Uniqueness of the stationary wave for the Extended Fisher Kolmogorov equation

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Abstract: The extended Fisher Kolmogorov equation, $u_t = -\beta u_{xxxx} + u_{xx} + u - u^3$, $\beta > 0$, models a binary system near the Lifshitz critical point and is known to exhibit a stationary heteroclinic solution joining the equilibria ± 1. For the classical case, $\beta = 0$, the heteroclinic is $u(x) = \tanh(x/\sqrt{2})$ and is unique up to the obvious symmetries. We prove the conjecture that the uniqueness persists all the way to $\beta = 1/8$, where the on-set of spatial chaos associated with the loss of monotonicity of the stationary wave is known to occur. Our methods are non-perturbative and employ a global cross-section to the Hamiltonian flow of the stationary fourth order equation on the energy level of ± 1. We also prove uniform a priori bounds on all bounded stationary solutions, valid for any $\beta > 0$.

1 Introduction

The Extended Fisher-Kolmogorov (EFK) equation

$$u_t = -\beta u_{xxxx} + u_{xx} + u - u^3, \quad x, u \in \mathbf{R}, \ \beta = \text{constant} > 0,$$

arises as the mezoscopic model of a phase transition in a binary system near the Lifshitz point [8] (also see [18]) and is frequently used as a model system

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for the study of pattern formation from an unstable spatially homogeneous state (see [4, 3]). The equation entered mathematical literature through a series of papers by Peletier and Troy. There is however a host of other related research, and the reader may consult the introduction in [9] for a broader perspective.

The case $\beta = 0$ corresponds to the classical Fisher-Kolmogorov equation and yields to a rather complete analysis [1, 7]. For $\beta > 0$, a difficulty already arises with classification of the equilibrium solutions determined by an ordinary differential equation

$$\beta u_{xxxx} - u_{xx} + F'(u) = 0, \tag{1}$$

where $F(u) = (1 - u^2)^2/4$ is the standard double-well potential. This is the Euler-Lagrange equation of the action functional

$$J[u(\cdot)] := \int \frac{\beta}{2} u_{xx}^2 + \frac{1}{2} u_x^2 + F(u)\, dx, \tag{2}$$

which corresponds to the Helmholtz free energy. The two homogeneous states $u = \pm 1$ are clearly the absolute minimizers of J and represent *the stable pure phases*. Of particular interest for the pattern formation study are *the stable organized states*, that is the spatially non-homogeneous local minimizers of J. These are contained together with $u = \pm 1$ in the zero level set of the Hamiltonian

$$H = \frac{\beta}{2} u_{xx}^2 + \frac{1}{2} u_x^2 - \beta u_{xxx} u_x - F(u). \tag{3}$$

Our main result is the following theorem about uniqueness.

Theorem 1 *For $\beta < 1/8$, up to the symmetry $u(\cdot) \mapsto -u(\cdot)$ and translations $u(\cdot) \mapsto u(\cdot + \tau)$ along the x-axis, $\tau \in \mathbf{R}$, there is a unique non-constant bounded solution $u(\cdot)$ to (1) on the level set $H = 0$.*

The unique solution is a monotonic odd *kink* connecting ± 1, similar to $u(x) = \tanh(x/\sqrt{2})$ for $\beta = 0$. Its existence has been already proved in [16] by using direct minimization and via a shooting method in [12], where also uniqueness in the class of monotonic odd functions is shown. Our contribution is a global grasp of all the solutions on $H = 0$, regardless of their monotonicity or symmetry. This problem is also resolved with quite different methods by the results announced recently in [17], where also the case $H < 0$ is considered.

Without the ambition of providing a comprehensive introduction to the EFK equation, let us mention that the range of β in the theorem is optimal: for every $\beta > 1/8$ there is a bewildering abundance of bounded solutions to (1); they exhibit localized, periodic, and chaotic patterns [14, 13, 10, 9]. The reason for qualitative change at $\beta = 1/8$ is a bifurcation of both equilibria ± 1 from saddle-nodes to saddle-foci, upon which *the tails at* $\pm\infty$ of the unique minimizer transform from monotonic to oscillatory. This allows for *combining* many translated copies of the minimizer into complicated patterns. Our theorem confirms the above scenario for the on-set of the spatial chaos in the EFK equation (tuned with the parameter β).

Perhaps even more then the result we should emphasize its proof as it develops a convenient framework to study (1) with more general nonlinearities F (satisfying merely $\liminf_{|u|\to\infty} F'(u)/|u|^{1+\epsilon} > 0$). Besides the Hamiltonian nature of the equation, the central role is played by a global two-dimensional Poincaré cross section to the flow of (1) on the level set of H. The section is simply taken at $u_x = 0$, and the return map is represented by a two dimensional map defined on a (u, E)-plane[3] with $\{(u, E) : F(u) \leq H\}$ removed. The map is area preserving and smooth with an exception of only two singular lines that get mapped to $\{(u, E) : F(u) = H\}$ — see Section 3 for details. The approach is a natural extension of that of Peletier and Troy, whose shooting method corresponds to looking at the restriction of the return map to the one-dimensional space of odd solutions (those satisfying $u_{xx}(0) = 0$).

The proof of the theorem, carried out in Section 4, essentially amounts to establishing three facts holding exactly for the range $\beta < 1/8$. First, no two consecutive iterates of a point under the return map are contained in the interior of the strip bounded by $u = \pm 1$ — which limits possible bounded solutions to monotonic heteroclinics between ± 1 (see Proposition 3). Second, also in the strip bounded by $u = \pm 1$, the cross-section map has a *twist property* — which implies that there is a unique monotonic heteroclinic (see Proposition 2). Third, all the points outside the strip are helplessly iterated out towards infinity — which eliminates the possibility of bounded solutions with $|u|$ exceeding one (see Proposition 4).

The proof of the three mechanisms above hinges on identification of a domain within which an appropriate time-change of the flow is order-preserving.

[3] E will be the appropriate momentum variable equal to $u_x - \beta u_{xxx}$.

This is exactly for $\beta < 1/8$ that this domain is essentially invariant (see Fact 1 in Section 2).

