9.7 Exponentially Weighted Moving Average Control Charts

- The **exponentially weighted moving average (EWMA)** chart was introduced by Roberts (*Technometrics* 1959) and was originally called a *geometric moving average chart.* The name was changed to reflect the fact that exponential smoothing serves as the basis of EWMA charts.

- Like a cusum chart, an EWMA chart is an alternative to a Shewhart individuals or $\overline{x}$ chart and provides quicker responses to shifts in the process mean then either an individuals or $\overline{x}$ chart because it incorporates information from all previously collected data.

- To construct an EWMA chart, we assume we have k samples of size $n \geq 1$ yielding k individual measurements $x_1, \ldots, x_k$ (if $n = 1$) or k sample means $\overline{x}_1, \ldots, \overline{x}_k$ (if $n > 1$).

- We will work with the simpler case of individual measurements ($n = 1$) when developing the formulas. To work with sample means, replace σ with $\sigma/\sqrt{n}$ in all formulas.

- Let z_i be the value of the exponentially weighted moving average at the i^{th} sample. That is,

$$z_i = \qquad (24)$$

  where $0 < \lambda \leq 1$. λ is called the **weighting constant**.

- We also need to define a **starting value** z_0 before the first sample is taken.
  - If a target value μ is specified, then $z_0 = \mu$.
  - Otherwise, it is typical to use the average of some preliminary data. That is, $z_0 = \overline{x}$.

- Note that the EWMA z_i is a weighted average of all observations that precede it. For example:

$$\begin{aligned}
i = 1 \quad z_1 &= \lambda x_1 \ + \ (1-\lambda) z_0 \\
i = 2 \quad z_2 &= \lambda x_2 \ + \ (1-\lambda) z_1 \ = \\
&= \\
&= \lambda(1-\lambda)^0 x_2 \ + \ (1-\lambda)^1 \lambda x_1 \ + \ (1-\lambda)^2 z_0 \\
i = 3 \quad z_3 &= \lambda x_3 \ + \ (1-\lambda) z_2 \\
&= \lambda x_3 \ + \ (1-\lambda) \\
&= \lambda x_3 \ + \ (1-\lambda) \\
&= (1-\lambda)^0 \lambda x_3 \ + \ (1-\lambda)^1 \lambda x_2 \ + \ (1-\lambda)^2 \lambda x_1 + (1-\lambda)^3 z_0
\end{aligned}$$

- In general, by repeated substitution in (24), we recursively can write each z_i (if $0 < \lambda < 1$) as

$$z_i \ = \ \lambda \sum_{j=0}^{i-1} (1-\lambda)^j x_{i-j} + (1-\lambda)^i z_0 \qquad (25)$$

- Recall $\displaystyle\sum_{j=0}^{i-1} p^j = \frac{1-p^i}{1-p}$ for $|p|<1$. If $p = 1-\lambda$, then the sum of the weights in (25) is

$$\begin{aligned}\lambda\sum_{j=0}^{i-1}(1-\lambda)^j + (1-\lambda)^i &= \\ \\ &= \\ &= \\ &= \end{aligned}$$

- The fact that the weights decrease exponentially is the reason it is called an exponentially weighted moving average chart.
- The weighting constant λ controls the amount of influence that previous observations have on the current EWMA z_i.
  - Values of λ near 1 put almost all weight on the current observation. That is, the closer λ is to 1, the more the EWMA chart resembles a Shewhart chart. (In fact, if $\lambda = 1$, the EWMA chart is a Shewhart chart).
  - For values of λ near 0, a small weight is applied to almost all of the past observations, and the performance of the EWMA chart parallels that of a cusum chart.
- Because the EWMA is a weighted average of the current and all past observations, it is generally insensitive to the normality assumption. Therefore, it can be a usefule control charting procedure to use with individual observations.
- If the observations x_i are independent with common variance σ^2, then the variance of z_i is

$$\begin{aligned}\sigma^2_{z_i} &= \mathrm{Var}\left(\lambda\sum_{j=0}^{i-1}(1-\lambda)^j x_{i-j} + (1-\lambda)^i z_0\right) \\ &= \lambda^2\sum_{j=0}^{i-1}(1-\lambda)^{2j}\sigma^2 \;+\; 0 \\ &= \lambda^2\frac{1-(1-\lambda)^{2i}}{1-(1-\lambda)^2}\sigma^2 \\ &= \lambda^2\frac{1-(1-\lambda)^{2i}}{2\lambda-\lambda^2}\sigma^2 \quad = \quad \frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2i}\right)\sigma^2\end{aligned}$$

- When μ_0 and σ^2 are known, the EWMA chart is constructed by plotting z_i versus the sample number i with control limits at:

$$\begin{aligned}\text{UCL} &= \mu_0 + L\sigma\sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2i}\right)} \\ \text{Centerline} &= \mu_0 \\ \text{LCL} &= \mu_0 - L\sigma\sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2i}\right)}\end{aligned}$$

  We will discuss the choice of L and λ later.

- Note that $\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2i}\right) \longrightarrow \frac{\lambda}{2-\lambda}$ as i increases. Thus, after the EWMA chart has been running for several samples, the control limits will approach the following steady-state values (called **asymptotic control limits**):

$$\text{UCL} = \mu_0 +$$

$$\text{LCL} = \mu_0 -$$

- It is recommended that exact control limits be used for small values of i because it will greatly improve the performance of the EWMA chart in detecting an off-target process very soon after the EWMA is started.

- *SAS* plots exact control limits by default. Plotting asymptotic control limits is an option.

- **Example**: Suppose $\lambda = .25, L = 3, \sigma = 1$, and $\mu_0 = 0$. Then, using the asymptotic variance, the control limits are

$$UCL = 0 + (3)(1)\sqrt{\frac{.25}{1.75}} \approx \qquad LCL = 0 - (3)(1)\sqrt{\frac{.25}{1.75}} \approx$$

The following table summarizes the EWMA calculation for 16 sample values (with comparison calculations for a tabular cusum with $h = 5$ and $k = .5$). Both the EWMA and the Cusum indicate an out-of-control signal on sample 16.

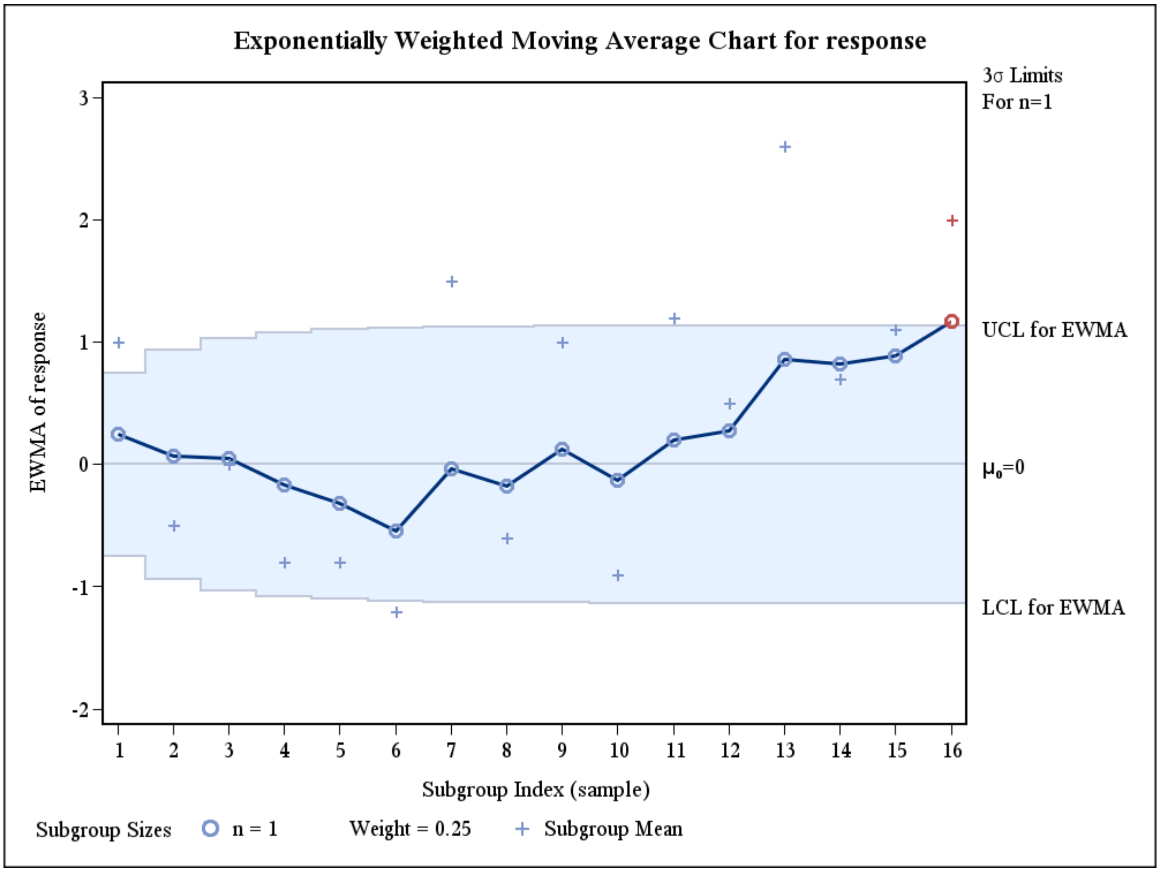
i	x_i	EWMA z_i $\lambda = .25$	Tabular Cusum with $h = 5$, $k = .5$	
0	—	$z_0 = 0$	$C_i^+ = 0$	$C_i^- = 0$
1	1.0	.250	0.5	0.0
2	-0.5	.063	0.0	0.0
3	0.0	.047	0.0	0.0
4	-0.8	-.165	0.0	0.3
5	-0.8	-.324	0.0	0.6
6	-1.2	-.543	0.0	1.3
7	1.5	-.032	1.0	0.0
8	-0.6	-.174	0.0	0.1
9	1.0	.120	0.5	0.0
10	-0.9	-.135	0.0	0.4
11	1.2	.199	0.7	0.0
12	0.5	.274	0.7	0.0
13	2.6	.855	2.8	0.0
14	0.7	.817	3.0	0.0
15	1.1	.887	3.6	0.0
16	2.0	**1.166**	**5.1**	0.0

