## 2 ONE-FACTOR COMPLETELY RANDOMIZED DESIGN (CRD)

An experiment is run to study the effects of one factor on a response. The levels of the factor can be

- **quantitative (numerical)** or **qualitative (categorical)**
- **fixed** with levels set by the experimenter or **random** with randomly chosen levels.

When random selection, random assignment, and a randomized run order of experimentation (when possible) can be applied then the experimental design is called a **completely randomized design (CRD)**.

### 2.1 Notation

Assume that the factor of interest has $a \geq 2$ levels with $n_i$ observations taken at level $i$ of the factor. Let $N$ be the total number of design observations.

**The General Sample Size Case**

| Treatments | 1 | 2 | 3 | $\cdots$ | $a$ |
|---|---|---|---|---|---|
| | $y_{11}$ | $y_{21}$ | $y_{31}$ | $\cdots$ | $y_{a1}$ |
| | $y_{12}$ | $y_{22}$ | $y_{32}$ | $\cdots$ | $y_{a2}$ |
| | $y_{13}$ | $y_{23}$ | $y_{33}$ | $\cdots$ | $y_{a3}$ |
| | . | . | . | . | . |
| | $y_{1n_1}$ | $y_{2n_2}$ | $y_{3n_3}$ | $\cdots$ | $y_{an_a}$ |
| treatment totals | $y_{1\cdot}$ | $y_{2\cdot}$ | $y_{3\cdot}$ | $\cdots$ | $y_{a\cdot}$ |
| treatment means | $\overline{y}_{1\cdot}$ | $\overline{y}_{2\cdot}$ | $\overline{y}_{3\cdot}$ | $\cdots$ | $\overline{y}_{a\cdot}$ |

Grand total $y_{\cdot\cdot} = \sum_{i=1}^{a}\sum_{j=1}^{n_i} y_{ij}$

Grand mean $\overline{y}_{\cdot\cdot} = \dfrac{\sum_{i=1}^{a}\sum_{j=1}^{n_i} y_{ij}}{\sum_{i=1}^{a} n_i} = \dfrac{y_{\cdot\cdot}}{N}$

Treatment total $y_{i\cdot} = \sum_{j=1}^{n_i} y_{ij}$

Treatment mean $\overline{y}_{i\cdot} = \dfrac{\sum_{j=1}^{n_i} y_{ij}}{n_i} = \dfrac{y_{i\cdot}}{n_i}$

**The Equal Sample Size Case ($n_i = n$ for $i = 1, 2, \ldots, a$)**

| Treatments | 1 | 2 | 3 | $\cdots$ | $a$ |
|---|---|---|---|---|---|
| | $y_{11}$ | $y_{21}$ | $y_{31}$ | $\cdots$ | $y_{a1}$ |
| | $y_{12}$ | $y_{22}$ | $y_{32}$ | $\cdots$ | $y_{a2}$ |
| | $y_{13}$ | $y_{23}$ | $y_{33}$ | $\cdots$ | $y_{a3}$ |
| | . | . | . | $\cdots$ | . |
| | $y_{1n}$ | $y_{2n}$ | $y_{3n}$ | $\cdots$ | $y_{an}$ |
| treatment totals | $y_{1\cdot}$ | $y_{2\cdot}$ | $y_{3\cdot}$ | $\cdots$ | $y_{a\cdot}$ |
| treatment means | $\overline{y}_{1\cdot}$ | $\overline{y}_{2\cdot}$ | $\overline{y}_{3\cdot}$ | $\cdots$ | $\overline{y}_{a\cdot}$ |

Grand total $y_{\cdot\cdot} = \sum_{i=1}^{a}\sum_{j=1}^{n} y_{ij}$

Grand mean $\overline{y}_{\cdot\cdot} = \dfrac{y_{\cdot\cdot}}{an}$

Treatment total $y_{i\cdot} = \sum_{j=1}^{n} y_{ij}$

Treatment mean $\overline{y}_{i\cdot} = \dfrac{y_{i\cdot}}{n}$

Notation related to TOTAL variability:

- $SS_T$ = the total (corrected) sum of squares = $\sum_{i=1}^{a}\sum_{j=1}^{n_i}(y_{ij} - \overline{y}_{\cdot\cdot})^2 = (N-1)s^2$

  where $s^2$ is the sample variance of the $N$ observations
- $N - 1$ = the degrees of freedom for total

Notation for variability WITHIN treatments: ("E" stands for "Error")

- $SS_E$ = the error sum of squares = $\sum_{i=1}^{a}\sum_{j=1}^{n_i}(y_{ij} - \overline{y}_{i\cdot})^2 = \sum_{i=1}^{a}(n_i - 1)s_i^2$

  where $s_i^2$ is the sample variance of the $n_i$ observations for the $i^{th}$ treatment
- $N - a$ = the error degrees of freedom
- $MS_E$ = the mean square error = $\dfrac{SS_E}{N-a}$

Notation for variability BETWEEN treatments:

- $SS_{Trt}$ = the treatment sum of squares = $\sum_{i=1}^{a}\sum_{j=1}^{n_i}(\overline{y}_{i\cdot} - \overline{y}_{\cdot\cdot})^2 \; = \; \sum_{i=1}^{a} n_i(\overline{y}_{i\cdot} - \overline{y}_{\cdot\cdot})^2$

  If all sample sizes are equal ($n_{ij} = n$), then $SS_{trt} = n\sum_{i=1}^{a}(\overline{y}_{i\cdot} - \overline{y}_{\cdot\cdot})^2$

- $a - 1$ = the treatment degrees of freedom

- $MS_{Trt}$ = the treatment mean square = $\dfrac{SS_{Trt}}{a-1}$

**Alternate Formulas**

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{n_i} y_{ij}^2 - \frac{y_{\cdot\cdot}^2}{N} \qquad SS_{Trt} = \sum_{i=1}^{a}\frac{y_{i\cdot}^2}{n_i} - \frac{y_{\cdot\cdot}^2}{N} \qquad SS_E = SS_T - SS_{Trt}$$

- $\dfrac{y_{\cdot\cdot}^2}{N}$ is called the **correction factor**.

**EXAMPLE**: Suppose a one-factor CRD has $a = 5$ treatments (5 factor levels) and $n = 6$ replicates per treatment ($N = 5 \times 6 = 30$). The following table summarizes the data:

| Treatment | | | | | |
|---|---|---|---|---|---|
| A | B | C | D | E | |
| 7 | 5 | 9 | 6 | 9 | |
| 8 | 4 | 11 | 12 | 6 | |
| 5 | 4 | 6 | 8 | 8 | |
| 9 | 6 | 8 | 5 | 12 | |
| 10 | 3 | 7 | 11 | 13 | |
| 11 | 5 | 8 | 9 | 12 | |
| $y_{1\cdot} =$ | $y_{2\cdot} =$ | $y_{3\cdot} =$ | $y_{4\cdot} =$ | $y_{5\cdot} =$ | $y_{\cdot\cdot} =$ |

$$\sum_{i=1}^{5}\sum_{j=1}^{6} y_{ij}^2 = 7^2 + 8^2 + 5^2 + \cdots + 12^2 + 12^2 + 13^2 =$$

$$SS_T = \sum_{i=1}^{5}\sum_{j=1}^{6} y_{ij}^2 - \frac{y_{\cdot\cdot}^2}{N} = 2091 - \frac{237^2}{30} = 2091 - 1872.3 =$$

$$SS_{trt} = \sum_{i=1}^{5}\frac{y_{i\cdot}^2}{n_i} - \frac{y_{\cdot\cdot}^2}{N} = \left(\frac{50^2}{6} + \frac{27^2}{6} + \frac{49^2}{6} + \frac{51^2}{6} + \frac{60^2}{6}\right) - \frac{237^2}{30}$$

$$= \frac{11831}{6} - 1872.3 = 1971.18\overline{3} - 1872.3 =$$

$$SS_E = SS_T - SS_{trt} = 218.7 - 99.5\overline{3} =$$

Degrees of freedom $df_T = N - 1 =$ $\qquad df_{trt} = a - 1 =$ $\qquad df_E = N - a =$

