r < q\}$ (and $p_* = 1$ when the product is vacuous). Let $\omega_0, \ldots, \omega_k$ be $k+1$ different elements of $\mathcal{G}_\mathcal{A}^q$ where $k = \dim(V)$; and take k_i's divisible by p_* and large enough to guarantee that $\mathrm{rank}(A(\beta_i)) = \mathrm{arank}(A(\beta_i))$ for $\beta_i := \omega_i^{k_i}$, $i = 0, \ldots, k$.

Observe that, for the sequence of matrices $\mathcal{B} := (A(\beta_i))_{i=0}^k$, we have $\#(\mathcal{G}_\mathcal{B}) = \infty$, since otherwise $\#(\mathcal{G}_\mathcal{B}) \leq k$ by Corollary 3.2 and $k + 1 = \#\mathcal{B} \leq \#(\mathcal{G}_\mathcal{B}) = k$ is a contradiction. Because $\#(\mathcal{G}_\mathcal{B}) = \infty$, there are $m \geq 2$ and $i_1, \ldots, i_m$ not all equal and such that $\tilde{A} := A(\beta_{i_1}) \cdots A(\beta_{i_m})$ is non-nilpotent. Set $\tilde{q} := \mathrm{arank}(\tilde{A})$; clearly $\tilde{q} \leq q$.

Suppose that $\tilde{q} < q$. Then $\beta := \beta_{i_1} \cdots \beta_{i_m} = \kappa^s$ for some $\kappa \in \mathcal{G}_\mathcal{A}^{\tilde{q}}$ and $s \in \mathbf{N}$. Because $p(\kappa)$ divides p_* and so $p(\kappa)$ also divides $p(\beta_{i_j})$, $\omega_{i_j} = \kappa^{s_j}$ for some $s_j \in \mathbf{N}$, $j = 1, \ldots, m$. We conclude that $\omega_{i_j} \in \mathcal{G}_\mathcal{A}^{\tilde{q}}$ — which contradicts the choice of ω_i's as elements of $\mathcal{G}_\mathcal{A}^q$. In this way, it must be that $\tilde{q} = q$ and $\mathrm{arank}(A(\omega_i)) = \mathrm{arank}\left(A(\omega_{i_1}^{k_1})A(\omega_{i_2}^{k_2}) \cdots A(\omega_{i_m}^{k_m})\right) = q$, which conludes the proof. ■

The following proposition provides for the existence of the desired pair of non-nilpotent matrices. Given a linear map M, let $\sigma^\times(M)$ denote the nonzero spectrum of M.

Proposition 3.4 *Let $A_0, A_1 : V \to V$ be linear maps such that*

$$\sigma^\times(A_0) \amalg \sigma^\times(A_1) \not\subset \sigma^\times(A_0 + A_1) \tag{4}$$

or given containment,

$$\sigma^\times(A_0 + A_1) \setminus (\sigma^\times(A_0) \amalg \sigma^\times(A_1)) \tag{5}$$

is not cyclic. Then, there exist a positive integer d and distinct sequences $\alpha, \beta \in \{0,1\}^{d+1}$ with the following property. Let

$$\begin{aligned} \Psi_0 &= A(\alpha) \\ \Psi_1 &= A(\beta), \end{aligned}$$

then given any positive integer n and any sequence $\omega \in \{0,1\}^n$, $\Psi(\omega)$ is not nilpotent.

PROOF. Assume that $\#(\mathcal{G}_{\{A_0,A_1\}}) < \infty$. Then

$$\det(I - zA) = \prod_{\omega\in\mathcal{G}} \det\left(I - z^{p(\omega)}A(\omega)\right)^{1/p(\omega)}. \tag{6}$$

The degree of the polynomials on the left and right hand sides of the equation must agree and hence $\sigma^\times(A_0) \amalg \sigma^\times(A_1) \subset \sigma^\times(A_0 + A_1)$. Obviously this is a contradiction if one assumes (4). If the condition (5) is satisfied, then equation (6) becomes

$$\prod_{\nu\in\Sigma}(1 - z\nu) = \prod_{\omega\in\mathcal{G}^*} \prod_{\lambda(\omega)\in\sigma^\times(A(\omega))} \left(1 - z^{p(\omega)}\lambda(\omega)\right)^{1/p(\omega)}, \tag{7}$$

where $\Sigma := \sigma^\times(A_0 + A_1) \setminus (\sigma^\times(A_0) \amalg \sigma^\times(A_1))$ and $\mathcal{G}^* := \mathcal{G} \setminus \{0, 1\}$. Notice that for each $\nu \in \Sigma$, there exits $\omega_\nu \in \mathcal{G}^*$ such that $\nu^{p(\omega_\nu)} \in \sigma^\times(A(\omega_\nu))$. Therefore, the reciprocal of all $p(\omega_\nu)$-th roots of $\nu^{p(\omega_\nu)}$ are roots of the polynomial in the right hand side of (7). This is also a contradiction. Hence $\#\mathcal{G}_{\{A_0,A_1\}} = \infty$.