Sample EWMA calculations of $z_i = .25x_i + .75z_{i-1}$

$$\begin{aligned}
z_1 &= (.25)(1) + (.75)(0) = .25 \\
z_2 &= (.25)(-.5) + (.75)(.25) = .0625 \approx .063 \\
z_3 &= (.25)(0) + (.75)(.0625) = .046875 \approx .047 \\
z_4 &= (.25)(-.8) + (.75)(.046875) = -.16484375 \approx -.165
\end{aligned}$$

EWMA Chart Example

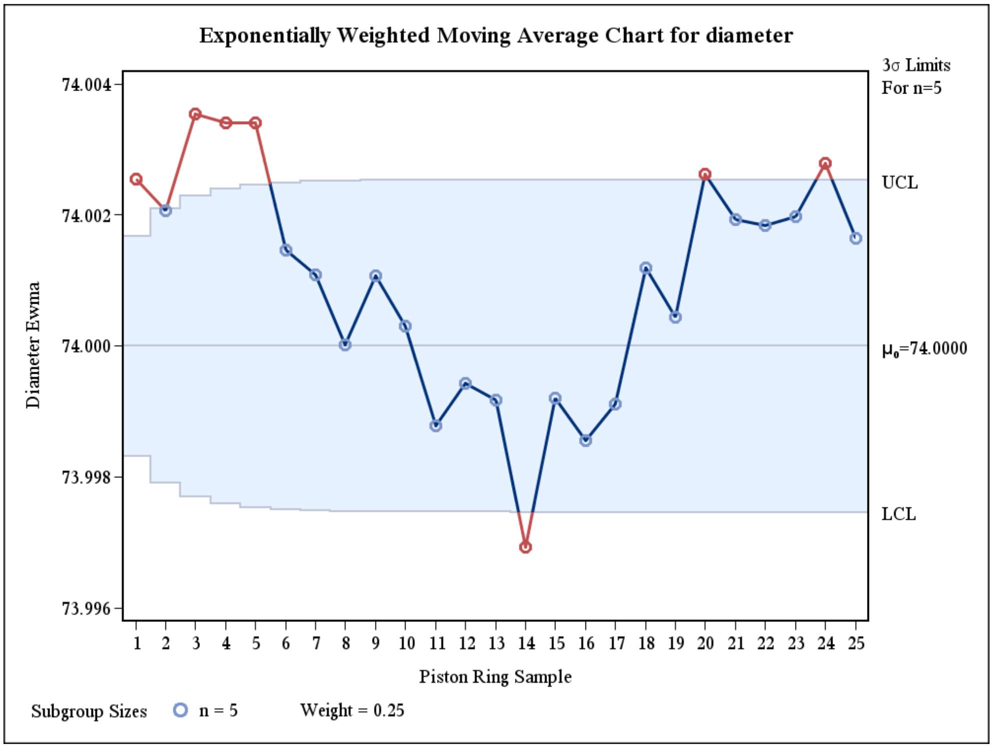
The MACONTROL Procedure



Exponentially Weighted Moving Average Chart Summary for response						
sample	Subgroup Sample Size	Lower Limit	EWMA	Subgroup Mean	Upper Limit	Limit Exceeded
1	1	-0.7500000	0.2500000	1.0000000	0.7500000	
2	1	-0.9375000	0.0625000	-0.5000000	0.9375000	
3	1	-1.0280490	0.0468750	0.0000000	1.0280490	
4	1	-1.0756383	-0.1648438	-0.8000000	1.0756383	
5	1	-1.1015041	-0.3236328	-0.8000000	1.1015041	
6	1	-1.1157901	-0.5427246	-1.2000000	1.1157901	
7	1	-1.1237462	-0.0320435	1.5000000	1.1237462	
8	1	-1.1281968	-0.1740326	-0.6000000	1.1281968	
9	1	-1.1306926	0.1194756	1.0000000	1.1306926	
10	1	-1.1320941	-0.1353933	-0.9000000	1.1320941	
11	1	-1.1328816	0.1984550	1.2000000	1.1328816	
12	1	-1.1333244	0.2738412	0.5000000	1.1333244	
13	1	-1.1335734	0.8553809	2.6000000	1.1335734	
14	1	-1.1337134	0.8165357	0.7000000	1.1337134	
15	1	-1.1337922	0.8874018	1.1000000	1.1337922	
16	1	-1.1338365	1.1655513	2.0000000	1.1338365	Upper

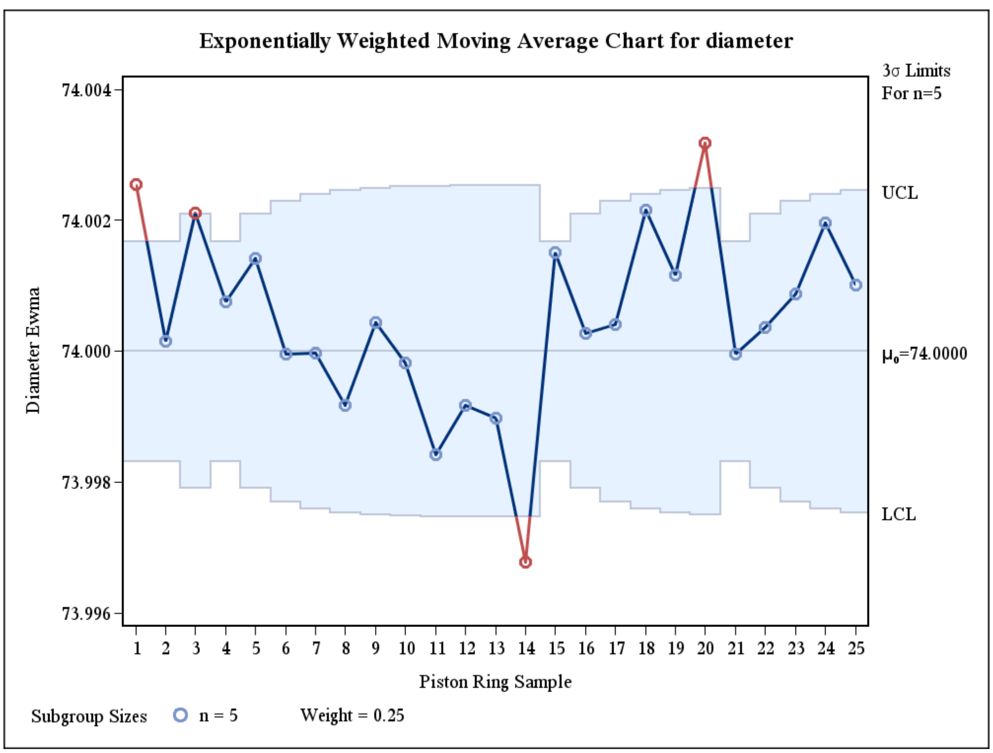
EWMA Chart for Piston-Ring Diameters
lambda weight=0.25 (mu, sigma known)

The MACONTROL Procedure



EWMA Chart for Piston-Ring Diameters with Resets
lambda weight=0.25 (mu, sigma known)

The MACONTROL Procedure



EWMA Chart for Piston-Ring Diameters lambda weight=0.25 (mu, sigma known)

The MACONTROL Procedure

EWMA Parameters	
Sigmas	3
Weight	0.25
Nominal Sample Size	5

Exponentially Weighted Moving Average Chart Summary for diameter						
sample	Subgroup Sample Size	Lower Limit	EWMA	Subgroup Mean	Upper Limit	Limit Exceeded
1	5	73.998323	74.002550	74.010200	74.001677	Upper
2	5	73.997904	74.002063	74.000600	74.002096	
3	5	73.997701	74.003547	74.008000	74.002299	Upper
4	5	73.997595	74.003410	74.003000	74.002405	Upper
5	5	73.997537	74.003408	74.003400	74.002463	Upper
6	5	73.997505	74.001456	73.995600	74.002495	
7	5	73.997487	74.001092	74.000000	74.002513	
8	5	73.997477	74.000019	73.996800	74.002523	
9	5	73.997472	74.001064	74.004200	74.002528	
10	5	73.997469	74.000298	73.998000	74.002531	
11	5	73.997467	73.998774	73.994200	74.002533	
12	5	73.997466	73.999430	74.001400	74.002534	
13	5	73.997465	73.999173	73.998400	74.002535	
14	5	73.997465	73.996929	73.990200	74.002535	Lower
15	5	73.997465	73.999197	74.006000	74.002535	
16	5	73.997465	73.998548	73.996600	74.002535	
17	5	73.997465	73.999111	74.000800	74.002535	
18	5	73.997465	74.001183	74.007400	74.002535	
19	5	73.997465	74.000437	73.998200	74.002535	
20	5	73.997465	74.002628	74.009200	74.002535	Upper
21	5	73.997465	74.001921	73.999800	74.002535	
22	5	73.997465	74.001841	74.001600	74.002535	
23	5	73.997465	74.001981	74.002400	74.002535	
24	5	73.997465	74.002785	74.005200	74.002535	Upper
25	5	73.997465	74.001639	73.998200	74.002535	