## 2.2 Linear Model Forms for Fixed Effects

- Assume the $a$ levels of the factor are fixed by the experimenter. This implies the levels are specifically chosen by the experimenter.
- For any observation $y_{ij}$ we can write: $y_{ij} = \overline{y}_{i\cdot} + (y_{ij} - \overline{y}_{i\cdot})$. Thus, an observation from treatment $i$ equals the observed treatment mean $\overline{y}_{i\cdot}$ plus a deviation from that observed mean $(y_{ij} - \overline{y}_{i\cdot})$.
- This deviation is called the **residual** for response $y_{ij}$, and it is denoted: $e_{ij} = y_{ij} - \overline{y}_{i\cdot}$.

The linear **effects model** is $\quad y_{ij} = \qquad\qquad\qquad\qquad$ where

- $\mu$ is the baseline mean and $\tau_i$ is the $i^{th}$ treatment effect $(i = 1, \ldots, a)$ relative to $\mu$.
- $\epsilon_{ij} \sim IIDN(0, \sigma^2)$. The random errors are **i**ndependent, **i**dentically **d**istributed following a **n**ormal distribution with mean 0 and variance $\sigma^2$.

The linear **means model** is $\quad y_{ij} = \qquad\qquad\qquad\qquad$ where $\mu_i = \mu + \tau_i$ is the mean associated with the $i^{th}$ treatment and $\epsilon_{ij} \sim IIDN(0, \sigma^2)$.

- The goal is to determine if there exist any differences in the set of $a$ treatment means (or effects) in a CRD. We want to check the null hypothesis that $\mu_1, \mu_2, \ldots, \mu_a$, are all equal against the alternative that they are not all equal,

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_a \qquad H_1 : \mu_i \neq \mu_j \text{ for } \text{ some } i \neq j.$$

  or, equivalently, that there are no significant treatment effects,

$$H_0 : \tau_1 = \tau_2 = \cdots = \tau_a \qquad H_1 : \tau_i \neq \tau_j \text{ for } \text{ some } i \neq j.$$

- To answer this question, we determine statistically whether any differences among the treatment means could reasonably have occurred based on the variation that occurs BETWEEN treatment $(MS_{Trt})$ and WITHIN each of the treatments $(MS_E)$.
- Our best estimate of the within treatment variability is the weighted average of the within treatment variances $(s_i^2, i = 1, 2, \ldots, a)$. The weights are the degrees of freedom $(n_i - 1)$ associated with each treatment:

$$\frac{\sum_{i=1}^{a}(n_i - 1)s_i^2}{\sum_{i=1}^{a}(n_i - 1)} = \frac{\sum_{i=1}^{a}\sum_{j=1}^{n_a}(y_{ij} - \overline{y}_{i\cdot})^2}{N - a} =$$

- If $\epsilon_{ij} \sim N(0, \sigma^2)$, then the $MSE$ is an unbiased estimate of $\sigma^2$. That is, $E(MS_{trt}) = \sigma^2$.
- If the null hypothesis $(H_0 : \mu_1 = \mu_2 = \cdots = \mu_a)$ is true then the $MS_{trt}$ is also an unbiased estimate of $\sigma^2$. That is, $(E(MS_{trt}) = \sigma^2$ **assuming all the means are equal**. This implies the ratio:

$$F_0 = \frac{MS_{Trt}}{MS_E}$$

  should be close to 1 because the numerator and denominator are both unbiased estimates of $\sigma^2$ when $H_0$ is true .

- If $F_0$ is too large, we will reject $H_0$ in favor of the alternative hypothesis $H_1$.
- When $H_0$ is true and the linear model assumptions are met, the test statistic $F_0$ follows an $F$ distribution with $(a - 1, N - a)$ degrees of freedom $(F_0 \sim F(a - 1, N - a))$.
- The formal statistical test is an **Analysis of Variance (ANOVA)** for a completely randomized design with one factor.

## Analysis of Variance (ANOVA) Table

| Source of Variation | Sum of Squares | d.f. | Mean Square | $F$-Ratio | $p$-value |
|---|---|---|---|---|---|
| Treatment | $SS_{Trt}$ | $a-1$ | $MS_{Trt}$ | $F_0 = MS_{Trt}/MS_E$ | $P\left[F(a-1, N-a) \ \geq \ F_0\right]$ |
| Error | $SS_E$ | $N-a$ | $MS_E$ | —— | |
| Total | $SS_T$ | $N-1$ | —— | —— | |

**EXAMPLE REVISITED**: Suppose a one-factor CRD has $a = 5$ treatments (5 factor levels) and $n = 6$ replicates per treatment ($N = 5 \times 6 = 30$). The following table summarizes the data:

| Treatment | | | | |
|---|---|---|---|---|
| A | B | C | D | E |
| 7 | 5 | 9 | 6 | 9 |
| 8 | 4 | 11 | 12 | 6 |
| 5 | 4 | 6 | 8 | 8 |
| 9 | 6 | 8 | 5 | 12 |
| 10 | 3 | 7 | 11 | 13 |
| 11 | 5 | 8 | 9 | 12 |

$$SS_T \ = \ 218.7$$

$$SS_{trt} \ = \ 99.5\overline{3}$$

$$SS_E \ = \ 119.1\overline{6}$$

Analysis of Variance (ANOVA) Table

| Source of Variation | Sum of Squares | d.f. | Mean Square | $F$-Ratio | $p$-value |
|---|---|---|---|---|---|
| Treatment | $99.5\overline{3}$ | 4 | $24.88\overline{3}$ | | |
| Error | $119.1\overline{6}$ | 25 | $4.7\overline{6}$ | $F_0 \approx 5.22$ | .0034 |
| Total | 218.7 | 29 | | | |

Hypotheses for Testing Equality of Means

$$H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 \qquad H_1 : \mu_i \neq \mu_j \text{ for } \quad \text{some } i \neq j.$$

Hypothesis for Testing Equality of Effects

$$H_0 : \tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau_5 \qquad H_1 : \tau_i \neq \tau_j \text{ for } \quad \text{some } i \neq j.$$

The Steps of the Hypothesis Test

- The **test statistic** is $F_0 = 5.22$.
- The **reference distribution** is the $F(4, 25)$ distribution.
- The $\alpha = .05$ **critical value** from the $F(4, 25)$ distribution is $F_{.05}(4, 25) = 2.76$.
- The **decision rule** is to reject $H_0$ if $F_0 \geq F_{.05}(4, 25)$ (or $p$-value $\leq .05$) **OR**
  fail to reject $H_0$ if $F_0 < F_{.05}(4, 25)$ (or $p$-value $> .05$)
- The **conclusion** is to reject $H_0$ because $F_0 \geq F_{.05}(4, 25)$ , i.e. $5.22 > 2.76$
  (or because $p$-value $.0034 \leq .05$).