We certainly hope that the two-dimensional dynamics of the return map will play further role in elucidating the solutions to the equilibrium EFK equation (1) also for $\beta > 1/8$.

To finish, we mention an a priori C^0-bound (Theorem 4) for all bounded solutions of (1), which is included in the appendix.

2 Hamiltonian preliminaries

If we cast u and $v := u_x$ as configuration variables, from the variation of J, the corresponding momenta are $p := \beta v_x$ and $E := v - p_x$. The Hamiltonian is a function on the phase space $\mathbf{R}^4$,

$$H = H(u, v, p, E) = \frac{1}{2\beta} p^2 - \frac{1}{2} v^2 + vE - F(u).$$

The solutions to (1) correspond to the integral curves of the symplectic gradient of H in $\mathbf{R}^4$ with the standard symplectic form $dp \wedge dv + dE \wedge du$. Explicitly, the Hamilton equations are

$$\begin{cases} u_x = H_E = v, \\ v_x = H_p = p/\beta, \\ p_x = -H_v = v - E, \\ E_x = -H_u = F'(u), \end{cases} \tag{4}$$

Note that this is not a mechanical system as *the kinetic part of* H is not positive definite. Nevertheless, J can be given a parameterization invariant form

$$J[u(\cdot)] = \int \left\{ \frac{\beta}{2} v \left(\frac{dv}{du} \right)^2 + \frac{1}{2} v + \frac{F(u)}{v} \right\} du, \tag{5}$$

which could be treated as a positive functional on the curves $(u(\cdot), v(\cdot))$ in the configuration space $\mathbf{R}^2$ with a constraint $u_x = v$. The somewhat awkward constraint decouples if u can be taken as an independent variable; and then J can be looked upon as action of a (non-autonomous) mechanical system with one degree of freedom (see (6) below). We should digress that, on a more fundamental level, the action of a solution is just the "symplectic area"

$$J[u(\cdot)] = \int p\,dv + E du - H dx,$$

or, more interestingly, the flux of degenerate forms

$$J[u(\cdot)] = \int 2E\,du - v\,du + \text{boundary terms} \; = \int 2p\,dv + v\,du + \text{boundary terms} \, .$$

To continue, on the level set $H = 0$, E becomes a function of u, v, and p,

$$E = E(u, v, p) = -\frac{1}{2\beta} p^2/v + \frac{1}{2} v + F(u)/v.$$

This is the Hamiltonian of the non-autonomous system (where u serves as time)

$$\begin{cases} p_u = E_v = \frac{1}{2} + \frac{1}{2\beta}\frac{p^2}{v^2} - \frac{F(u)}{v^2}\,, \\ v_u = -E_p = \frac{p}{\beta v}\,. \end{cases} \tag{6}$$

Clearly, we can utilize (6) only to follow $u(\cdot)$ between any two consecutive critical points. Since the equations are equivariant with respect to $v \mapsto -v$, we will also restrict our attention to the positive halfplane $\{(v, p) : v > 0\}$.

A reader familiar with [12] can verify that the system (6) is equivalent to the equation (1.6a) in that work by abandoning the Hamiltonian variables (u, v, p, E).

The following simple fact is of key importance. The part (i) brings out the privileged role of the conjugate pair of variables (v, p), and the part (ii) can be thought of as the ultimate reason why the behavior of (1) is so tame for $\beta < 1/8$.

Fact 1 (order preserving strip) *(i) The flow of (6) in the positive half-plane $v > 0$ is strictly order preserving in the strip*

$$S(u) := \left\{ (v, p) : \frac{p^2}{2\beta} < F(u) \right\}$$

That is, given $(v, p), (\tilde{v}, \tilde{p}) : [u_0, u_1) \to \mathbf{R}^2$ that solve (6) and take values in $S(u)$ for all $u \in [u_0, u_1)$, if

$$v(u_0) \geq \tilde{v}(u_0) \text{ and } p(u_0) \geq \tilde{p}(u_0),$$

then

$$v(u) \geq \tilde{v}(u) \text{ and } p(u) \geq \tilde{p}(u) \text{ for all } u > u_0;$$

and both inequalities are strict for $u > u_0$ *if* $(v(u_0), p(u_0)) \neq (\tilde{v}(u_0), \tilde{p}(u_0))$.

(ii) For $\beta < 1/8$, *if* $(v, p) : (-1, 1) \to \mathbf{R}^2$ *is a solution to (6), then either* $(v(u), p(u)) \in S(u)$ *for all* $u \in (-1, 1)$, *or there is* $u_0 \in (-1, 1)$ *such that either* $p(u) > \sqrt{2\beta F(u)}$ *for all* $u > u_0$ *or* $p(u) < -\sqrt{2\beta F(u)}$ *for all* $u < u_0$.

Since the lines $p = \pm\sqrt{2\beta F(u)}$ are the boundary of $S(u)$, (ii) effectively says that the solution cannot leave and then come back to $S(u)$.

Proof. (ii): At the upper boundary of $S(u)$, $p = \sqrt{2\beta F(u)}$, and $p_u = 1/2$ from (6); therefore,

$$(p^2/(2\beta))_u = p/(2\beta) = \sqrt{F(u)/(2\beta)} > F'(u), \tag{7}$$

where for the last inequality we use $\beta < 1/8$ to secure

$$\sqrt{F(u)} = \frac{1}{2}(u^2 - 1) > \sqrt{2\beta}u(u^2 - 1) = \sqrt{2\beta}F'(u), \quad u \in (-1, 1).$$

Thus once on the upper boundary at some u_0, the solution must leave $S(u)$ and a subsequent return to $S(u)$ is impossible by (7). The situation at the lower boundary is analogous under time reversion, $u \mapsto -u$.

(i): This hinges on the linear variational equation of (6) preserving the positive cone in the $\mathbf{R}^2$, for which one just needs positivity of the off-diagonal entries $-E_{pp}$ and E_{vv}:

$$-E_{pp} = (\beta v)^{-1} > 0,$$

$$E_{vv} = -\frac{2}{v^3}\left(\frac{p^2}{2\beta} - F(u)\right) > 0.$$

The proof is completed via a standard argument traced below because of the minor complication caused by $S(u)$ being a proper and *time* dependent subset of $\mathbf{R}^2$.