EWMA Chart for Piston-Ring Diameters with Resets lambda weight=0.25 (mu, sigma known)

The MACONTROL Procedure

EWMA Parameters	
Sigmas	3
Weight	0.25
Nominal Sample Size	5

Exponentially Weighted Moving Average Chart Summary for diameter						
sample	Subgroup Sample Size	Lower Limit	EWMA	Subgroup Mean	Upper Limit	Limit Exceeded
1	5	73.998323	74.002550	74.010200	74.001677	Upper
2	5	73.998323	74.000150	74.000600	74.001677	
3	5	73.997904	74.002113	74.008000	74.002096	Upper
4	5	73.998323	74.000750	74.003000	74.001677	
5	5	73.997904	74.001413	74.003400	74.002096	
6	5	73.997701	73.999959	73.995600	74.002299	
7	5	73.997595	73.999970	74.000000	74.002405	
8	5	73.997537	73.999177	73.996800	74.002463	
9	5	73.997505	74.000433	74.004200	74.002495	
10	5	73.997487	73.999825	73.998000	74.002513	
11	5	73.997477	73.998418	73.994200	74.002523	
12	5	73.997472	73.999164	74.001400	74.002528	
13	5	73.997469	73.998973	73.998400	74.002531	
14	5	73.997467	73.996780	73.990200	74.002533	Lower
15	5	73.998323	74.001500	74.006000	74.001677	
16	5	73.997904	74.000275	73.996600	74.002096	
17	5	73.997701	74.000406	74.000800	74.002299	
18	5	73.997595	74.002155	74.007400	74.002405	
19	5	73.997537	74.001166	73.998200	74.002463	
20	5	73.997505	74.003175	74.009200	74.002495	Upper
21	5	73.998323	73.999950	73.999800	74.001677	
22	5	73.997904	74.000363	74.001600	74.002096	
23	5	73.997701	74.000872	74.002400	74.002299	
24	5	73.997595	74.001954	74.005200	74.002405	
25	5	73.997537	74.001015	73.998200	74.002463	

```
DM 'LOG; CLEAR; OUT; CLEAR;';
ODS PRINTER PDF file='C:\COURSES\ST528\ewma1.pdf';
* ODS LISTING;
OPTIONS LS=76 PS=100 NONUMBER NODATE;

DATA piston;
     DO sample=1 TO 25;
     DO item=1 TO 5;
        INPUT diameter @@;
        diameter = diameter + 70; OUTPUT;
     END; END;
LINES;
 4.030 4.002 4.019 3.992 4.008   3.995 3.992 4.001 4.011 4.004
   :     :     :     :     :
 3.982 3.984 3.995 4.017 4.013
 ;
SYMBOL1 V=dot WIDTH=3 ;

PROC MACONTROL DATA=piston;
      EWMACHART diameter*sample='1' / WEIGHT  = 0.25
                          mu0 = 74 sigma0=.005
                          XSYMBOL = mu0 COUT
                          HAXIS = 1 to 25
                          TABLE TABLEOUTLIM; * MEANSYMBOL=plus;
     LABEL diameter='Diameter Ewma'
           sample = 'Piston Ring Sample';
TITLE 'EWMA Chart for Piston-Ring Diameters';
TITLE2 'lambda weight=0.25 (mu, sigma known)';

PROC MACONTROL DATA=piston;
      EWMACHART diameter*sample='1' / WEIGHT  = 0.25
                          mu0 = 74 sigma0=.005
                          XSYMBOL = mu0 COUT RESET
                          HAXIS = 1 to 25
                          TABLE TABLEOUTLIM; * MEANSYMBOL=plus;
     LABEL diameter='Diameter Ewma'
           sample = 'Piston Ring Sample';
TITLE 'EWMA Chart for Piston-Ring Diameters with Resets';
TITLE2 'lambda weight=0.25 (mu, sigma known)';
RUN;
```

9.8 Estimating σ for a EWMA chart

- If unknown, the process standard deviation σ must be estimated from the data. In *SAS*, σ is estimated for the EWMA chart by a method similar to the MSSD used for cusum charts.
- Montgomery provides two alternative formulas for estimating σ:
  1. If μ is specified, $\widehat{\sigma^2}$ is the **exponentially weighted mean square error (EWMS)**:

     $$S_i^2 =$$

     – S_i^2 is consistent ($E(S_i^2) \to \sigma^2$ as $i \to \infty$).
     – S_i^2 is approximately χ^2 distributed with $\nu = (2 - \lambda)/\lambda$ degrees of freedom.
     – Thus, if σ_0 is the in-control σ, we could plot $\sqrt{S_i^2}$ versus the sample and set up control limits for an exponentially weighted root mean square error control chart by:

     $$\text{UCL} = \sigma_0 \sqrt{\frac{\chi^2_{\nu,\alpha/2}}{\nu}} \qquad \text{and} \qquad \text{LCL} = \sigma_0 \sqrt{\frac{\chi^2_{\nu,1-\alpha/2}}{\nu}}$$

2. If μ is not specified, $\widehat{\sigma^2}$ is the **exponentially weighted moving variance (EWMV)**:

$$S_i^2 =$$

- Because the points on the EWMA chart are weighted averages of the previous observations, successive EWMA points tend to be highly correlated with one another. As a result the out-of-control modified rules used with Shewhart charts cannot be applied to an EWMA chart because these rules apply to points that are statistically independent.
- From a SPC viewpoint, the EWMA is roughly equivalent to the cusum in its ability to *monitor* a process and to detect assignable causes that result in a process shift. The EWMA, however, also provides a *forecast* of where the process mean will be at time or sample $i+1$. Thus, z_i is a forecast of μ at time $i+1$.
- Thus, with an EWMA, we dynamically update our forecast as each new observation x arrives.
- The EWMA chart control limits can be used to determine *when* an adjustment in the process is necessary. We can determine *how much* to adjust the process at time $i+1$ by the difference $z_i - \mu$ (the difference between our current estimate and the target value).

```
DM 'LOG; CLEAR; OUT; CLEAR;';
ODS PRINTER PDF file='C:\COURSES\ST528\ewma2.pdf';
* ODS LISTING;
OPTIONS LS=76 PS=100 NONUMBER NODATE;

DATA piston;
     DO sample=1 TO 25;
     DO item=1 TO 5;
        INPUT diameter @@;
        diameter = diameter + 70; OUTPUT;
     END; END;
LINES;
 4.030 4.002 4.019 3.992 4.008   3.995 3.992 4.001 4.011 4.004
 3.988 4.024 4.021 4.005 4.002   4.002 3.996 3.993 4.015 4.009
 3.992 4.007 4.015 3.989 4.014   4.009 3.994 3.997 3.985 3.993
 3.995 4.006 3.994 4.000 4.005   3.985 4.003 3.993 4.015 3.988
 4.008 3.995 4.009 4.005 4.004   3.998 4.000 3.990 4.007 3.995
 3.994 3.998 3.994 3.995 3.990   4.004 4.000 4.007 4.000 3.996
 3.983 4.002 3.998 3.997 4.012   4.006 3.967 3.994 4.000 3.984
 4.012 4.014 3.998 3.999 4.007   4.000 3.984 4.005 3.998 3.996
 3.994 4.012 3.986 4.005 4.007   4.006 4.010 4.018 4.003 4.000
 3.984 4.002 4.003 4.005 3.997   4.000 4.010 4.013 4.020 4.003
 3.988 4.001 4.009 4.005 3.996   4.004 3.999 3.990 4.006 4.009
 4.010 3.989 3.990 4.009 4.014   4.015 4.008 3.993 4.000 4.010
 3.982 3.984 3.995 4.017 4.013
 ;
SYMBOL1 V=dot WIDTH=3 ;

PROC MACONTROL DATA=piston;
      EWMACHART diameter*sample='1' / WEIGHT  = 0.25
                          mu0 = 74  RESET
                          XSYMBOL = mu0 COUT
                          HAXIS = 1 to 25  OUTLIMITS=limset
                          TABLE TABLEOUTLIM;
     LABEL diameter='Diameter Ewma'
           sample = 'Piston Ring Sample';
TITLE 'EWMA Chart for Piston-Ring Diameters';
TITLE2 'lambda weight=0.25 (sigma unknown)';

PROC PRINT DATA = limset;
RUN;
```

EWMA Chart for Piston-Ring Diameters
lambda weight=0.25 (sigma unknown)