### 2.3 Expected Mean Squares

If we assume the constraint $\sum_{i=1}^{a} n_i\tau_i = 0$, then the expected values of the mean squares are

- $E(MS_{Trt}) = E\left[\frac{\sum_{i=1}^{a} n_i(\overline{y}_{i\cdot} - \overline{y}_{\cdot\cdot})^2}{a-1}\right] = \quad \sigma^2 +$
- $E(MS_E) \ = E\left[\frac{\sum_{i=1}^{a}\sum_{j=1}^{n_i}(y_{ij} - \overline{y}_{i\cdot})^2}{N-a}\right] = \sigma^2$

If $H_0$ is true then $\tau_i = 0$ for $i = 1, 2, \ldots, a$. This implies

$$E(MS_{Trt}) = \sigma^2 + \frac{\sum_{i=1}^{a} n_i \cdot 0}{a-1} = \sigma^2 + 0 = \sigma^2.$$

If $H_0$ is not true then $\tau_i \neq 0$ for at least one $i$. This implies

$$E(MS_{Trt}) = \sigma^2 + (positive\ quantity) \quad \Longrightarrow \quad E(MS_{Trt}) > \sigma^2.$$

As $|\tau_i|$ increases, the $E(MS_{Trt})$ also increases. This implies the $F-$ratio of the expected mean squares

$$F = \frac{E(MS_{Trt})}{E(MS_E)} = \frac{\sigma^2 + \sum_{i=1}^{a} n_i\tau_i^2/(a-1)}{\sigma^2}$$

increases. This summarizes part of the statistical theory behind using $F_0 = \frac{MS_{Trt}}{MS_E}$ to estimate $F = \frac{E(MS_{Trt})}{E(MS_E}$ and reject $H_0$ for large values of $F_0$.

### 2.4 Estimation of Model Parameters under Constraints

- For the effects model, $\mu$ and $\tau_1, \ldots, \tau_a$ cannot be uniquely estimated without imposing a constraint on the model effects.
- If we assume the linear constraint (i) $\sum_{i=1}^{a} n_i\tau_i = 0$, (ii) $\tau_a = 0$ (SAS default), or (iii) $\tau_1 = 0$ (R default), then $\mu, \tau_1, \ldots, \tau_a$ can be uniquely estimated from the grand $\overline{y}_{\cdot\cdot}$ and the treatment means $\overline{y}_{1\cdot}, \ldots, \overline{y}_{a\cdot}$. The **least-squares estimates**:

  assuming $\sum_{i=1}^{a} n_i\tau_i = 0$: $\widehat{\mu} = \overline{y}_{\cdot\cdot}$ and $\widehat{\tau}_i = \overline{y}_{i\cdot} - \overline{y}_{\cdot\cdot}$ for $i = 1, 2, \ldots, a$

  assuming $\tau_a = 0$: $\widehat{\mu} = \overline{y}_a$ and $\widehat{\tau}_i = \overline{y}_{i\cdot} - \overline{y}_a$ for $i = 1, 2, \ldots, a$

  assuming $\tau_1 = 0$: $\widehat{\mu} = \overline{y}_1$ and $\widehat{\tau}_i = \overline{y}_{i\cdot} - \overline{y}_1$ for $i = 1, 2, \ldots, a$

ONEWAY CLASSIFICATION

EFFECTS MODEL: $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$ $\quad i = 1, 2, \ldots, a$; $j = 1, 2, \ldots, n_i$

ASSUMPTIONS: (1) $\varepsilon_{ij} \sim IIDN(0, \sigma^2)$

(2) FIXED EFFECTS (LEVELS 1, 2, ..., a ARE SET BY THE EXPERIMENTER)

(3) COMPLETE RANDOMIZATION

(4) A CONSTRAINT IS PLACED ON THE $\tau_i$'s TO ALLOW FOR ESTIMATION.

REGARDING (4): TO UNIQUELY ESTIMATE THE a+1 PARAMETERS $\mu, \tau_1, \ldots, \tau_a$ WE USE THE NORMAL EQUATIONS WITH THE CONSTRAINT TO GET ESTIMATES $\hat{\mu}, \hat{\tau}_1, \ldots, \hat{\tau}_a$.

## ESTIMATION OF EFFECTS

THE CRITERION FOR ESTIMATING EFFECTS IS TO MINIMIZE THE SSE (OTHERWISE KNOWN AS THE LEAST SQUARES METHOD). MATHEMATICALLY, WE WANT TO FIND $\hat{\mu}, \hat{\tau}_1, \dots, \hat{\tau}_a$ THAT MINIMIZE

$$L = \sum_{i=1}^{a} \sum_{j=1}^{n_i} (y_{ij} - \hat{\mu} - \hat{\tau}_i)^2$$

A SOLUTION CAN BE FOUND BY USING THE NORMAL EQUATIONS WHICH ARE FOUND BY EQUATING THE PARTIAL DERIVATIVES TO Ø AND THEN SOLVING:

$$\frac{\partial L}{\partial \hat{\mu}} = -2 \sum_{i=1}^{a} \sum_{j=1}^{n_i} (y_{ij} - \hat{\mu} - \hat{\tau}_i) = 0$$

$$\Rightarrow \sum_{i=1}^{a} \sum_{j=1}^{n_i} y_{ij} = \sum_{i=1}^{a} \sum_{j=1}^{n_i} \hat{\mu} + \sum_{i=1}^{a} \sum_{j=1}^{n_i} \hat{\tau}_i$$

$$\Rightarrow y_{..} = N\hat{\mu} + \sum_{i=1}^{a} n_i \hat{\tau}_i \qquad (1)$$

$$\frac{\partial L}{\partial \hat{\tau}_i} = -2 \sum_{j=1}^{n_i} (y_{ij} - \hat{\mu} - \hat{\tau}_i) = 0 \qquad i = 1, 2, \dots, a$$

$$\Rightarrow \sum_{j=1}^{n_i} y_{ij} = \sum_{j=1}^{n_i} \hat{\mu} + \sum_{j=1}^{n_i} \hat{\tau}_i$$

$$\Rightarrow y_{i.} = n_i \hat{\mu} + n_i \hat{\tau}_i \qquad \text{FOR } i = 1, 2, \dots, a$$

NOTE THERE ARE $a+1$ NORMAL EQUATIONS BUT THEY ARE NOT LINEARLY INDEPENDENT:

$$y_{1.} = n_1 \hat{\mu} + n_1 \hat{\tau}_1$$
$$\vdots$$
$$y_{a.} = n_a \hat{\mu} + n_a \hat{\tau}_a$$
$$\overline{y_{..} = N\hat{\mu} + \sum_{i=1}^{a} n_i \hat{\tau}_i} \longleftarrow \text{EQ (1)}$$

TO GET UNIQUE SOLUTIONS FOR $\hat{\mu}$ AND $\hat{\tau}_i$ WE NEED TO IMPOSE A CONSTRAINT. THE CONSTRAINT USED IN MANY TEXT BOOKS IS

[CONSTRAINT I]

$$\sum_{i=1}^{a} n_i \hat{\tau}_i = 0 \qquad \text{FOR UNEQUAL } n_i$$

OR

$$\sum_{i=1}^{a} \hat{\tau}_i = 0 \qquad \text{FOR EQUAL } n_i$$

USING THIS CONSTRAINT INTO (1) YIELDS $y_{..} = N\hat{\mu}$ OR $\boxed{\hat{\mu} = \bar{y}_{..}}$

THUS $y_{i.} = n_i \hat{\mu} + n_i \hat{\tau}_i$ BECOMES $y_{i.} = n_i \bar{y}_{..} + n_i \hat{\tau}_i$.