Consider (6) with a family of initial conditions

$$(\lambda v(u_0) + (1 - \lambda)\tilde{v}(u_0), \lambda p(u_0) + (1 - \lambda)\tilde{p}(u_0)), \quad \lambda \in [0, 1],$$

and consider a family of the corresponding solutions $\gamma(u,\lambda) := (v(u,\lambda), p(u,\lambda))$, $u \in [u_0, u_1(\lambda))$. The solutions cannot blow up while in $S(u)$; p is obviously bounded there, and so is v^2 because $(\beta v^2/2)_u = p$ from (6). Hence, we may take above $u_1(\lambda) := \sup\{u \in (u_0, u_1) : \ \gamma(u,\lambda) \in S(u)\}$. Let $u_* := \inf\{u_1(\lambda) : \lambda \in [0,1]\}$. For $u \in (u_0, u_*)$, $\gamma(u,\lambda) \in S(u)$ and $\partial\gamma/\partial\lambda$ is in the positive quadrant by the property of the variational equation. By integrating we get then, for any $u \in (u_0, u_*)$ and $\lambda \in (0,1)$,

$$v(0,u) \le v(u,\lambda) \le v(u,1) \quad \text{and} \quad p(0,u) \le p(u,\lambda) \le p(u,1);$$

and the inequalities are strict when $\partial\gamma/\partial\lambda$ is not zero at u_0. Finally, $u_* = u_1$, because the last inequality guarantees

$$\frac{p(u_*,\lambda)^2}{2\beta} \le \max_{\lambda=0,1}\left\{\frac{p(u_*,\lambda)^2}{2\beta}\right\} < F(u_*).$$

□

3 The Poincaré cross-section

The Hamiltonian vectorfield of (4) has a disadvantage of not being complete. It admits however a convenient global cross-section $\{v = 0, H = 0\} \subset \mathbf{R}^4$, as elucidated below.

The surface $\Sigma = \{v = 0, H = 0\} \subset \mathbf{R}^4$ solves in the (u, E, p)-space

$$\frac{1}{2\beta}p^2 - F(u) = 0$$

and the map $(u, v, E, p) \mapsto (u, E)$ is a 2-to-1 covering, with a fold along $p = 0$, from Σ onto $\Omega_0 := \{(u,E) : F(u) \ge 0\} \subset \mathbf{R}^2$. We will work mostly in the (u,E)-plane and for $p > 0$; as the other case, $p < 0$, is totally symmetric for even F (see the remarks after Lemma 1). Consider then

$$\Omega := \{(u,E) : F(u) > 0\} \subset \mathbf{R}^2.$$

For each point $(u_0, E_0) \in \Omega$, the corresponding initial condition is $(u_0, v_0 = 0, E_0, p_0 = \sqrt{2\beta F(u_0)})$, and it determines a unique maximal solution to (4), $u : [0,a) \to \mathbf{R}$. Let $x_c := \sup\{x \in (0,a) : u_x|_{(0,x)} > 0\}$.

Definition 1 *The* **return map** *T has domain $\Omega = \{(u,E) : F(u) > 0\} \subset \mathbf{R}^2$ and maps (u_0, E_0) to $(u_1, E_1) \in \Omega_0$, where*

$$u_1 := \lim_{x \to x_c^-} u(x) \quad \textit{and} \quad E_1 := \lim_{x \to x_c^-} E(x).$$

The following lemma brings more meaning to the definition.

Lemma 1 *(i) Any (maximal) solution $u(\cdot)$ of the initial value problem to (4) with $H = 0$ either has a critical point at some finite $x = x_c$ or it is monotonic and asymptotic to a constant, i.e. $\lim_{x\to\infty} u(x) = u_1 \in \mathbf{R}$.*
(ii) In the later case $F'(u_1) = 0$ (so $u_1 = \pm 1$), and $u(\cdot)$ has a "critical point at $x_c = \infty$", i.e. $\lim_{x\to\infty} v = 0$. Also, $\lim_{x\to\infty} p, E = 0$.

A few remarks about T are in order. We defined T using an increasing lap of $u(\cdot)$ (with $p_0 = \sqrt{2\beta F(u_0)}$). For decreasing laps (with $p_0 = -\sqrt{2\beta F(u_0)}$), we have $\tilde{T}$ that is conjugated to T via the symmetry $R : (u,E) \mapsto (-u,-E)$, i.e. $\tilde{T} = R \circ T \circ R$. Following solutions of (4) from one critical point to the next corresponds to iterating $T \circ R$, of which $T \circ \tilde{T}$ is the second iterate. Because any smooth bounded function — and so also a bounded solution to (1) — can be a priori decomposed into monotonic laps between its critical points, a reader interested only in the proof of Theorem 1 will notice that (i) of Lemma 1 could be circumvented.

Also note that the map T is smooth on $\Omega \cap T^{-1}\Omega$, where both (u_0, E_0) and (u_1, E_1) are in Ω and x_c is a finite transversal ($u_{xx}(x_c) \neq 0$) zero of u_x. Nevertheless, due to the possibility of critical inflection points in $u(\cdot)$, T is not globally continuous: for orbits hitting the set $\Omega_0 \setminus \Omega = \{(u,E) : F(u) = 0\}$, T is undefined. This happens for relatively few orbits and will be of no relevance to our considerations. Finally, it is expected that a cross-section to the Hamiltonian flow yields a measure preserving return map. After the proof of Lemma 1 below, we will digress and show that the measure preserved by T is simply the Lebesgue area $du \wedge dE$.

Proof of Lemma 1. Let $u : [0,a) \to \mathbf{R}$ be a maximal solution. To fix attention assume that u is initially increasing (otherwise consider $-u$), and set $x_c := \sup\{x : v|_{(0,x)} > 0\}$. Clearly, x_c is a finite critical point if $x_c < a$. We will show that otherwise we have a "critical point at $x_c = a = \infty$". A priori there are two possibilities of which the second will be eliminated.