From Proposition 3.3 we have $k_1, \ldots, k_m \in \mathbf{N}$ and $\omega_1, \ldots, \omega_m \in \mathcal{G}_\mathcal{A}$ not all equal ($m \geq 2$) such that

$$\operatorname{arank}(A(\omega_i)) = \operatorname{arank}\left(A(\omega_1^{k_1})A(\omega_2^{k_2}) \cdots A(\omega_m^{k_m})\right) = q > 0 \tag{8}$$

for $i = 1, \ldots, m$. Let $\tilde{\alpha} := \omega_1^{k_1}$ and $\tilde{\beta} := \omega_1^{k_1} \cdots \omega_m^{k_m}$. Finally, define

$$\begin{aligned} \alpha &:= \tilde{\alpha}^{p(\tilde{\beta})} \\ \beta &:= \tilde{\beta}^{p(\tilde{\alpha})} \end{aligned}$$

and $d = p(\tilde{\alpha})p(\tilde{\beta})$.

The conclusion now follows from (8) because, taking $\Psi_0 = A(\alpha)$ and $\Psi_1 = A(\beta)$, we have that $\operatorname{rank}(\Psi(\sigma)) = q > 0$ for any finite binary sequence $\sigma \in \{0, 1\}^n$. ■

4 Proof of Theorem 1.1

The proof of Theorem 1.1 is now a simple compilation of the results of the previous section.

Let $F_i : (Y, \bar{y}) \to (Y, \bar{y})$ be as defined in Section 2. Switching to the notation of Section 3, let

$$A_i := F_i^n : H^n(Y, \bar{y}) \to H^n(Y, \bar{y}), \quad i = 0, 1.$$

Then $A_0 + A_1 = A := F^n : H^n(Y, \bar{y}) \to H^n(Y, \bar{y})$. By Proposition 2.3

$$\sigma^\times(A_i) = \sigma(\chi^n(S_i)) \quad i = 0, 1$$

and

$$\sigma^{\times}(A) = \sigma(\chi^n(S)).$$

Choose a positive integer d and distinct sequences $\alpha, \beta \in \{0,1\}^d$ according to Proposition 3.4. Let $\Psi_0 := A(\alpha)$ and $\Psi_1 := A(\beta)$.

Since α and β are distinct there exists $a \in \{0, \ldots, d-1\}$ such that $\alpha_a \neq \beta_a$. Define $\rho : S \to \Sigma_2^+$ by

$$\rho(x) = (\Theta(f^a(x)), \Theta(f^{a+d}(x)), \Theta(f^{a+2d}(x)), \Theta(f^{a+3d}(x)), \ldots).$$

It needs to be shown that ρ is surjective. Since S is compact it is sufficient to prove that $\rho(S)$ is a dense subset of Σ_2^+. To this end, let $s = (s_0, s_1, \ldots, s_{m-1}) \in \{0,1\}^m$. It will be shown that there exists $x \in S$ such that the first m terms of $\rho(x)$ are s.

Define a sequence

$$\omega := (c_0^0, \ldots, c_{d-1}^0, c_0^1, \ldots, c_{d-1}^1, \ldots, c_0^{m-1}, \ldots, c_{d-1}^{m-1}) \in \{0,1\}^{md}$$

according to whether (i) $\alpha_a = 0$ and $\beta_a = 1$, or (ii) $\alpha_a = 1$ and $\beta_a = 0$. In the first case (i) let

$$(c_0^i, \ldots, c_{d-1}^i) = \begin{cases} (\alpha_0, \alpha_1, \ldots, \alpha_{d-1}) & \text{if } s_i = 0, \\ (\beta_0, \beta_1, \ldots, \beta_{d-1}) & \text{if } s_i = 1. \end{cases}$$

If the latter case (ii) let

$$(c_0^i, \ldots, c_{d-1}^i) = \begin{cases} (\beta_0, \beta_1, \ldots, \beta_{d-1}) & \text{if } s_i = 0, \\ (\alpha_0, \alpha_1, \ldots, \alpha_{d-1}) & \text{if } s_i = 1. \end{cases}$$

Observe that for each $i = 0, \ldots, m-1$,

$$\Psi_{s_i} = A(c_0^i, \ldots, c_{d-1}^i).$$

By Propositions 2.2 and 3.4

$$h^*(S_\omega, f^{md}) = \left[H^*(Y, \bar{y}), \Psi_{s_0} \circ \Psi_{s_1} \circ \ldots \circ \Psi_{s_{m-1}}\right] \neq 0.$$

By Theorem 2.1,

$$S_\omega \neq \emptyset.$$

Let $x \in S_\omega$. By definition

$$x \in \bigcap_{i=0}^{m-1} \bigcap_{j=0}^{d-1} f^{-(j+di)}(N_{c_j^i}).$$

Therefore, $f^{a+di}(x) \in N_{c_a^i}$. However, in case (i)

$$c_a^i = \begin{cases} \alpha_a = 0 & \text{if } s_i = 0 \\ \beta_a = 1 & \text{if } s_i = 1, \end{cases}$$

while in case (ii)

$$c_a^i = \begin{cases} \alpha_a = 1 & \text{if } s_i = 1 \\ \beta_a = 0 & \text{if } s_i = 0. \end{cases}$$

Therefore, the first m terms of $\rho(x)$ agree with s.

■

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