SOLVING FOR $\hat{\tau}_i$ YIELDS $\boxed{\hat{\tau}_i = \bar{y}_{i.} - \bar{y}_{..}}$ FOR $i = 1, 2, \dots, a$

[CONSTRAINT II]

ANOTHER COMMON CONSTRAINT USED (E.G., BY SAS) IS $\hat{\tau}_a = 0$. THEN THE LAST NORMAL EQUATION $y_{a.} = n_a \hat{\mu} + n_a \hat{\tau}_a$ BECOMES

$$y_{a.} = n_a \hat{\mu} \Rightarrow \boxed{\hat{\mu} = \bar{y}_{a.}}$$

THUS, THE OTHER NORMAL EQUATIONS $y_{i.} = n_i \hat{\mu} + n_i \hat{\tau}_i$ BECOME

$$y_{i.} = n_i \bar{y}_{a.} + n_i \hat{\tau}_i \Rightarrow \bar{y}_{i.} = \bar{y}_{a.} + \hat{\tau}_i$$

$$\Rightarrow \boxed{\hat{\tau}_i = \bar{y}_{i.} - \bar{y}_{a.}} \qquad \text{FOR } i = 1, 2, \dots, a$$

## 2.5 Sleep Deprivation Example ($n_i$ are equal)

A study was conducted to determine the effects of sleep deprivation on hand-steadiness. The four levels of sleep deprivation of interest are 12, 18, 24, and 30 hours. 32 subjects were randomly selected and assigned to the four levels of sleep deprivation such that 8 subjects were randomly assigned to each level. The response is the reaction time to the onset of a light cue. The results (in hundredths of a second) are contained in the following table:

Treatment (in hours)

| 12 | 18 | 24 | 30 |
|---|---|---|---|
| 20 | 21 | 25 | 26 |
| 20 | 20 | 23 | 27 |
| 17 | 21 | 22 | 24 |
| 19 | 22 | 23 | 27 |
| 20 | 20 | 21 | 25 |
| 19 | 20 | 22 | 28 |
| 21 | 23 | 22 | 26 |
| 19 | 19 | 23 | 27 |

Note: subscripts 1, 2, 3, 4 correspond to the 12, 18, 24, and 30 hour sleep deprivation treatments.

- $\overline{y}_{\cdot\cdot} = 22.25$, $\overline{y}_{1\cdot} = 19.375$ $\overline{y}_{2\cdot} = 20.75$ $\overline{y}_{3\cdot} = 22.625$ $\overline{y}_{4\cdot} = 26.25$
- Assuming Constraint II: $\tau_a = 0$ where $a = 4$.

$\widehat{\mu} = \overline{y}_{4\cdot} =$

$\widehat{\tau}_1 = \overline{y}_{1\cdot} - \overline{y}_{4\cdot} = 19.375 - 26.25 =$

$\widehat{\tau}_2 = \overline{y}_{2\cdot} - \overline{y}_{4\cdot} = 20.75 - 26.25 =$

$\widehat{\tau}_3 = \overline{y}_{3\cdot} - \overline{y}_{4\cdot} = 22.625 - 26.25 =$

$\widehat{\tau}_4 = \overline{y}_{4\cdot} - \overline{y}_{4\cdot} = 26.25 - 26.25 =$

- Thus, our estimates $\widehat{\mu}_1$, $\widehat{\mu}_2$, $\widehat{\mu}_3$, and $\widehat{\mu}_4$ under Constraint II are:

$\widehat{\mu}_1 = \widehat{\mu} + \widehat{\tau}_1 = 26.25 - 6.875 =$

$\widehat{\mu}_2 = \widehat{\mu} + \widehat{\tau}_2 = 26.25 - 5.50 =$

$\widehat{\mu}_3 = \widehat{\mu} + \widehat{\tau}_3 = 26.25 - 3.625 =$

$\widehat{\mu}_4 = \widehat{\mu} + \widehat{\tau}_4 = 26.25 - 0 =$

- What if we assume Constraint I: $\sum_{i=1}^{4} \widehat{\tau}_i = 0$ (because all $n_i = 8$)? The parameter estimates are:

$\widehat{\mu} = \overline{y}_{\cdot\cdot} =$

$\widehat{\tau}_1 = \overline{y}_{1\cdot} - \overline{y}_{\cdot\cdot} = 19.375 - 22.25 =$

$\widehat{\tau}_2 = \overline{y}_{2\cdot} - \overline{y}_{\cdot\cdot} = 20.75 - 22.25 =$

$\widehat{\tau}_3 = \overline{y}_{3\cdot} - \overline{y}_{\cdot\cdot} = 22.625 - 22.25 =$

$\widehat{\tau}_4 = \overline{y}_{4\cdot} - \overline{y}_{\cdot\cdot} = 26.25 - 22.25 =$

- Thus, our estimates $\widehat{\mu}_1$, $\widehat{\mu}_2$, $\widehat{\mu}_3$, and $\widehat{\mu}_4$ under Constraint I are:

$$\widehat{\mu}_1 = \widehat{\mu} + \widehat{\tau}_1 = 22.25 - 2.875 =$$

$$\widehat{\mu}_2 = \widehat{\mu} + \widehat{\tau}_2 = 22.25 - 1.5 =$$

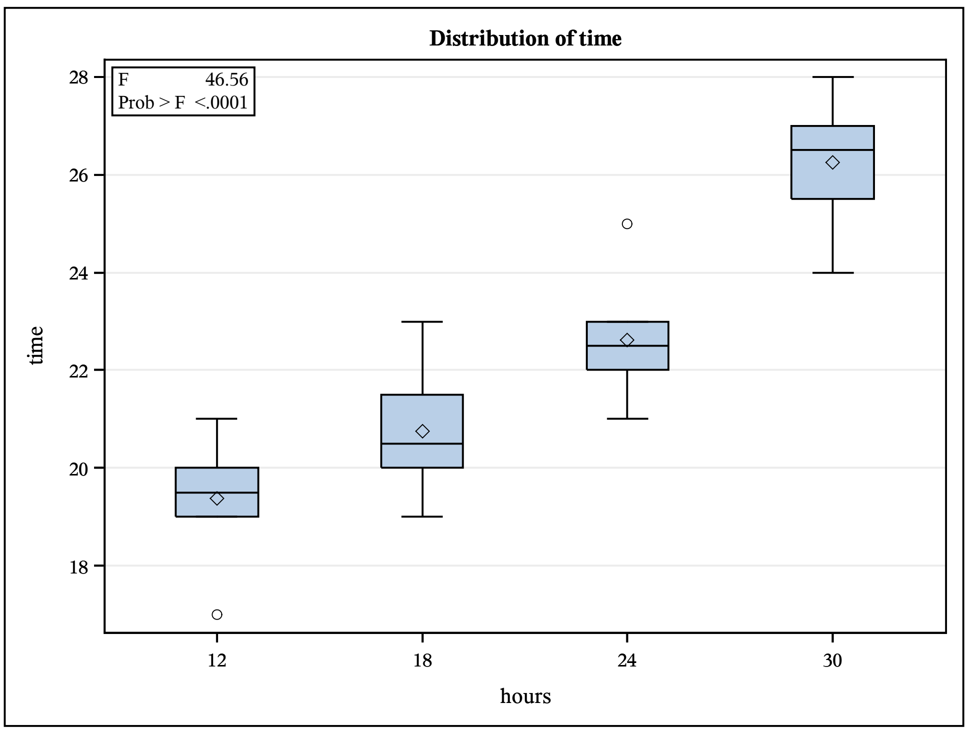
$$\widehat{\mu}_3 = \widehat{\mu} + \widehat{\tau}_3 = 22.25 + 0.375 =$$

$$\widehat{\mu}_4 = \widehat{\mu} + \widehat{\tau}_4 = 22.25 + 4.0 =$$

- Note that both constraints yield the same $\widehat{\mu}_i$ estimates even though the $\widehat{\mu}$ and $\widehat{\tau}_i$ estimates differ between constraints.
- A function that is uniquely estimated regardless of which constraint is used is said to be **estimable**.
- For a oneway ANOVA, $\mu + \tau_i$ for $i = 1, 2, \ldots, a$ are estimable functions, while individually $\mu$, $\tau_1$, $\tau_2$ $\ldots$, $\tau_a$ are not estimable.