Case 1: u is bounded, i.e. $u_1 := \lim_{x\to a=x_c} u(x)$ is finite.
First observe that $a = \infty$. Indeed, u_{xxx} cannot blow up because $p = \beta u_{xx}$ obeys a linear equation extracted from (4)

$$p_{xx} = \beta^{-1}p - F'(u(x)). \tag{8}$$

By inspection of the vectorfield in the (p, p_x)-plane, we observe that the positive quadrant $\{p, p_x \geq 0\}$ is invariant under the flow of (8) if $F'(u(\cdot)) < 0$, and the negative quadrant $\{p, p_x \leq 0\}$ is invariant if $F'(u(\cdot)) > 0$. Clearly $F'(u(x))$ has eventually (as $x \to \infty$) a fixed sign, and therefore so does p (otherwise consider x_0 with $p_x(x_0) = 0$ and $p(x_0) \neq 0$ of the appropriate sign). Because u is bounded, $p \geq 0$ is excluded and we have $p \leq 0$; so $\lim_{x\to a} v = 0$.

By using the equation (8) again, $\lim_{x\to a} v = 0$ already implies that $\lim_{x\to a} p, p_x = 0$; as a result, $\lim_{x\to a} E = 0$ because $E = v - p_x$. The argument is an easy application of Lemma 4 in Appendix, so we skip it.

To finish with (ii), it is left to see that $a = x_c = \infty$ implies that $F'(u_1) = 0$. Note that, for any $\epsilon > 0$, since $E_x = F'(u)$ from (4), integration yields $(F'(u_1) - \epsilon)x < E < (F'(u_1) + \epsilon)x$ for large x. Thus unless $F'(u_1) = 0$, $|E|$ tends to infinity; and so does $|p|$ because $p = u - \int^x E$ from (4). This contradicts boundedness of u.

Case 2: u is unbounded over $[0, x_c = a)$, i.e. $\lim_{x\to a=x_c} u(x) = \infty$.
Because eventually $F'(u(x)) > 0$, the sign of p stabilizes for $x \to a$ as in Case 1. If eventually $p < 0$, then $p_{xx} = \beta^{-1}p - F'(u) << 0$, which clearly forces a critical point in a finite time $x_c < a$. Thus we face the possibility that eventually $p > 0$, which is eliminated by estimating *the kinetic energy* as follows. In (6) we drop the decelerating force $-F(u)/v^2 < 0$ and consider the truncated Hamiltonian

$$e(p, v) := -\frac{p^2}{2\beta v} + \frac{v}{2},$$

so that (6) gives the inequalities

$$\begin{cases} v_u = -e_p = \frac{p}{\beta v} < 0, \\ p_u = E_v = \frac{1}{2} + \frac{1}{2\beta}p^2/v^2 - F(u)/v^2 \leq e_v. \end{cases} \tag{9}$$

Consequently, $\frac{de}{du} = -e_v v_u + e_p p_u \geq -e_v e_p + e_p e_v = 0$. From the monotonicity of e, writing C for a constant depending on the initial conditions, we estimate for all x close to a:

$$\frac{p^2}{2\beta v} \leq \frac{v}{2} + C,$$

so $p \leq \sqrt{\beta}v + C$, that is $v_u \leq 1/\sqrt{\beta} + C/v$. By integrating with respect to $du = vdx$, this finally yields

$$v \leq u/\sqrt{\beta} + Cx \leq Cu.$$

It follows that $a = \infty$, in fact $u(x) \leq C\exp(x/\sqrt{\beta})$. Also [4],

$$\frac{dE}{du} = \frac{F'(u)}{v} \geq C\frac{F'(u)}{u} \geq Cu^2.$$

Hence $E \geq Cu^3$, and so $p_x = v - E \leq C(u - u^3) \to -\infty$, which contradicts $p > 0$. $\Box$

One can actually see that the blow up for the solutions to the initial value problem of (1) occurs as a series of oscillations with dramatically increasing amplitude and frequency.

Proposition 1 *The area $du \wedge dE$ is preserved by the return map T.*

Sketch of Proof. A minor difficulty to be bypassed is the singularity of (6) at $v = 0$. In the neighborhood of every $(u_0, E_0) \in \Omega$, for $\epsilon > 0$ sufficiently small, we can factor T into three (area preserving) cross-section maps:
1. from (u, E) at $v = 0$, $p > 0$ to (u, E) at $v = \epsilon$, $p > 0$,
2. from (u, E) at $v = \epsilon$, $p > 0$ to (u, E) at $v = \epsilon$, $p < 0$,
3. from (u, E) at $v = \epsilon$, $p < 0$ to (u, E) at $v = 0$, $p < 0$.

We indicate the sign of p only because, from $H = 0$, p^2 is an explicit function of u, v, and E; namely, $p^2/(2\beta) = F(u) + v^2/2 - Ev$.

As long as v is sufficiently small, $v \in [0, \epsilon]$, p is positive and thus v can be treated as an independent variable. From (4), we get then the system

$$\begin{cases} u_v = \beta v/p, \\ E_v = \beta F'(u)/p. \end{cases} \tag{10}$$

[4]All we need here is that $\liminf_{|u|\to\infty} F'(u)/|u|^{1+\epsilon} > 0$, and this is the only instance when the growth condition on F needs to be used.

Here $p = p(u, v, E)$ satisfies (from the formula for p^2)

$$\frac{\partial p}{\partial u} = \frac{\beta F'(u)}{p} \quad \text{and} \quad \frac{\partial p}{\partial E} = -\frac{\beta v}{p},$$

so p is the Hamiltonian of the system (10)[5]. The cross-section map from $v = 0$ to $v = \epsilon$ conserves the area $du \wedge dE$, as the symplectic form of (10).