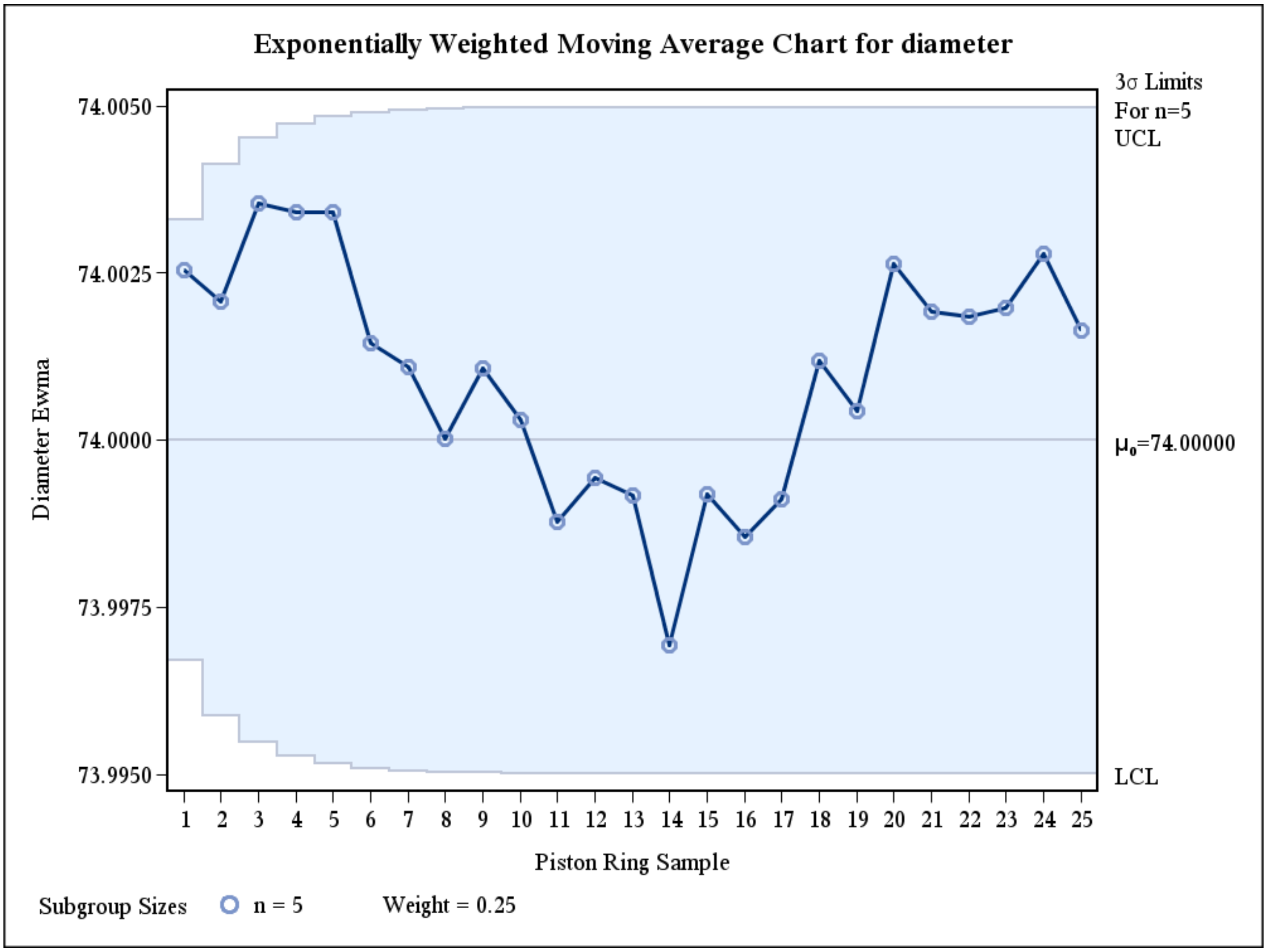
The MACONTROL Procedure

EWMA Parameters	
Sigmas	3
Weight	0.25
Nominal Sample Size	5

Exponentially Weighted Moving Average Chart Summary for diameter						
sample	**Subgroup Sample Size**	**Lower Limit**	**EWMA**	**Subgroup Mean**	**Upper Limit**	**Limit Exceeded**
1	5	73.996703	74.002550	74.010200	74.003297	
2	5	73.995879	74.002063	74.000600	74.004121	
3	5	73.995481	74.003547	74.008000	74.004519	
4	5	73.995271	74.003410	74.003000	74.004729	
5	5	73.995158	74.003408	74.003400	74.004842	
6	5	73.995095	74.001456	73.995600	74.004905	
7	5	73.995060	74.001092	74.000000	74.004940	
8	5	73.995040	74.000019	73.996800	74.004960	
9	5	73.995029	74.001064	74.004200	74.004971	
10	5	73.995023	74.000298	73.998000	74.004977	
11	5	73.995020	73.998774	73.994200	74.004980	
12	5	73.995018	73.999430	74.001400	74.004982	
13	5	73.995017	73.999173	73.998400	74.004983	
14	5	73.995016	73.996929	73.990200	74.004984	
15	5	73.995016	73.999197	74.006000	74.004984	
16	5	73.995016	73.998548	73.996600	74.004984	
17	5	73.995015	73.999111	74.000800	74.004985	
18	5	73.995015	74.001183	74.007400	74.004985	
19	5	73.995015	74.000437	73.998200	74.004985	
20	5	73.995015	74.002628	74.009200	74.004985	
21	5	73.995015	74.001921	73.999800	74.004985	
22	5	73.995015	74.001841	74.001600	74.004985	
23	5	73.995015	74.001981	74.002400	74.004985	
24	5	73.995015	74.002785	74.005200	74.004985	
25	5	73.995015	74.001639	73.998200	74.004985	

EWMA Chart for Piston-Ring Diameters
lambda weight=0.25 (sigma unknown)