We will now analyze the data using SAS. The analysis will include

- Side-by-side boxplots of the time response across sleep deprivation treatments.




- ANOVA table with parameter estimates assuming the constraint $\tau_4 = 0$. This is the default using SAS.
- A table of treatment means and standard deviations.
- Parameter estimates assuming the constraint $\sum_{i=1}^{4} \tau_i = 0$. This are calculated using ESTIMATE statements in SAS.

## *SLEEP DEPRIVATION EXAMPLE*
## *CONTRASTS AND MULTIPLE COMPARISONS*

### *The GLM Procedure*

| Class Level Information | | |
|---|---|---|
| **Class** | **Levels** | **Values** |
| **hours** | 4 | 12 18 24 30 |

| | |
|---|---|
| **Number of Observations Read** | 32 |
| **Number of Observations Used** | 32 |

### *The GLM Procedure*

***Dependent Variable: time***

| Source | DF | Sum of Squares | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| **Model** | 3 | 213.2500000 | 71.0833333 | 46.56 | <.0001 |
| **Error** | 28 | 42.7500000 | 1.5267857 | | |
| **Corrected Total** | 31 | 256.0000000 | | | |

| R-Square | Coeff Var | Root MSE | time Mean |
|---|---|---|---|
| 0.833008 | 5.553401 | 1.235632 | 22.25000 |

| Source | DF | Type III SS | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| **hours** | 3 | 213.2500000 | 71.0833333 | 46.56 | <.0001 |

| Parameter | Estimate | | Standard Error | t Value | Pr > \|t\| | 95% Confidence Limits | |
|---|---|---|---|---|---|---|---|
| **Intercept** | 26.25000000 | B | 0.43686178 | 60.09 | <.0001 | 25.35512921 | 27.14487079 |
| **hours 12** | -6.87500000 | B | 0.61781585 | -11.13 | <.0001 | -8.14053841 | -5.60946159 |
| **hours 18** | -5.50000000 | B | 0.61781585 | -8.90 | <.0001 | -6.76553841 | -4.23446159 |
| **hours 24** | -3.62500000 | B | 0.61781585 | -5.87 | <.0001 | -4.89053841 | -2.35946159 |
| **hours 30** | 0.00000000 | B | . | . | . | . | . |

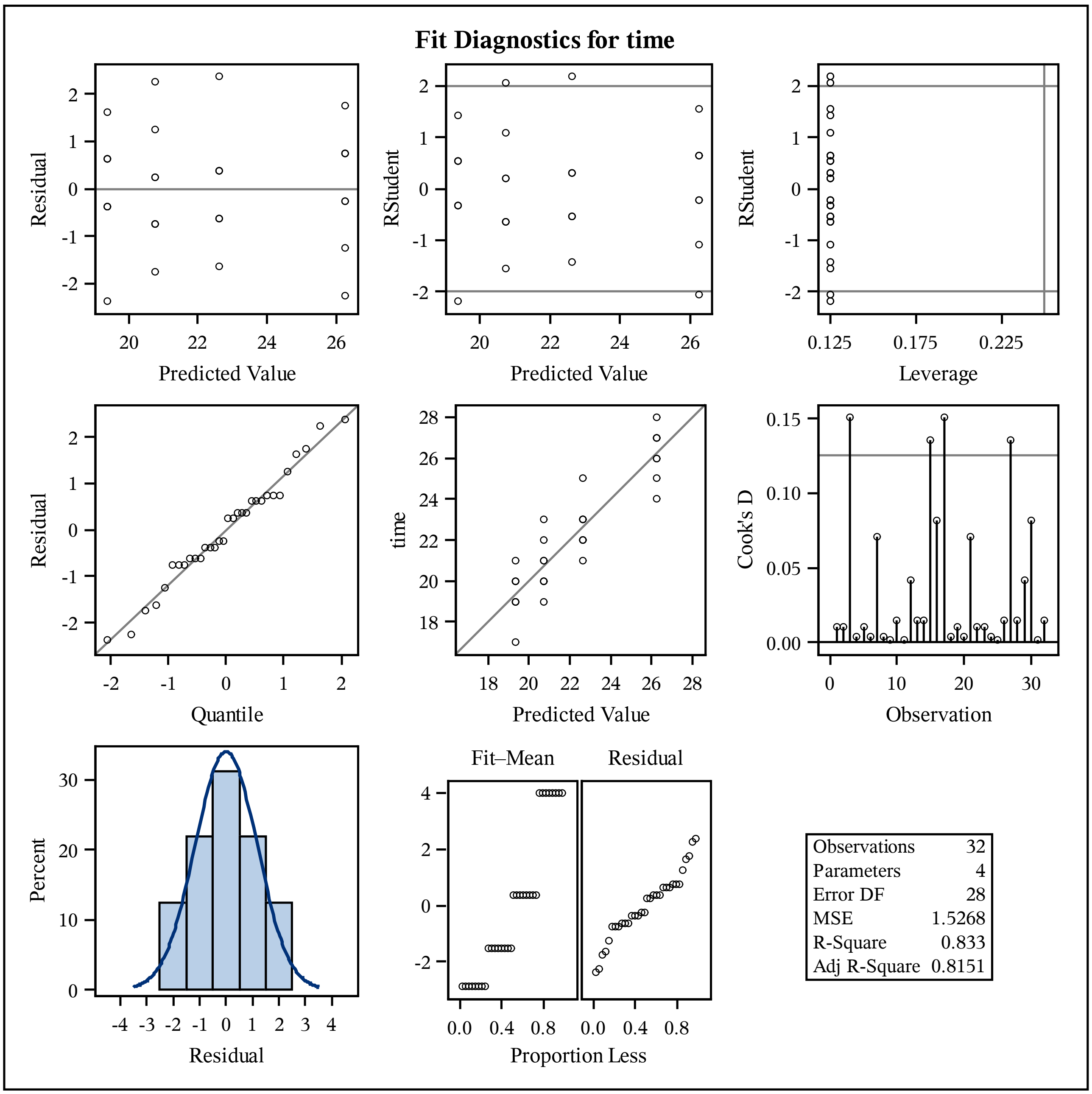
**Note:** The X'X matrix has been found to be singular, and a generalized inverse was used to solve the normal equations. Terms whose estimates are followed by the letter 'B' are not uniquely estimable.