For the second cross-section from $v = \epsilon$ back to $v = \epsilon$ (with p changing sign), we use u as an independent coordinate. The flow of (6), being Hamiltonian, preserves the volume $du \wedge dv \wedge dp$. Also, the manifold $v = \epsilon$ is of co-dimension one in $H = 0$, and it is transversal to the flow: the normal component is $p/(\beta\epsilon)$. Thus the cross-section map preserves the induced two form

$$\frac{p}{\beta\epsilon} du \wedge dp = dE \wedge du,$$

where we used $E = \epsilon^{-1}(-p^2/(2\beta) + F(u) + \epsilon^2/2)$ deriving from $H = 0$.

The third map can be treated in an analogous way to the first. □

4 Proof of Theorem 1.

4.1 Twist property from order preservation.

Given $-1 < u_0 < u_1 < 1$ one can consider the boundary value problem seeking $l \in (0, \infty]$ and a monotonic function $u : (0, l) \to \mathbf{R}$ satisfying (1) and increasing from u_0 to u_1 with vanishing derivative at the endpoints. We show below uniqueness of solutions to this problem for $\beta < 1/8$. (In particular, there is a unique monotonic heteroclinic u for which $u \in (-1, 1)$.) In terms of T, this means that if (u_0, E_0) maps to (u_1, E_1) and $u_0, u_1 \in (-1, 1)$, then one can recover (E_0, E_1) from (u_0, u_1). This is the essence of *the twist property* as known from the theory of annulus homeomorphisms (see [2]).

Proposition 2 (twist property) *For $\beta < 1/8$, if $u : (a, b) \to (-1, 1)$ and $\tilde{u} : (\tilde{a}, \tilde{b}) \to (-1, 1)$ are both monotonic solutions to (1) with $H = 0$ that agree*

[5] This actually follows automatically from the fact that (u, E) and (v, p) are conjugate pairs of symplectic coordinates: once we treat one variable as independent, its conjugate becomes the Hamiltonian.

at the corresponding boundary points and have vanishing first derivatives at these points, i.e.

$$\lim_{x\to a^+} u(x) = \lim_{x\to \tilde{a}^+} \tilde{u}(x) =: u_- \quad \text{and} \quad \lim_{x\to b^-} u(x) = \lim_{x\to \tilde{b}^-} \tilde{u}(x) =: u_+$$

$$\lim_{x\to a^+} u_x(x) = \lim_{x\to \tilde{a}^+} \tilde{u}_x(x) =: v_- = 0 \quad \text{and} \quad \lim_{x\to b^-} u_x(x) = \lim_{x\to \tilde{b}^-} \tilde{u}_x(x) =: v_+ = 0$$

then $u(\cdot)$ and $\tilde{u}(\cdot)$ coincide up to translation[6].

Proof. Say $u_- < u_+$. If $u_{xx} > 0$ for x in the neighborhood of b, then u_x could not decrease to 0 at b. Similarly, we eliminate the possibility that $u_{xx} < 0$ for x in the neighborhood of a (by running u backwards). As a result $(v,p) \in S(u)$ for all times; otherwise, $p = \pm\sqrt{2\beta F}$ at some u_0, and by (ii) of Fact 1, either $p > \sqrt{2\beta F} \geq 0$ for $u > u_0$, or $p < -\sqrt{2\beta F} \leq 0$ for $u < u_0$ — u_{xx} has the wrong sign around either b or a, respectively.

Analogously, $(\tilde{v},\tilde{p}) \in S(u)$, and we can use the order preservation given by (i) of Fact 1.

Consider $v = u_x$ and $\tilde{v} = \tilde{u}_x$ as functions of $u \in (u_-, u_+)$. Suppose that $p(u_0) = \tilde{p}(u_0)$ for some u_0. If also $v(u_0) = \tilde{v}(u_0)$, then $v(\cdot)$ and $\tilde{v}(\cdot)$ coincide and so do $u(\cdot)$ and $\tilde{u}(\cdot)$ up to translation. Assume then that $v(u_0) > \tilde{v}(u_0)$ — swap u and $\tilde{u}$ if necessary. Then by (i) of Fact 1, we have $p > \tilde{p}$ for all $u > u_0$. By using $p_u = \beta v v_u = \beta(v^2/2)_u$, we integrate to get a contradiction

$$\frac{\beta}{2}(v_+^2 - v(u_0)^2) = \int_{u_0}^{u(b)} p\, du > \int_{u_0}^{u(b)} \tilde{p}\, du = \frac{\beta}{2}(v_+^2 - \tilde{v}(u_0)^2).$$

Thus p and $\tilde{p}$ are never equal, that is $p > \tilde{p}$ or $p < \tilde{p}$ for all $u \in (u_-, u_+)$; and we can integrate to get a contradiction again,

$$\frac{\beta}{2}(v_+^2 - v_-^2) = \int_{u_-}^{u_+} p\, du \neq \int_{u_-}^{u_+} \tilde{p}\, du = \frac{\beta}{2}(v_+^2 - v_-^2).$$

□

[6]One can drop $v_- = v_+ = 0$ in favor of a weaker assumption that $p_- \geq 0$ and $p_+ \geq 0$.

4.2 Non-monotonic solutions within ± 1 do not exist.

By building on (ii) of Fact 1, we will prove that the only solutions to (1) with $\beta < 1/8$ on the level set $H = 0$ that satisfy $-1 < u < 1$ for all times are monotonic heteroclinics joining ± 1.

Proposition 3 (monotonicity) *For $\beta < 1/8$, all solutions $u : \mathbf{R} \to (-1, 1)$ to (1) with $H = 0$ are monotonic and asymptotic to ± 1, i.e. $\lim_{x\to-\infty} u(x) = \pm 1$ and $\lim_{x\to\infty} u(x) = \pm 1$.*

In terms of T, this is the following lemma that stands behind the proposition.