The MACONTROL Procedure

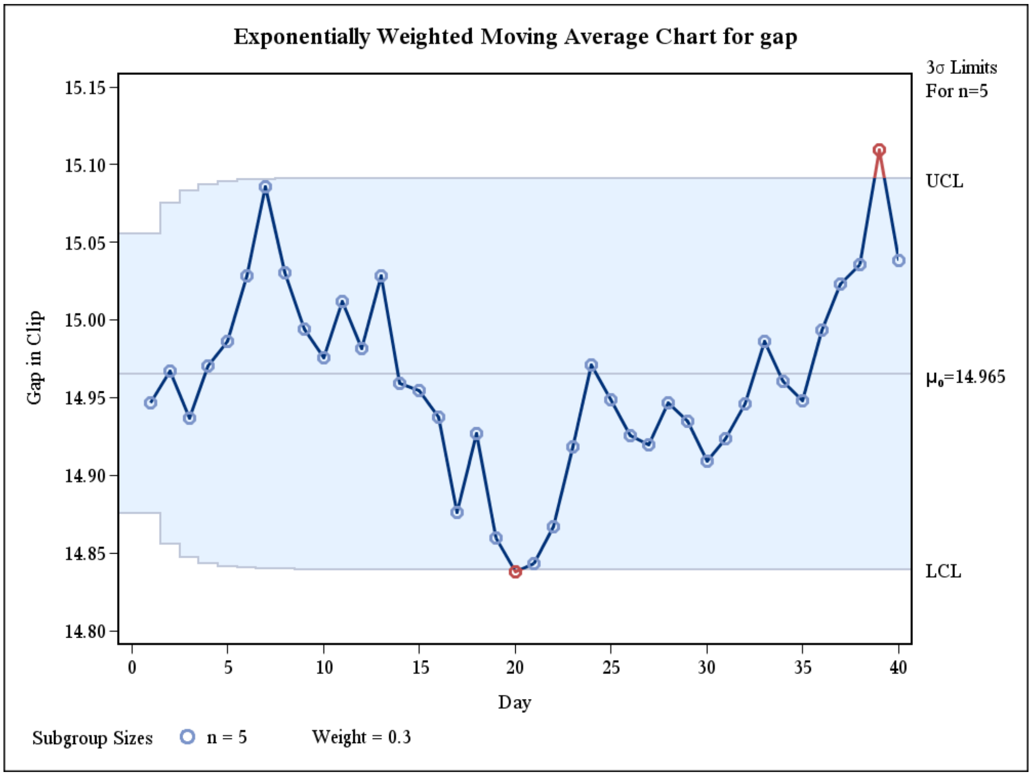


Obs	_VAR_	_SUBGRP_	_TYPE_	_LIMITN_	_ALPHA_	_SIGMAS_	_MEAN_	_STDDEV_	_WEIGHT_
1	diameter	sample	STDMU	5	.002699796	3	74	.009829977	0.25

```
**************************************************************************;
*** In the manufacture of a metal clip, the gap between the ends of the ***;
*** clip is a critical dimension. To monitor the process for change in  ***;
*** the average gap, subgroup samples of five clips are selected daily. ***;
**************************************************************************;
DM 'LOG; CLEAR; OUT; CLEAR;';
ODS LISTING;
* ODS PRINTER PDF file='C:\COURSES\ST528\ewma3.PDF';
OPTIONS NODATE NONUMBER LS=76 PS=54;

DATA clips1;
    INPUT day @@;
    DO i=1 TO 5;
       INPUT gap @@ ; OUTPUT;
    END;
LINES;
 1 14.76 14.82 14.88 14.83 15.23    2 14.95 14.91 15.09 14.99 15.13
 3 14.50 15.05 15.09 14.72 14.97    4 14.91 14.87 15.46 15.01 14.99
 5 14.73 15.36 14.87 14.91 15.25    6 15.09 15.19 15.07 15.30 14.98
 7 15.34 15.39 14.82 15.32 15.23    8 14.80 14.94 15.15 14.69 14.93
 9 14.67 15.08 14.88 15.14 14.78   10 15.27 14.61 15.00 14.84 14.94
11 15.34 14.84 15.32 14.81 15.17   12 14.84 15.00 15.13 14.68 14.91
13 15.40 15.03 15.05 15.03 15.18   14 14.50 14.77 15.22 14.70 14.80
15 14.81 15.01 14.65 15.13 15.12   16 14.82 15.01 14.82 14.83 15.00
17 14.89 14.90 14.60 14.40 14.88   18 14.90 15.29 15.14 15.20 14.70
19 14.77 14.60 14.45 14.78 14.91   20 14.80 14.58 14.69 15.02 14.85
21 14.86 15.01 14.67 14.67 15.07   22 14.93 14.53 15.07 15.10 14.98
23 15.27 14.90 15.12 15.10 14.80   24 15.02 15.21 14.93 15.11 15.20
25 14.90 14.81 15.26 14.57 14.94   26 14.78 15.29 15.13 14.62 14.54
27 14.78 15.15 14.61 14.92 15.07   28 14.92 15.31 14.82 14.74 15.26
29 15.11 15.04 14.61 15.09 14.68   30 15.00 15.04 14.36 15.20 14.65
31 14.99 14.76 15.18 15.04 14.82   32 14.90 14.78 15.19 15.06 15.06
33 14.95 15.10 14.86 15.27 15.22   34 15.03 14.71 14.75 14.99 15.02
35 15.38 14.94 14.68 14.77 14.83   36 14.95 15.43 14.87 14.90 15.34
37 15.18 14.94 15.32 14.74 15.29   38 14.91 15.15 15.06 14.78 15.42
39 15.34 15.34 15.41 15.36 14.96   40 15.12 14.75 15.05 14.70 14.74
;
SYMBOL1 V=dot WIDTH=3;

PROC MACONTROL DATA=clips1 ;
     EWMACHART  gap*day='1' / WEIGHT  = 0.3 COUT
                          TABLE TABLEID TABLEOUTLIM
                          XSUMBOL = mu0;
      LABEL gap = 'Gap in Clip'
            day = 'Day';
TITLE 'EWMA Chart -- weight=0.3 (mu, sigma unknown)';
RUN;

PROC MACONTROL DATA=clips1 ;
     EWMACHART  gap*day='1' / WEIGHT  = 0.5 COUT
                          TABLE TABLEID TABLEOUTLIM
                          XSUMBOL = mu0;
      LABEL gap = 'Gap in Clip'
            day = 'Day';
TITLE 'EWMA Chart -- weight=0.5 (mu, sigma unknown)';
RUN;
```

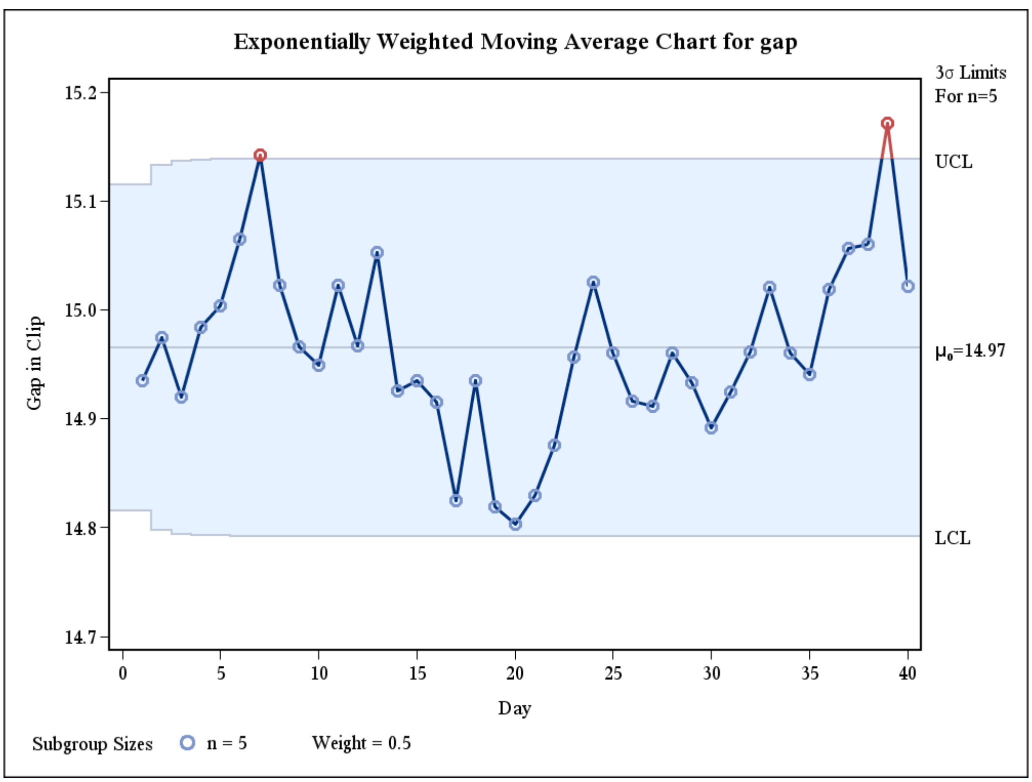
EWMA Chart -- weight=0.3 (mu, sigma unknown)

The MACONTROL Procedure



EWMA Chart -- weight=0.5 (mu, sigma unknown)

The MACONTROL Procedure



EWMA Chart -- weight=0.3 (mu, sigma unknown)

The MACONTROL Procedure

EWMA Parameters

Sigmas	3
Weight	0.3
Nominal Sample Size	5

Exponentially Weighted Moving Average Chart Summary for gap

day	Subgroup Sample Size	Lower Limit	EWMA	Subgroup Mean	Upper Limit	Limit Exceeded
1	5	14.875545	14.946980	14.904000	15.055255	
2	5	14.855718	14.967086	15.014000	15.075082	
3	5	14.847211	14.936760	14.866000	15.083589	
4	5	14.843258	14.970132	15.048000	15.087542	
5	5	14.841368	14.986292	15.024000	15.089432	
6	5	14.840452	15.028205	15.126000	15.090348	
7	5	14.840005	15.085743	15.220000	15.090795	
8	5	14.839787	15.030620	14.902000	15.091013	
9	5	14.839680	14.994434	14.910000	15.091120	
10	5	14.839628	14.975704	14.932000	15.091172	
11	5	14.839602	15.011793	15.096000	15.091198	
12	5	14.839590	14.981855	14.912000	15.091210	
13	5	14.839584	15.028698	15.138000	15.091216	
14	5	14.839581	14.959489	14.798000	15.091219	
15	5	14.839579	14.954842	14.944000	15.091221	
16	5	14.839578	14.937190	14.896000	15.091222	
17	5	14.839578	14.876233	14.734000	15.091222	
18	5	14.839578	14.927163	15.046000	15.091222	
19	5	14.839578	14.859614	14.702000	15.091222	
20	5	14.839578	14.838130	14.788000	15.091222	Lower
21	5	14.839578	14.843491	14.856000	15.091222	
22	5	14.839578	14.867044	14.922000	15.091222	
23	5	14.839578	14.918331	15.038000	15.091222	
24	5	14.839578	14.971031	15.094000	15.091222	
25	5	14.839578	14.948522	14.896000	15.091222	
26	5	14.839578	14.925565	14.872000	15.091222	
27	5	14.839578	14.919696	14.906000	15.091222	
28	5	14.839578	14.946787	15.010000	15.091222	
29	5	14.839578	14.934551	14.906000	15.091222	
30	5	14.839578	14.909186	14.850000	15.091222	
31	5	14.839578	14.923830	14.958000	15.091222	
32	5	14.839578	14.946081	14.998000	15.091222	
33	5	14.839578	14.986257	15.080000	15.091222	
34	5	14.839578	14.960380	14.900000	15.091222	
35	5	14.839578	14.948266	14.920000	15.091222	
36	5	14.839578	14.993186	15.098000	15.091222	
37	5	14.839578	15.023430	15.094000	15.091222	
38	5	14.839578	15.035601	15.064000	15.091222	
39	5	14.839578	15.109521	15.282000	15.091222	Upper
40	5	14.839578	15.038265	14.872000	15.091222	

EWMA Chart -- weight=0.5 (mu, sigma unknown)