| Level of hours | N | time Mean | Std Dev |
|---|---|---|---|
| 12 | 8 | 19.3750000 | 1.18773494 |
| 18 | 8 | 20.7500000 | 1.28173989 |
| 24 | 8 | 22.6250000 | 1.18773494 |
| 30 | 8 | 26.2500000 | 1.28173989 |

| Contrast | DF | Contrast SS | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| Linear Trend | 1 | 202.5000000 | 202.5000000 | 132.63 | <.0001 |
| Quadratic Trend | 1 | 10.1250000 | 10.1250000 | 6.63 | 0.0156 |
| Cubic Trend | 1 | 0.6250000 | 0.6250000 | 0.41 | 0.5275 |

| Parameter | Estimate | Standard Error | t Value | Pr > \|t\| | 95% Confidence Limits | |
|---|---|---|---|---|---|---|
| 12 hour effect | -2.8750000 | 0.37833340 | -7.60 | <.0001 | -3.6499808 | -2.1000192 |
| 18 hour effect | -1.5000000 | 0.37833340 | -3.96 | 0.0005 | -2.2749808 | -0.7250192 |
| 24 hour effect | 0.3750000 | 0.37833340 | 0.99 | 0.3301 | -0.3999808 | 1.1499808 |
| 30 hour effect | 4.0000000 | 0.37833340 | 10.57 | <.0001 | 3.2250192 | 4.7749808 |
| 12 hour mean | 19.3750000 | 0.43686178 | 44.35 | <.0001 | 18.4801292 | 20.2698708 |
| 18 hour mean | 20.7500000 | 0.43686178 | 47.50 | <.0001 | 19.8551292 | 21.6448708 |
| 24 hour mean | 22.6250000 | 0.43686178 | 51.79 | <.0001 | 21.7301292 | 23.5198708 |
| 30 hour mean | 26.2500000 | 0.43686178 | 60.09 | <.0001 | 25.3551292 | 27.1448708 |
| 12 vs 18 hrs | 1.3750000 | 0.61781585 | 2.23 | 0.0343 | 0.1094616 | 2.6405384 |
| 12 vs 30 hrs | 6.8750000 | 0.61781585 | 11.13 | <.0001 | 5.6094616 | 8.1405384 |
| 18 vs 24 hrs | 1.8750000 | 0.61781585 | 3.03 | 0.0052 | 0.6094616 | 3.1405384 |
| Linear Trend | 22.5000000 | 1.95370527 | 11.52 | <.0001 | 18.4980162 | 26.5019838 |
| Quadratic Trend | 2.2500000 | 0.87372356 | 2.58 | 0.0156 | 0.4602584 | 4.0397416 |
| Cubic Trend | 1.2500000 | 1.95370527 | 0.64 | 0.5275 | -2.7519838 | 5.2519838 |

- Diagnostic plots of the residuals to assess if any model assumptions are seriously violated. These include:
  - A **normal probability (NP) plot** and a **histogram** of the residuals. These plot assesses the assumption that the errors are normally distributed. The pattern in NP plot should be close to linear when the residuals are approximately normally distributed while the histogram should be bell-shaped (assuming there are a reasonable number of residuals. Any serious deviations from linearity suggests the normality assumption has been violated.
  - **Residual versus predicted (fitted) value plot**. This plot assesses the homogeneity of variance (HOV) assumption that the errors have the same variance for each treatment. The residuals should be centered about 0 and the spread of the residuals should be similar for each treatment.



### 2.5.1 SAS Code for Sleep Deprivation Example

```
DM 'LOG; CLEAR; OUT; CLEAR;';

ODS GRAPHICS ON;
ODS PRINTER PDF file='C:\COURSES\ST541\SLEEP.PDF';
OPTIONS NODATE NONUMBER;

*******************************;
*** Sleep deprivation example ***;
*******************************;

DATA in;
     DO hours = 12 to 30 by 6;
     DO rep = 1 to 8;
        INPUT time @@; OUTPUT;
     END; END;
CARDS;
20 20 17 19 20 19 21 19    21 20 21 22 20 20 23 19
25 23 22 23 21 22 22 23    26 27 24 27 25 28 26 27
;
PROC GLM DATA=in PLOTS = (ALL);
     CLASS hours;
     MODEL time = hours / SS3 SOLUTION CLPARM ALPHA=.05;
     MEANS hours ;
     * OUTPUT OUT=diag P=pred R=resid;

     ESTIMATE '12 hour effect' hours  3 -1 -1 -1 / DIVISOR=4;
     ESTIMATE '18 hour effect' hours -1  3 -1 -1 / DIVISOR=4;
     ESTIMATE '24 hour effect' hours -1 -1  3 -1 / DIVISOR=4;
     ESTIMATE '30 hour effect' hours -1 -1 -1  3 / DIVISOR=4;

     ESTIMATE '12 hour mean' INTERCEPT 1 hours 1 0 0 0;
     ESTIMATE '18 hour mean' INTERCEPT 1 hours 0 1 0 0;
     ESTIMATE '24 hour mean' INTERCEPT 1 hours 0 0 1 0;
     ESTIMATE '30 hour mean' INTERCEPT 1 hours 0 0 0 1;

     ESTIMATE '12 vs 18 hrs' hours -1  1  0  0;
     ESTIMATE '12 vs 30 hrs' hours -1  0  0  1;
     ESTIMATE '18 vs 24 hrs' hours  0 -1  1  0;

     ESTIMATE 'Linear Trend'    hours  -3  -1  1 3;
     ESTIMATE 'Quadratic Trend' hours   1  -1 -1 1;
     ESTIMATE 'Cubic Trend'     hours  -1   3 -3 1;

     CONTRAST 'Linear Trend'    hours  -3  -1  1 3;
     CONTRAST 'Quadratic Trend' hours   1  -1 -1 1;
     CONTRAST 'Cubic Trend'     hours  -1   3 -3 1;
TITLE 'SLEEP DEPRIVATION EXAMPLE';
TITLE2 'CONTRASTS AND MULTIPLE COMPARISONS';
RUN;
```

### 2.5.2 R Analysis for Sleep Deprivation Example

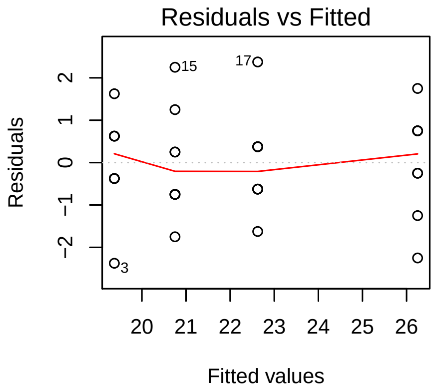
## R Output for Sleep Deprivation Example

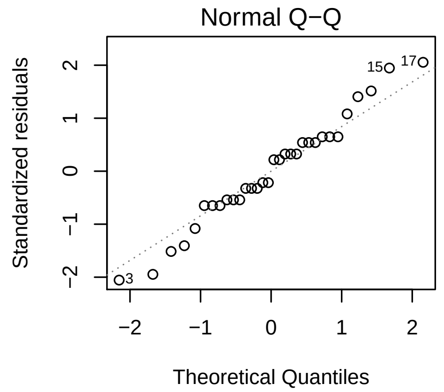
```
> #---------- Treatment means and std dev ----------------
>
> tapply(time,hours,mean)

    12     18     24     30
19.375 20.750 22.625 26.250

> tapply(time,hours,sd)

      12       18       24       30
1.187735 1.281740 1.187735 1.281740


> #----------- Generate ANOVA results ----------------
> summary (f1)
              Df Sum Sq Mean Sq F value   Pr(>F)
factor(hours)  3 213.25   71.08   46.56 5.22e-11 ***
Residuals     28  42.75    1.53
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

> summary(f2)
Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)      19.3750     0.4369  44.350  < 2e-16 ***
factor(hours)18   1.3750     0.6178   2.226   0.0343 *
factor(hours)24   3.2500     0.6178   5.260 1.36e-05 ***
factor(hours)30   6.8750     0.6178  11.128 8.64e-12 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1.236 on 28 degrees of freedom
Multiple R-squared: 0.833,      Adjusted R-squared: 0.8151
F-statistic: 46.56 on 3 and 28 DF,  p-value: 5.222e-11
```