Lemma 2 *For $\beta < 1/8$, if $-1 < u_0 < 1$ and $T(u_0, E_0) \in \{(u, E) : |u| \leq 1\}$, then $E_0 > 0$.*

Proof of Proposition 3. Suppose that u is not constant and has a finite critical point at x_0. Let (x_{-1}, x_0) and (x_0, x_1) be the maximal intervals (perhaps infinite) over which u_x is of constant sign. We may assume for convenience that $u|_{[x_0,x_1)}$ is increasing. If $u_0 := u(x_0)$ and $E_0 := E(x_0)$, then clearly $T(u_0, E_0) \in \{(u, E) : |u| \leq 1\}$, and $E_0 > 0$ by the lemma. Analogously, consider the mirror image of $u|_{(x_{-1},x_0]}$, namely $\tilde{u} : [x_0, -x_{-1}) \to (-1, 1)$, $\tilde{u}(x) := u(-x)$, to see that $\tilde{E}_0 = -E_0 > 0$. This is a contradiction which proves that $u_x > 0$. That $\lim_{x\to\pm\infty} u(x) = \pm 1$ is assured by (ii) of Lemma 1. □

Proof of Lemma 2. Let $u : [0, x_c) \to (-1, 1)$ where $u_x = 0$ and $x_c = \sup\{x : u_x|_{x\in(0,x)} > 0\}$ be as in the definition of T in Section 3. The key observation is that if $E_0 < 0$ then $p(x) > \sqrt{2\beta F(u(x))}$ for all $x \in [0, x_c)$. Indeed, initially $p(0) = \sqrt{2\beta F(u_0)}$ and $p_x(0) = -E_0 > \left(\sqrt{2\beta F(u)}\right)_x = 0$; therefore, $p(x) > \sqrt{2\beta F(u(x))}$ for all sufficiently small $x > 0$, and in fact for all $x \in (0, x_c)$ by (ii) of Fact 2.

In this way, if $E_0 < 0$ then $u|_{(0,x_c)}$ is convex up ($p > 0$) which contradicts $\lim_{x\to x_c^-} u_x(x) = 0$. The convexity clearly persists also for $E_0 = 0$, so there must actually be $E_0 > 0$. □

4.3 Acceleration outside ± 1.

For the stationary Fischer Kolmogorov equation ($\beta = 0$), the solutions not contained in the band $|u| < 1$ must escape to infinity monotonically, which is no longer the case for the EFK equation (1) as we saw in the proof of Lemma 1. Nevertheless, we will uncover an "acceleration" mechanism largely responsible for destroying boundedness of solutions wandering outside $|u| < 1$. One could say that the return map, $R \circ T$, has a large basin of attraction to infinity containing at least $\{(u, E) : |u| > 1\}$.

Proposition 4 *For $\beta < 1/8$, any bounded solution $u : \mathbf{R} \to \mathbf{R}$ to (1) on $H = 0$ satisfies $-1 < u < 1$ or is constant, $u = \pm 1$.*

We notice, as it has already been done in [15], the following fact, which is actually a key feature of a broader class of elliptic PDE's — see Lemma 1 in [6] and the references there in.

Fact 2 *The partial Lagrangian density,*

$$\mathcal{L} := \frac{\beta}{2} u_{xx}^2 + F(u) = \frac{p^2}{2\beta} + F(u),$$

satisfies along the solution to (1)

$$\mathcal{L}_{xx} = \beta u_{xxx}^2 + u_{xx}^2 + F''(u) u_x^2. \tag{11}$$

Thus $\mathcal{L}$ is a convex up function of x as long $u \notin (-1, 1)$ (or just $F''(u) > 0$).

For a solution $u : (x_0 - \epsilon, x_0 + \epsilon) \to \mathbf{R}$ to (1) with $H = 0$, we call x_0 an *exit* iff $|u| - 1$ changes sign to positive at x_0. An exit x_0 is *accelerated* iff $\mathcal{L}_x(x_0) \geq 0$.

Fact 3 *x_0 is an accelerated exit iff $u(x_0) \cdot p(x_0) \geq 0$.*

Proof. Without any loss of generality we can assume that $u(x_0) = +1$. At the exit x_0, $\mathcal{L}_x = pp_x/\beta$. In the case $v(x_0) = 0$, $p(x_0) = 0$ from $H = 0$; and so $l_x(x_0) = 0$ — the exit is accelerated. In the case $v(x_0) > 0$, $p_x(x_0) = p_u(x_0)v(x_0) > v(x_0)/2 > 0$ from the Hamiltonian system (6); thus $p_x(x_0)$ and $l_x(x_0)$ are of the same sign. □

Note that, in view of the Taylor expansion for $u(x)$,

$$u(x_0+\delta x) = u(x_0) + v(x_0)\delta x + \frac{p(x_0)}{2\beta}\delta x^2 + \frac{v(x_0)-E(x_0)}{6\beta}\delta x^3 + ..., \quad (12)$$

the first possibility in the proof above when $v(x_0) = 0$ implies that $-E(x_0) > 0$; otherwise, x_0 would not be an exit.

Lemma 3 (acceleration) *If* x_0 *is an* accelerated exit *for* $u(\cdot)$*, then*
(i) $u(\cdot)$ *has another exit at some* $x_3 > x_0$*, i.e.* $u(\cdot)$ *continues as a solution over* $[x_0, x_3)$ *where* x_3 *is an exit;*
(ii) for $\beta < 1/8$*,* x_3 *is also an* accelerated exit *and* $|p(x_3)| \geq |p(x_0)| + 1$.

We will apply (ii) by bootstrapping it to get infinitely many accelerated exits with $|p|$ increasing to infinity.

Parenthetically, the proof will actually show that that x_3 is the exit immediately following x_0 and that $u|_{[x_0,x_3]}$ has exactly two monotonicity intervals and a unique inflection point in the first one.

Proof of Proposition 4. By Theorem 3 in Appendix, since $u(\cdot)$ is bounded so is $p(\cdot)$. Suppose that $|u(x_0)| = 1$ for some x_0. To fix attention, assume that $u(x_0) = -1$. Consider the Taylor expansion (12) for $u(\cdot)$ around x_0. Either $v(x_0) = 0$ and $E(x_0) = 0$, and then also $p(x_0) = \sqrt{2\beta F(u(x_0))} = 0$ — which yields $u(\cdot) = -1$ by the ODE uniqueness. Or $v(x_0)$ and $E(x_0)$ do not vanish simultaneously, and then x_0 must be an exit either for $u(x)$ or for its reversion $u(-x)$. We can assume that x_0 is an exit for $u(-x)$.