The MACONTROL Procedure

EWMA Parameters

Sigmas	3
Weight	0.5
Nominal Sample Size	5

Exponentially Weighted Moving Average Chart Summary for gap

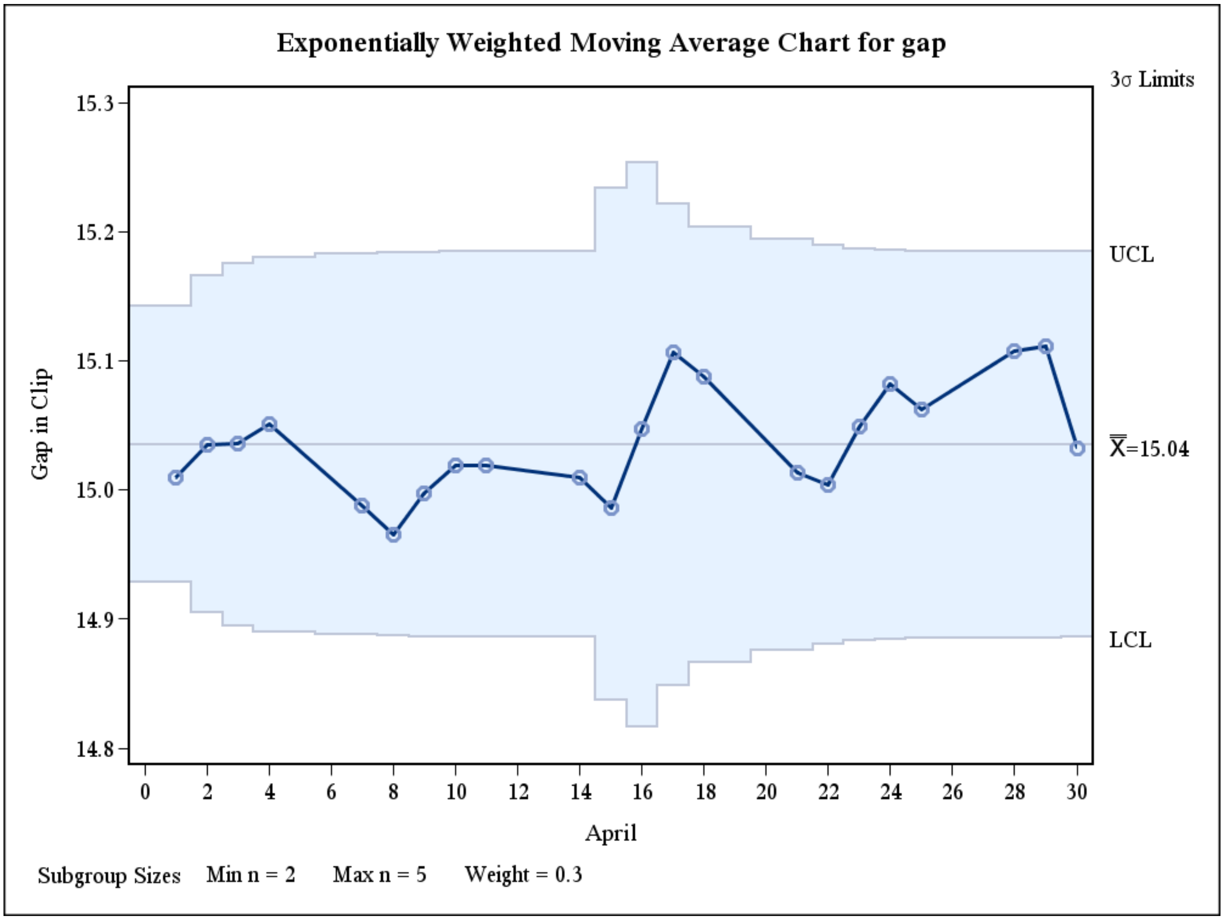
day	Subgroup Sample Size	Lower Limit	EWMA	Subgroup Mean	Upper Limit	Limit Exceeded
1	5	14.815642	14.934700	14.904000	15.115158	
2	5	14.797965	14.974350	15.014000	15.132835	
3	5	14.793830	14.920175	14.866000	15.136970	
4	5	14.792812	14.984088	15.048000	15.137988	
5	5	14.792558	15.004044	15.024000	15.138242	
6	5	14.792495	15.065022	15.126000	15.138305	
7	5	14.792479	15.142511	15.220000	15.138321	Upper
8	5	14.792475	15.022255	14.902000	15.138325	
9	5	14.792474	14.966128	14.910000	15.138326	
10	5	14.792474	14.949064	14.932000	15.138326	
11	5	14.792474	15.022532	15.096000	15.138326	
12	5	14.792474	14.967266	14.912000	15.138326	
13	5	14.792474	15.052633	15.138000	15.138326	
14	5	14.792474	14.925316	14.798000	15.138326	
15	5	14.792474	14.934658	14.944000	15.138326	
16	5	14.792474	14.915329	14.896000	15.138326	
17	5	14.792474	14.824665	14.734000	15.138326	
18	5	14.792474	14.935332	15.046000	15.138326	
19	5	14.792474	14.818666	14.702000	15.138326	
20	5	14.792474	14.803333	14.788000	15.138326	
21	5	14.792474	14.829667	14.856000	15.138326	
22	5	14.792474	14.875833	14.922000	15.138326	
23	5	14.792474	14.956917	15.038000	15.138326	
24	5	14.792474	15.025458	15.094000	15.138326	
25	5	14.792474	14.960729	14.896000	15.138326	
26	5	14.792474	14.916365	14.872000	15.138326	
27	5	14.792474	14.911182	14.906000	15.138326	
28	5	14.792474	14.960591	15.010000	15.138326	
29	5	14.792474	14.933296	14.906000	15.138326	
30	5	14.792474	14.891648	14.850000	15.138326	
31	5	14.792474	14.924824	14.958000	15.138326	
32	5	14.792474	14.961412	14.998000	15.138326	
33	5	14.792474	15.020706	15.080000	15.138326	
34	5	14.792474	14.960353	14.900000	15.138326	
35	5	14.792474	14.940176	14.920000	15.138326	
36	5	14.792474	15.019088	15.098000	15.138326	
37	5	14.792474	15.056544	15.094000	15.138326	
38	5	14.792474	15.060272	15.064000	15.138326	
39	5	14.792474	15.171136	15.282000	15.138326	Upper
40	5	14.792474	15.021568	14.872000	15.138326	

EWMA Example with Unequal Sample Sizes

```
DM 'LOG; CLEAR; OUT; CLEAR;';
* ODS PRINTER PDF file='C:\COURSES\ST528\EWMA4.PDF';
OPTIONS NODATE NONUMBER;

***********************************;
*** UNEQUAL SAMPLE SIZE EXAMPLE  ***;
***********************************;

DATA clips4;
    INPUT day @@;
    DO i=1 TO 5;
       INPUT gap @@; OUTPUT;
    END;
LINES;
  1  14.93  14.65  14.87  15.11  15.18
  2  15.06  14.95  14.91  15.14  15.41
  3  14.90  14.90  14.96  15.26  15.18
  4  15.25  14.57  15.33  15.38  14.89
  7  14.68  14.63  14.72  15.32  14.86
  8  14.48  14.88  14.98  14.74  15.48
  9  14.99  15.16  15.02  15.53  14.66
 10  14.88  15.44  15.04  15.10  14.89
 11  15.14  15.33  14.75  15.23  14.64
 14  15.46  15.30  14.92  14.58  14.68
 15  15.23  14.63    .      .      .
 16  15.13  15.25    .      .      .
 17  15.06  15.25  15.28  15.30  15.34
 18  15.22  14.77  15.12  14.82  15.29
 21  14.95  14.96  14.65  14.87  14.77
 22  15.01  15.11  15.11  14.79  14.88
 23  14.97  15.50  14.93  15.13  15.25
 24  15.23  15.21  15.31  15.07  14.97
 25  15.08  14.75  14.93  15.34  14.98
 28  15.07  14.86  15.42  15.47  15.24
 29  15.27  15.20  14.85  15.62  14.67
 30  14.97  14.73  15.09  14.98  14.46
 ;
SYMBOL1 V=dot WIDTH=3;

PROC MACONTROL DATA=clips4;
     EWMACHART gap*day='1' / WEIGHT  = 0.3 COUT
               TABLE TABLEID TABLEOUTLIM;
     LABEL gap = 'Gap in Clip'
           day = 'April';
TITLE 'EWMA Chart -- weight=0.3 (unequal sample sizes)';
RUN;
```

EWMA Chart -- weight=0.3 (unequal sample sizes)

The MACONTROL Procedure



Exponentially Weighted Moving Average Chart Summary for gap

day	Subgroup Sample Size	Lower Limit	EWMA	Subgroup Mean	Upper Limit	Limit Exceeded
1	5	14.928714	15.009169	14.948000	15.142055	
2	5	14.905177	15.034618	15.094000	15.165593	
3	5	14.895077	15.036233	15.040000	15.175692	
4	5	14.890385	15.050563	15.084000	15.180384	
7	5	14.888141	14.987994	14.842000	15.182629	
8	5	14.887053	14.965196	14.912000	15.183716	
9	5	14.886523	14.997237	15.072000	15.184246	
10	5	14.886264	15.019066	15.070000	15.184505	
11	5	14.886138	15.018746	15.018000	15.184632	
14	5	14.886075	15.009522	14.988000	15.184694	
15	2	14.836965	14.985666	14.930000	15.233804	
16	2	14.816894	15.046966	15.190000	15.253875	
17	5	14.848917	15.106676	15.246000	15.221852	
18	5	14.866814	15.087873	15.044000	15.203955	
21	5	14.876317	15.013511	14.840000	15.194452	
22	5	14.881187	15.003458	14.980000	15.189582	
23	5	14.883631	15.049221	15.156000	15.187139	
24	5	14.884842	15.081854	15.158000	15.185927	
25	5	14.885440	15.062098	15.016000	15.185330	
28	5	14.885733	15.107069	15.212000	15.185036	
29	5	14.885877	15.111548	15.122000	15.184892	
30	5	14.885948	15.031884	14.846000	15.184821	