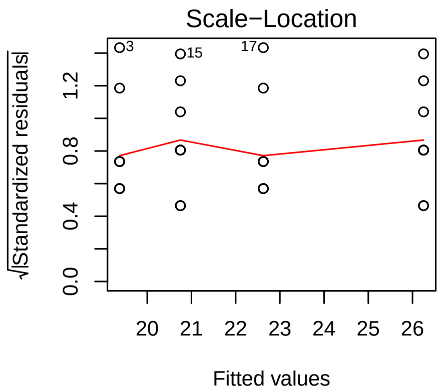
## R Code for Sleep Deprivation Example

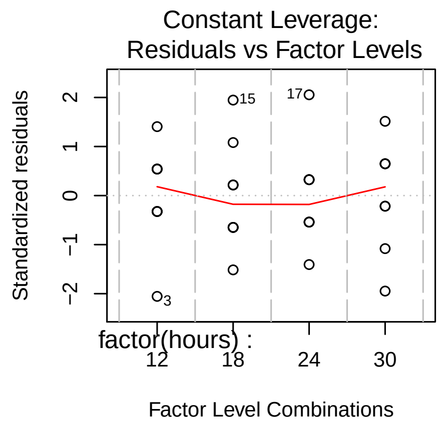
```
hours <- c(rep(12,8),rep(18,8),rep(24,8),rep(30,8))

time <- c(20,20,17,19,20,19,21,19,21,20,21,22,20,20,23,19,
25,23,22,23,21,22,22,23,26,27,24,27,25,28,26,27)

#---------- Treatment means and std dev ----------------
tapply(time,hours,mean)
tapply(time,hours,sd)

#----------- Generate ANOVA results ----------------
f1 <- aov(time~factor(hours))
summary (f1)
f2 <- lm(time~factor(hours))
summary(f2)

#----------- Generate diagnostic plots ----------------
windows()
par(mfrow=c(2,2))
plot(f1)
windows()
par(mfrow=c(2,2))
stripchart(time~hours,vertical=TRUE,main="Response Time vs Treatment")
plot(fitted(f1),resid(f1),main="Residuals vs Predicted Values")
qqnorm(resid(f1),main="Normal Probability Plot")
hist(resid(f1),nclass=8,main="Histogram of Residuals")
```

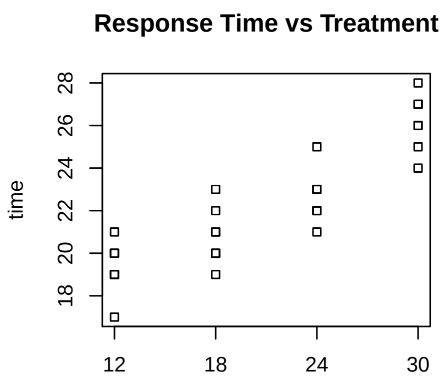
## R Diagnostic Plots

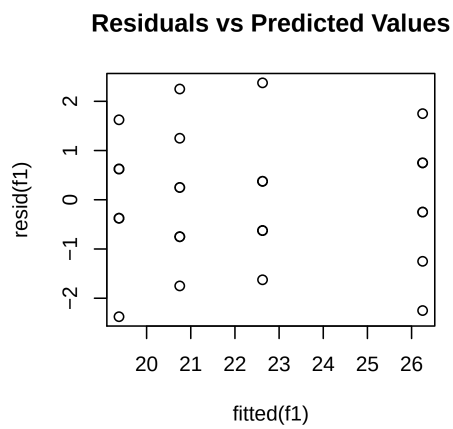


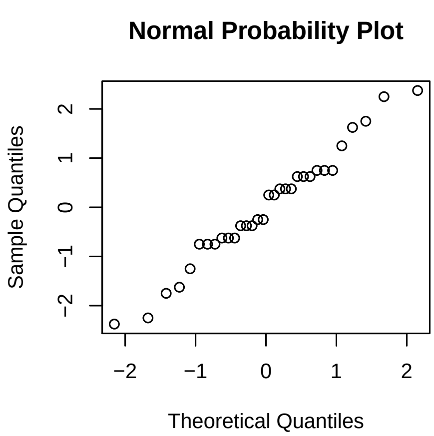



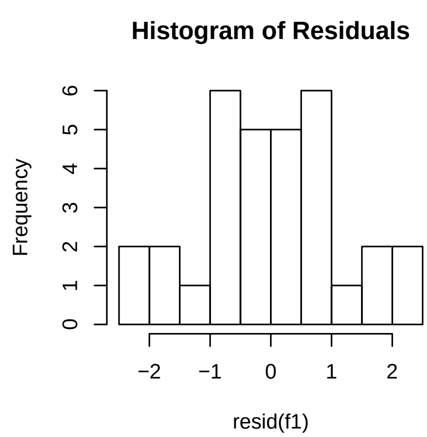



### 2.6 CRD Matrix Form Example

Suppose there are $a = 3$ treatments and $n = 3$ observations per treatment. The data were:

| Treatment | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 10 | 7 |
| 5 | 12 | 8 |
| 6 | 14 | 12 |

| Summary Statistics | | | |
|---|---|---|---|
| $y_{1\cdot} = 15$ | $y_{2\cdot} = 36$ | $y_{3\cdot} = 27$ | $y_{\cdot\cdot} = 78$ |
| $\overline{y}_{1\cdot} = 5$ | $\overline{y}_{2\cdot} = 12$ | $\overline{y}_{3\cdot} = 9$ | $\overline{y}_{\cdot\cdot} = 78/9 = 26/3$ |

**CONSTRAINT I**: $\sum_{i=1}^{a} \tau_i = 0$ (equal $n_i$ case)

- Model: $y_{ij} = \mu + \tau_i + \epsilon_{ij}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$
- In matrix form $y = X'\theta + \epsilon$ where $\theta' = [\mu, \tau_1, \tau_2]$.
- Goal: Find $\widehat{\theta}' = [\widehat{\mu}, \widehat{\tau}_1, \widehat{\tau}_2]$, and assuming $\sum_{i=1}^{a} \tau_i = 0$ we will get $\widehat{\tau}_3 = -\widehat{\tau}_1 - \widehat{\tau}_2$

$$X = \overset{\begin{matrix}\mu & \tau_1 & \tau_2\end{matrix}}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ \hline 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ \hline 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}} \qquad y = \begin{bmatrix} 4 \\ 5 \\ 6 \\ \hline 10 \\ 12 \\ 14 \\ \hline 7 \\ 8 \\ 12 \end{bmatrix}$$

$$X'X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 3 \\ 0 & 3 & 6 \end{bmatrix} \qquad (X'X)^{-1} = \frac{1}{9} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$X'y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \\ \hline 10 \\ 12 \\ 14 \\ \hline 7 \\ 8 \\ 12 \end{bmatrix} = \begin{bmatrix} 78 \\ 15\text{-}27 \\ 36 - 27 \end{bmatrix} = \begin{bmatrix} 78 \\ -12 \\ 9 \end{bmatrix} = \begin{bmatrix} y_{\cdot\cdot} \\ y_{1\cdot} - y_{3\cdot} \\ y_{2\cdot} - y_{3\cdot} \end{bmatrix}$$

$$\widehat{\theta} = (X'X)^{-1}X'y = \frac{1}{9}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 78 \\ -12 \\ 9 \end{bmatrix} = \begin{bmatrix} 78/9 \\ (-24-9)/9 \\ (12+18)/9 \end{bmatrix} = \begin{bmatrix} 26/3 \\ -11/3 \\ 10/3 \end{bmatrix} = \begin{bmatrix} \widehat{\mu} \\ \widehat{\tau}_1 \\ \widehat{\tau}_2 \end{bmatrix}$$