If $p(x_0) \leq 0$, this is an accelerated exit by Fact 3; and $p(\cdot)$ is unbounded by repeated application of Lemma 3. If on the other hand $p(x_0) > 0$, then clearly $p(x_0) > \sqrt{2\beta F(u(x_0))} = 0$ and, from (ii) of Fact 1, we must have $p > \sqrt{2\beta F(u)} \geq 0$ for all $x > x_0$ as long as $u(x) \in (-1, 1)$. It follows that $u(x)$ reaches 1 at some point $x_1 > x_0$, and $p(x_1) > 0$. This makes x_1 an accelerated exit for $u(x)$, which yields once again a contradiction via Lemma 3. □

Proof of Lemma 3. We can assume that $u(x_0) = +1$. By Lemma 1 there exists a smallest critical point $x_1 > x_0$ and $u(x_1)$ is a local maximum, $p(x_1) < 0$. Because $\mathcal{L}_x(x_0) \geq 0$, Fact 3 guarantees that $\mathcal{L}(x)$ is increasing as long as $u(x) > 1$. Hence, once $x > x_1$ and $F(u(x))$ starts to decrease, $|p(x)|$ must increase — see the definition of $\mathcal{L}$. Thus $u(x)$ descends back down to $u = +1$ at some $x_2 > x_1$, and it has no inflection points over (x_1, x_2). Moreover, since $\mathcal{L}(x)$ is increasing over (x_0, x_2), necessarily $-p(x_2) > p(x_0)$.

Now, we claim that since $p(x_2) < 0$, we have $p(x) < 0$ for $x > x_2$ at least until $u(x)$ reaches -1 at some exit point $x_3 > x_2$. Indeed, consider the reflection $\tilde{u}(\cdot) := -u(\cdot)$ of $u(\cdot)$ to put us in the situation of Fact 1 in Section 2. Since $\tilde{p}(x_2) > 0 = \sqrt{2\beta F(\tilde{u}(x_2))}$, by (ii) of Fact 1, we have $\tilde{p} > \sqrt{2\beta F(\tilde{u})}$ as $\tilde{u}$ increases through $(-1, 1)$. Therefore, $\tilde{p}_{\tilde{u}} \geq 1/2$ from the Hamiltonian equation (6), so $\tilde{p}(x_3) \geq \tilde{p}(x_2) + 2 \cdot 1/2$. It follows that $-p(x_3) \geq p(x_0) + 1$; and x_3 is an accelerated exit by Fact 3. □

4.4 Conclusion of the proof of Theorem.

At this point we are ready to prove Theorem 1 as stated in Introduction. Recall that the existence of a heteroclinic solution has been already proved in [12] and [16], so we need not dwell on it. For uniqueness, we consider any bounded non-constant solution $u(\cdot)$ to (1) on $H = 0$. By Proposition 4, $-1 < u < +1$. By Proposition 3, u is monotonic and asymptotic to ± 1. By (ii) of Lemma 1, $\lim_{x \to \pm\infty} u_x(x) = 0$, and we can use Proposition 2 with $a = \tilde{a} = -\infty$ and $b = \tilde{b} = \infty$ to conclude that there is a unique such $u(\cdot)$ up to a translation. This ends the proof.

5 Appendix: Universal Bounds

Apart from some technicalities needed for the proof of Theorem 1, we provide a uniform a priori bound on the bounded stationary solutions to the EFK equation.

The results are no longer restricted to $\beta < 1/8$, and we will find it more convenient to use the following rescaled version of (1)

$$u_{xxxx} - \gamma u_{xx} + F'(u) = 0, \quad x \in \mathbf{R}, \quad \gamma > 0. \tag{13}$$

We start with a simple Liouville type result which is actually a version of a more delicate and general theorem holding for elliptic fourth order PDE's (see [6] and the references there in).

Theorem 2 *Let $F : \mathbf{R} \to \mathbf{R}$ be C^2, convex up, and proper (i.e. $F'' \geq 0$ and $\lim_{|u|\to\infty} F(u) = \infty$). If $u : \mathbf{R} \to \mathbf{R}$ is bounded and $u_{xxxx} + F'(u) = 0$ for $x \in \mathbf{R}$, then u is constant.*

Proof: As in Fact 2 in Section 4, the Lagrangian density $L := \frac{1}{2}u_{xx}^2 + F(u)$ is convex up because

$$L_{xx} = u_{xxx}^2 + F''(u)u_x^2 + u_{xx}(u_{xxxx} + F'(u)) = u_{xxx}^2 + F''(u)u_x^2 \geq 0. \quad (14)$$

We have two a priori cases:
Case 1: L is unbounded, i.e. $\lim_{|x|\to\infty} L = \lim_{|x|\to\infty} \frac{1}{2}u_{xx}^2 = \infty$. Then however $u_x \geq 1$ over an infinite interval and so u is unbounded — a contradiction.
Case 2: L is bounded, and thus it is constant. Then $L_{xx} = 0$ and $u_{xxx} = 0$ from (14), which makes u a bounded quadratic, a constant. □

The following are a priori bounds naturally expected due to the ellipticity of (13).

Theorem 3 *For any $\gamma_0 > 0$, there is $K > 0$ such that if $0 < \gamma \leq \gamma_0$ and $u : \mathbf{R} \to \mathbf{R}$ satisfies (13) and is (uniformly) bounded, then*

$$\|u_x\|, \|u_{xx}\|, \|u_{xxx}\| \leq K(\|u\| + \|F'(u)\|),$$

where $\|\cdot\|$ stands for the supremum norm over $\mathbf{R}$.