9.9 Design of the EWMA Control Chart

- The design parameters of the EWMA chart are L (the multiple of σ_{z_i} used in the control limits) and λ (the weighting constant).
- It is possible to choose L and λ so that the ARL performance of the EWMA chart closely approximates the performance of a cusum ARL for detecting small shifts.
- The optimal EWMA design procedure would be to specify a desired in-control ARL, the magnitude of the shift to be detected quickly, and an out-of-control ARL corresponding to this shift. Then, determine if there exist L and λ values satisfying these conditions.
- Typically, we take the same approach taken with determining cusum parameters h and k. That is, we use EWMA ARL tables to find reasonable L and λ parameter values that best meet our desired ARL conditions. You have been supplied with tables of ARLs taken from the *SAS QC* manual ($r = \lambda$, $k = L$).
- Montgomery recommends the following:
  1. Use smaller values of λ to detect smaller shifts.
  2. Values of λ in the interval $0.05 \leq \lambda \leq 0.25$ work well in practice (with $\lambda = 0.05, 0.10, 0.20$ being commonly used values).
  3. Using $L = 3$ works reasonably well with larger values of λ.
  4. Using $2.6 \leq L \leq 2.8$ works reasonably well with smaller values of λ ($\lambda \leq 0.10$).
  5. $\lambda = .10$ and $L = 2.7$ produces a EWMA chart approximately equivalent to a cusum chart with $h = 5$ and $k = .5$.
  6. If $\lambda > 0.10$, the EWMA is often superior to a cusum for large shifts.
  7. To improve the sensitivity of the EWMA chart (or cusum chart) to detect large shifts without sacrificing the ability to detect small shifts quickly, combine a Shewhart chart with the EWMA (or cusum). The combined Shewhart-EWMA (or combined Shewhart-Cusum) procedures are effective against both large and small shifts.
- There is also the EWMAARL function in *SAS* that will generate ARLs for a given shift $\delta\sigma$, L, and λ values.

```
OPTIONS LS=72 PS=54 NODATE NONUMBER;

*** Designing an EWMA Chart;
DATA ewmaarl;
DO L = 3 TO 3.1 BY .1;
DO lambda = .05 to 1 BY .05;
   arl0 = EWMAARL(0,lambda,L);
   arl1 = EWMAARL(1,lambda,L);
   arl2 = EWMAARL(2,lambda,L);  *** 2 sigma shift for comparison;
   * IF (148 le arl0 le 152) and (8.5 le arl1 le 9.5) THEN OUTPUT;
   OUTPUT;
END; END;

PROC PRINT DATA=ewmaarl;
RUN;
```

Obs	L	lambda	arl0	arl1	arl2
1	3.0	0.05	1383.62	13.5161	6.00467
2	3.0	0.10	842.15	11.3840	4.66950
3	3.0	0.15	655.01	10.8303	4.10896
4	3.0	0.20	559.87	10.8359	3.80085
5	3.0	0.25	502.90	11.1543	3.61677
6	3.0	0.30	465.55	11.6986	3.50635
7	3.0	0.35	439.70	12.4349	3.44553
8	3.0	0.40	421.16	13.3518	3.42226
9	3.0	0.45	407.57	14.4500	3.43069
10	3.0	0.50	397.46	15.7378	3.46850
11	3.0	0.55	389.89	17.2287	3.53564
12	3.0	0.60	384.21	18.9408	3.63376
13	3.0	0.65	379.96	20.8959	3.76603
14	3.0	0.70	376.81	23.1192	3.93714
15	3.0	0.75	374.50	25.6391	4.15346
16	3.0	0.80	372.85	28.4873	4.42336
17	3.0	0.85	371.70	31.6983	4.75762
18	3.0	0.90	370.95	35.3093	5.16998
19	3.0	0.95	370.53	39.3605	5.67780
20	3.0	1.00	.	.	.
21	3.1	0.05	1856.77	14.1078	6.21240
22	3.1	0.10	1131.69	11.9924	4.83882
23	3.1	0.15	885.42	11.5272	4.26644
24	3.1	0.20	760.52	11.6638	3.95539
25	3.1	0.25	685.96	12.1514	3.77323
26	3.1	0.30	637.32	12.9044	3.66892
27	3.1	0.35	603.82	13.8922	3.61822
28	3.1	0.40	579.96	15.1086	3.60900
29	3.1	0.45	562.60	16.5599	3.63558
30	3.1	0.50	549.80	18.2612	3.69610
31	3.1	0.55	540.31	20.2336	3.79126
32	3.1	0.60	533.26	22.5033	3.92374
33	3.1	0.65	528.06	25.1014	4.09809
34	3.1	0.70	524.25	28.0630	4.32078
35	3.1	0.75	521.49	31.4272	4.60045
36	3.1	0.80	519.55	35.2369	4.94830
37	3.1	0.85	518.22	39.5387	5.37866
38	3.1	0.90	517.37	44.3824	5.90969
39	3.1	0.95	516.89	49.8214	6.56426
40	3.1	1.00	.	.	.

EWMAARL Function

computes the average run length for an exponentially weighted moving average.

Syntax

EWMAARL(δ, r, k)

where

- δ is the shift to be detected, expressed as a multiple of the process standard deviation (σ), where $\delta \geq 0$.
- *r* is the weight factor for the current subgroup mean in the EWMA, where $0 < r \leq 1$. If *r*=1, the EWMAARL function returns the average run length for a Shewhart chart for means. Refer to Wadsworth and others (1986). If $r \leq 0.05$, $k \geq 3$, and $\delta < 0.10$, the algorithm used is unstable. However, note that the EWMA behaves like a cusum when $r \to 0$, and in this case the CUSUMARL function is applicable.
- *k* is the multiple of σ used to define the control limits, where $k \geq 0$. Typically *k*=3.

Description

The EWMAARL function computes the average run length for an exponentially weighted moving average (EWMA) scheme using the method of Crowder (1987a,b). The notation used in the preceding list is consistent with that used in the MACONTROL procedure.

For a specified shift δ, you can use the EWMAARL function to design an exponentially weighted moving average scheme by first calculating average run lengths for a range of values of *r* and *k* and then choosing the combination of *r* and *k* that yields a desired average run length.

Examples

The following statements specify a shift of 1σ, a weight factor of 0.25, and 3σ control limits. The EWMAARL function returns an average run length of 11.154267016.

```
data;
   arl=ewmaarl(1.00,0.25,3.0);
   put arl;
run;
```

Table 19.18. Average Run Lengths for Two-Sided EWMA Charts

		r (weight parameter)					
k	δ	0.05	0.10	0.25	0.50	0.75	1.00
2.0	0.00	127.53	73.28	38.56	26.45	22.88	21.98
2.0	0.25	43.94	34.49	24.83	20.12	18.86	19.13
2.0	0.50	18.97	15.53	12.74	11.89	12.34	13.70
2.0	0.75	11.64	9.36	7.62	7.29	7.86	9.21
2.0	1.00	8.38	6.62	5.24	4.91	5.26	6.25
2.0	1.25	6.56	5.13	3.96	3.59	3.76	4.40
2.0	1.50	5.41	4.20	3.19	2.80	2.84	3.24
2.0	1.75	4.62	3.57	2.68	2.29	2.26	2.49
2.0	2.00	4.04	3.12	2.32	1.95	1.88	2.00
2.0	2.25	3.61	2.78	2.06	1.70	1.61	1.67
2.0	2.50	3.26	2.52	1.85	1.51	1.42	1.45
2.0	2.75	2.99	2.32	1.69	1.37	1.29	1.29
2.0	3.00	2.76	2.16	1.55	1.26	1.19	1.19
2.0	3.25	2.56	2.03	1.43	1.18	1.13	1.12
2.0	3.50	2.39	1.93	1.32	1.12	1.08	1.07
2.0	3.75	2.26	1.83	1.24	1.08	1.05	1.04
2.0	4.00	2.15	1.73	1.17	1.05	1.03	1.02
2.5	0.00	379.09	223.35	124.18	91.17	82.49	80.52
2.5	0.25	73.98	66.59	59.66	58.33	61.07	65.77
2.5	0.50	26.63	23.63	23.28	27.16	33.26	41.49
2.5	0.75	15.41	12.95	11.96	13.96	18.05	24.61
2.5	1.00	10.79	8.75	7.52	8.27	10.57	14.92
2.5	1.25	8.31	6.60	5.39	5.52	6.75	9.46
2.5	1.50	6.78	5.31	4.18	4.03	4.65	6.30
2.5	1.75	5.75	4.46	3.43	3.14	3.43	4.41
2.5	2.00	5.00	3.86	2.92	2.57	2.67	3.24
2.5	2.25	4.43	3.42	2.56	2.18	2.17	2.49
2.5	2.50	4.00	3.07	2.29	1.90	1.83	2.00
2.5	2.75	3.64	2.80	2.08	1.69	1.59	1.67
2.5	3.00	3.36	2.57	1.91	1.52	1.41	1.45
2.5	3.25	3.12	2.39	1.77	1.39	1.29	1.29
2.5	3.50	2.92	2.24	1.64	1.28	1.19	1.19
2.5	3.75	2.74	2.13	1.52	1.20	1.13	1.12
2.5	4.00	2.58	2.04	1.42	1.13	1.08	1.07

(continued)