Proof: If $D := \frac{d}{dx}$ then the EFK equation (13) can be written as

$$(D^2 - \gamma/2)^2 u = -F'(u) + \frac{\gamma^2}{4}u =: g.$$

The C^0 semi-group generated by $A := D^2 - \gamma/2$ is clearly

$$T(t) = e^{-\gamma/2}U(t)$$

where $U(t)$ is the heat equation semigroup, so $\|T(t)\| \leq e^{-\gamma t/2}$. By (elementary) Lemma 2.8, page 7, in [11], we have that

$$\|Au\| \leq \frac{2e^{-\gamma t/2}}{t}\|u\| + \frac{e^{-\gamma t/2}t}{2}\|A^2 u\|.$$

For $t = 1$, we get a useful bound on u_{xx}

$$\|u_{xx} - \frac{\gamma}{2}u\| \leq K(\|u\| + \|g\|)$$

with $K = K(\gamma) > 0$. Via Landau's inequality $\|u_x\| \le 4\sqrt{\|u\|\|u_{xx}\|}$ (see [11]),

$$\|u_x\| \le 4\sqrt{\|u\|\|u_{xx}\|} \le 2\|u\| + 2\|u_{xx}\| \le K(\|u\| + \|F'(u)\|),$$

and similarly, using also $u_{xxxx} = \gamma u_{xx} - F'(u)$,

$$\|u_{xxx}\| \le 4\sqrt{\|u_{xx}\|\|u_{xxxx}\|} \le 2\|u_{xx}\| + 2\|u_{xxxx}\| \le K(\|u\| + \|F'(u)\|),$$

perhaps with a slightly different K. □

It is well known that, if a PDE resembles in the large length scales another PDE governed by a Liouville type result, one expects uniform a priori bounds for the original PDE (see e.g. [5]). The following result is an instance of this situation.

Theorem 4 *Suppose that F is C^2 and $\lim_{|u|\to\infty} F'(u)/u^3 = 1$. There is $M > 0$ such that any bounded $u : \mathbf{R} \to \mathbf{R}$ solving the EFK equation (13) satisfies $\|u\| \le M$.*

Proof: We follow the method of [5]. Suppose that solutions $u_n : \mathbf{R} \to \mathbf{R}$ are such that $a_n := \|u_n\| \to \infty$. Let $\lambda_n := a_n^{-1/2}$, and translate u_n's so that $u_n(0) \ge \|u_n\|/2$. The functions $v_n(s) := u_n(\lambda_n s)/a_n$, i.e. $u_n(x) = a_n v_n(\lambda_n^{-1} x)$, are of norm one, $\|v_n\| = 1$, and they satisfy

$$a_n \lambda_n^{-4} {v_n}'''' - \gamma a_n \lambda_n^{-2} v_n'' + F'(a_n v_n) = 0,$$

$$v_n'''' - \gamma \lambda_n^2 v_n'' + \lambda_n^4 a_n^{-1} F'(a_n v_n) = 0.$$

Note that

$$F_n'(v) := \lambda_n^4 a_n^{-1} F'(a_n v) = a_n^{-3} F'(a_n v) = v^3 + o(1).$$

From Theorem 2, $\|v_n'\|$, $\|v_n''\|$, $\|v_n'''\|$ are bounded by $K(\|v_n\| + \|F_n'(v_n)\|) \le C$. From the equation, also $\|v_n''''\|$ and $\|v_n'''''\|$ are universally bounded so that we can select a subsequence uniformly convergent with four derivatives to some v_*, which must then satisfy the limiting equation

$$v_*'''' + v_*^3 = 0.$$

From Theorem 2, v_* is constant, $v_* = 0$, which contradicts the fact that $v_*(0) \ge 1/2$ as forced by the normalization of u_n's. □

The proof can be generalized to $F'(u)$ behaving as $|u|^{1+\epsilon}$ at $u \approx \infty$.

We finish with a technical lemma needed in the proof of (ii) in Lemma 1 in Section 3 to show that p, $p_x \to 0$ whenever $p < 0$, $u \to \pm 1$, and $v \to 0$, all for $x \to \infty$. The lemma should be applied to p over $[x_* - 1, x_* + 1]$ with $x_* \to \infty$ (and $\gamma := -F'(u)$). It is only for convenience that we shift below the x variable back to $[-1, 1]$.

Lemma 4 *If $p_{xx}(x) = p(x) + \gamma(x)$ and $p(x) > 0$ for all $x \in [-1, 1]$, then*

$$\frac{p(0)^2}{3|p(0)| + 2\sup|\gamma|} \leq \int_{-1}^{1} |p|$$

and

$$p'(0) \leq 3p(0)/2 + \sup|\gamma|/2.$$

Proof. Set $\delta := \sup|\gamma|$. We can assume that both $h := p(0) > 0$ and $s := -p'(0) > 0$; otherwise consider $p(-x)$. We start with the second inequality. Suppose it fails; then $s - \frac{h+\delta}{2} > h$. Let $a := \sup\{x \in [0, 1] : \ p|_{[0,x]} \leq h\}$. By integrating p_{xx} twice, we get, for $x \in [0, a]$,

$$p'(x) = -s + \int_0^x (p + \gamma) \leq -s + (h + \delta)x,$$

$$p(x) \leq h - sx + \frac{h+\delta}{2}x^2 \leq h - \left(s - \frac{h+\delta}{2}\right)x.$$

The last inequality guarantees then that $p(x) \leq h$ for all $x \in [0, a]$, that $a = 1$, and that $p(x_0) = 0$ for some $x_0 \leq 1$. Thus we contradicted $p > 0$.

For the first inequality we estimate analogously for all $x \in [0, 1]$:

$$p'(x) \geq -s + \int_0^x \gamma \geq -s - \delta x,$$

$$p(x) \geq h - sx - \frac{\delta}{2}x^2 \geq h - \left(s + \frac{\delta}{2}\right)x.$$

An inspection of the area underneath the linear function above yields

$$\int_0^1 p \geq \min\left\{\frac{h^2}{2s+\delta}, \frac{h}{2}\right\},$$

with the two cases depending on whether the function has a zero in $[0,1]$. Since the second inequality is already established, and so $s \leq 3h/2 + \delta/2$, we recover the first inequality of the lemma

$$\int_0^1 p \geq \min\left\{\frac{h^2}{3h+2\delta}, \frac{h}{2}\right\} = \frac{h^2}{3h+2\delta}.$$

□

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