Table 19.18 *(continued)*

		r (weight parameter)					
k	δ	0.05	0.10	0.25	0.50	0.75	1.00
3.0	0.00	1383.62	842.15	502.90	397.46	374.50	370.40
3.0	0.25	133.61	144.74	171.09	208.54	245.76	281.15
3.0	0.50	37.33	37.41	48.45	75.35	110.95	155.22
3.0	0.75	19.95	17.90	20.16	31.46	50.92	81.22
3.0	1.00	13.52	11.38	11.15	15.74	25.64	43.89
3.0	1.25	10.24	8.32	7.39	9.21	14.26	24.96
3.0	1.50	8.26	6.57	5.47	6.11	8.72	14.97
3.0	1.75	6.94	5.45	4.34	4.45	5.80	9.47
3.0	2.00	6.00	4.67	3.62	3.47	4.15	6.30
3.0	2.25	5.30	4.10	3.11	2.84	3.16	4.41
3.0	2.50	4.76	3.67	2.75	2.41	2.52	3.24
3.0	2.75	4.32	3.32	2.47	2.10	2.09	2.49
3.0	3.00	3.97	3.05	2.26	1.87	1.79	2.00
3.0	3.25	3.67	2.82	2.09	1.69	1.57	1.67
3.0	3.50	3.42	2.62	1.95	1.53	1.41	1.45
3.0	3.75	3.22	2.45	1.84	1.41	1.29	1.29
3.0	4.00	3.04	2.30	1.73	1.31	1.20	1.19
3.5	0.00	12851.0	4106.4	2640.16	2227.34	2157.99	2149.34
3.5	0.25	281.09	381.29	625.78	951.18	1245.90	1502.76
3.5	0.50	53.58	64.72	123.43	267.36	468.68	723.81
3.5	0.75	25.62	25.33	38.68	88.70	182.12	334.40
3.5	1.00	16.65	14.79	17.71	35.97	78.05	160.95
3.5	1.25	12.36	10.37	10.48	17.64	37.15	81.80
3.5	1.50	9.86	8.00	7.25	10.19	19.63	43.96
3.5	1.75	8.22	6.54	5.52	6.70	11.46	24.96
3.5	2.00	7.07	5.55	4.47	4.86	7.33	14.97
3.5	2.25	6.21	4.83	3.77	3.78	5.08	9.47
3.5	2.50	5.55	4.29	3.28	3.10	3.76	6.30
3.5	2.75	5.03	3.87	2,91	2.63	2.94	4.41
3.5	3.00	4.60	3.54	2.63	2.30	2.40	3.24
3.5	3.25	4.25	3.26	2.41	2.05	2.03	2.49
3.5	3.50	3.95	3.03	2.23	1.85	1.76	2.00
3.5	3.75	3.70	2.84	2.10	1.69	1.56	1.67
3.5	4.00	3.47	2.66	1.99	1.55	1.40	1.45

Constructing EWMA Charts

The following notation is used in this section:

E_i	exponentially weighted moving average for the i^{th} subgroup
r	EWMA weight parameter ($0 < r \leq 1$)
μ	process mean (expected value of the population of measurements)
σ	process standard deviation (standard deviation of the population of measurements
x_{ij}	j^{th} measurement in i^{th} subgroup, with $j = 1, 2, 3, \ldots, n_i$
n_i	sample size of i^{th} subgroup
$\overline{X}_i$	mean of measurements in i^{th} subgroup. If $n_i = 1$, then the subgroup mean reduces to the single observation in the subgroup
$\overline{\overline{X}}$	weighted average of subgroup means
$\Phi^{-1}(\cdot)$	inverse standard normal function

Plotted Points

Each point on the chart indicates the value of the exponentially weighted moving average (EWMA) for that subgroup. The EWMA for the i^{th} subgroup (E_i) is defined recursively as

$$E_i = r\overline{X}_i + (1 - r)E_{i-1}\,, \quad i > 0$$

where r is a weight parameter ($0 < r \leq 1$). Some authors (for example, Hunter 1986 and Crowder 1987a,b) use the symbol λ instead of r for the weight. You can specify the weight with the WEIGHT= option in the EWMACHART statement or with the variable _WEIGHT_ in a LIMITS= data set. If you specify a known value (μ_0) for μ, $E_0 = \mu_0$; otherwise, $E_0 = \overline{\overline{X}}$.

The preceding equation can be rewritten as

$$E_i = E_{i-1} + r(\overline{X}_i - E_{i-1})$$

which expresses the current EWMA as the previous EWMA plus the weighted error in the prediction of the current mean based on the previous EWMA.

The EWMA for the i^{th} subgroup can also be written as

$$E_i = r\sum_{j=0}^{i-1}(1 - r)^j\overline{X}_{i-j} + (1 - r)^i E_0$$

which expresses the EWMA as a weighted average of past subgroup means, where the weights decline exponentially, and the heaviest weight is assigned to the most recent subgroup mean.

Central Line

By default, the central line on an EWMA chart indicates an estimate for μ, which is computed as

$$\hat{\mu} = \overline{\overline{X}} = \frac{n_1 \bar{X}_1 + \cdots + n_N \bar{X}_N}{n_1 + \cdots + n_N}$$

If you specify a known value (μ_0) for μ, the central line indicates the value of μ_0.

Control Limits

You can compute the limits in the following way

- as a specified multiple (k) of the standard error of E_i above and below the central line. The default limits are computed with $k = 3$ (these are referred to as 3σ *limits*).

Table 19.17. Limits for an EWMA Chart

Control Limits
LCL = lower limit = $\overline{\overline{X}} - k\hat{\sigma}r\sqrt{\sum_{j=0}^{i-1}(1-r)^{2j}/n_{i-j}}$
UCL = upper limit = $\overline{\overline{X}} + k\hat{\sigma}r\sqrt{\sum_{j=0}^{i-1}(1-r)^{2j}/n_{i-j}}$

These formulas assume that the data are normally distributed. If standard values μ_0 and σ_0 are available for μ and σ, respectively, replace $\overline{\overline{X}}$ with μ_0 and $\hat{\sigma}$ with σ_0 in Table 19.17. Note that the limits vary with both n_i and i.

If the subgroup sample sizes are constant ($n_i = n$), the formulas for the control limits simplify to

$$\text{LCL} = \overline{\overline{X}} - k\hat{\sigma}\sqrt{r(1-(1-r)^{2i})/n(2-r)}$$

$$\text{UCL} = \overline{\overline{X}} + k\hat{\sigma}\sqrt{r(1-(1-r)^{2i})/n(2-r)}$$

Consequently, when the subgroup sample sizes are constant, the width of the control limits increases monotonically with i. For probability limits, replace k with $\Phi^{-1}(1-\alpha/2)$ in the previous equations. Refer to Roberts (1959) and Montgomery (1991).

As i becomes large, the upper and lower control limits approach constant values:

$$\text{LCL} = \overline{\overline{X}} - k\hat{\sigma}\sqrt{r/n(2-r)}$$

$$\text{UCL} = \overline{\overline{X}} + k\hat{\sigma}\sqrt{r/n(2-r)}$$

Some authors base the control limits for EWMA charts on the asymptotic expressions in the two previous equations.

Choosing the Value of the Weight Parameter

Various approaches have been proposed for choosing the value of r.

- Hunter (1986) states that the choice "can be left to the judgment of the quality control analyst" and points out that the smaller the value of r, "the greater the influence of the historical data."
- Hunter (1986) also discusses a least squares procedure for estimating r from the data, **assuming an exponentially weighted moving average model for the data**. In this context, the fitted EWMA model provides a forecast of the process that is the basis for dynamic process control. You can use the ARIMA procedure in SAS/ETS software to compute the least squares estimate of r. (Refer to *SAS/ETS User's Guide: Version 6, Second Edition* for information on PROC ARIMA.) Also see "Autocorrelation in Process Data" on page 1522.
- A number of authors have studied the design of EWMA control schemes based on average run length (ARL) computations. The ARL is the expected number of points plotted before a shift is detected. Ideally, the ARL should be short when a shift occurs, and it should be long when there is no shift (the process is in control.) The effect of r on the ARL was described by Roberts (1959), who used simulation methods. The ARL function was approximated and tabulated by Robinson and Ho (1978), and a more general method for studying run-length distributions of EWMA charts was given by Crowder (1987a,b). Unlike Hunter (1986), these authors assume the data are independent and identically distributed; typically the normal distribution is assumed for the data, although the methods extend to nonnormal distributions. A more detailed discussion of the ARL approach follows.

Average run lengths for two-sided EWMA charts are shown in Table 19.18, which is patterned after Table 1 of Crowder (1987a,b). The ARLs were computed using the EWMAARL DATA step function (see page 1602 for details on the EWMAARL function). Note that Crowder (1987a,b) uses the notation L in place of k and the notation λ in place of r.

You can use Table 19.18 to find a combination of k and r that yields a desired ARL for an in-control process ($\delta = 0$) and for a specified shift of δ. Note that δ is assumed to be standardized; in other words, if a shift of Δ is to be detected in the process mean μ, and if σ is the process standard deviation, you should select the table entry with

$$\delta = \Delta/(\sigma/\sqrt{n})$$

where n is the subgroup sample size. Thus, δ can be regarded as the shift in the sampling distribution of the subgroup mean.

For example, suppose you want to construct an EWMA scheme with an in-control ARL of 90 and an ARL of 9 for detecting a shift of $\delta = 1$. Table 19.18 shows that the combination $r = 0.5$ and $k = 2.5$ yields an in-control ARL of 91.17 and an ARL of 8.27 for $\delta = 1$.