Then $\widehat{\tau}_3 = -\widehat{\tau}_1 - \widehat{\tau}_2 = \frac{1}{3}$. Because $\widehat{\mu}_i = \widehat{\mu} + \widehat{\tau}_i$, we get $\widehat{\mu}_1 = 5$, $\widehat{\mu}_2 = 12$, and $\widehat{\mu}_3 = 9$,

**CONSTRAINT II**: $\tau_3 = 0$ (equal $n_i$ case) Goal: Find $\widehat{\theta}' = [\widehat{\mu}, \widehat{\tau}_1, \widehat{\tau}_2]$ with $\widehat{\tau}_3 = 0$ because of the constraint.

$$\begin{matrix} \mu & \tau_1 & \tau_2 \end{matrix}$$

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ \hline 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad y = \begin{bmatrix} 4 \\ 5 \\ 6 \\ \hline 10 \\ 12 \\ 14 \\ \hline 7 \\ 8 \\ 12 \end{bmatrix}$$

$$X'X = \begin{bmatrix} 9 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix} \qquad (X'X)^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

$$X'y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} y = \begin{bmatrix} y_{\cdot\cdot} \\ y_{1\cdot} \\ y_{2\cdot} \end{bmatrix}$$

$$\widehat{\theta} = (X'X)^{-1}X'y = \begin{bmatrix} (1/3)(y_{\cdot\cdot} - y_{1\cdot} - y_{2\cdot}) \\ (1/3)(-y_{\cdot\cdot} + 2y_{1\cdot} + y_{2\cdot}) \\ (1/3)(-y_{\cdot\cdot} + y_{1\cdot} + 2y_{2\cdot}) \end{bmatrix} \quad \text{Note: } y_{\cdot\cdot} = y_{1\cdot} + y_{2\cdot} + y_{3\cdot}$$

$$= \begin{bmatrix} (1/3)y_{3\cdot} \\ (1/3)(y_{1\cdot} - y_{3\cdot}) \\ (1/3)(y_{2\cdot} - y_{3\cdot}) \end{bmatrix} = \begin{bmatrix} \overline{y}_{3\cdot} \\ \overline{y}_{1\cdot} - \overline{y}_{3\cdot} \\ \overline{y}_{2\cdot} - \overline{y}_{3\cdot} \end{bmatrix} = \begin{bmatrix} 9 \\ 5-9 \\ 12-9 \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \\ 3 \end{bmatrix}$$

Then $\widehat{\mu} = 9$, $\widehat{\tau}_1 = -4, \widehat{\tau}_2 = 3$, and $\widehat{\tau}_3 = 0$

- The estimates of the 3 means are

$$\begin{aligned} \widehat{\mu}_1 &= \widehat{\mu} + \widehat{\tau}_1 = 9 - 4 &&= 5 \\ \widehat{\mu}_2 &= \widehat{\mu} + \widehat{\tau}_2 = 9 + 3 &&= 12 \\ \widehat{\mu}_3 &= \widehat{\mu} + \widehat{\tau}_3 &&= 9 \end{aligned}$$

which are the same as those using Constraint I.

## Alternate Matrix Form Solutions

We can retain all $a+1$ parameter columns and still find the least squares solutions for $\mu,\ \tau_1,\ldots,\tau_a$ if we append a row to matrix $X$ and a value $c$ to vector $y$ based on the based on the linear constraint, and then follow the same procedure as before.

**CONSTRAINT I**: $\sum_{i=1}^{a}\tau_i = 0$ (equal $n_i$ case) In matrix form $E(y_1) = X_1'\theta$ where:

$$X_1 = \begin{array}{c} \begin{matrix} \mu & \tau_1 & \tau_2 & \tau_3 \end{matrix} \\ \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 1 \end{array}\right] \end{array} \qquad y_1 = \left[\begin{array}{c} 4 \\ 5 \\ 6 \\ \hline 10 \\ 12 \\ 14 \\ \hline 7 \\ 8 \\ 12 \\ \hline 0 \end{array}\right] \qquad \theta = \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$$

$$X_1'X_1 = \begin{bmatrix} 1&1&1&1&1&1&1&1&1&0 \\ 1&1&1&0&0&0&0&0&0&1 \\ 0&0&0&1&1&1&0&0&0&1 \\ 0&0&0&0&0&0&1&1&1&1 \end{bmatrix} \begin{bmatrix} 1&1&0&0 \\ 1&1&0&0 \\ 1&1&0&0 \\ 1&0&1&0 \\ 1&0&1&0 \\ 1&0&1&0 \\ 1&0&0&1 \\ 1&0&0&1 \\ 1&0&0&1 \\ 0&1&1&1 \end{bmatrix} = \begin{bmatrix} 9&3&3&3 \\ 3&4&1&1 \\ 3&1&4&1 \\ 3&1&1&4 \end{bmatrix} \qquad (X_1'X_1)^{-1} = \frac{1}{9}\begin{bmatrix} 2&-1&-1&-1 \\ -1&3&0&0 \\ -1&0&3&0 \\ -1&0&0&3 \end{bmatrix}$$

$$X_1'y_1 = \begin{bmatrix} 1&1&1&1&1&1&1&1&1&0 \\ 1&1&1&0&0&0&0&0&0&1 \\ 0&0&0&1&1&1&0&0&0&1 \\ 0&0&0&0&0&0&1&1&1&1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \\ 10 \\ 12 \\ 14 \\ 7 \\ 8 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 78 \\ 15 \\ 36 \\ 27 \end{bmatrix} = \begin{bmatrix} y_{..} \\ y_{1.} \\ y_{2.} \\ y_{3.} \end{bmatrix}$$

$$\widehat{\theta} = (X_1'X_1)^{-1}X_1'y_1 = \frac{1}{9}\begin{bmatrix} 2&-1&-1&-1 \\ -1&3&0&0 \\ -1&0&3&0 \\ -1&0&0&3 \end{bmatrix} \begin{bmatrix} 78 \\ 15 \\ 36 \\ 27 \end{bmatrix} = \begin{bmatrix} 26/3 \\ -11/3 \\ 10/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \widehat{\mu} \\ \widehat{\tau}_1 \\ \widehat{\tau}_2 \\ \widehat{\tau}_3 \end{bmatrix}$$

**CONSTRAINT II**: $\tau_3 = 0$ (equal $n_i$ case)

$$X_2 = \begin{array}{c} \begin{array}{cccc} \mu & \tau_2 & \tau_2 & \tau_3 \end{array} \\ \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \end{array} \qquad y_2 = \left[\begin{array}{c} 4 \\ 5 \\ 6 \\ \hline 10 \\ 12 \\ 14 \\ \hline 7 \\ 8 \\ 12 \\ \hline 0 \end{array}\right] \qquad \theta = \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$$

$$X_2'X_2 = \begin{bmatrix} 1&1&1&1&1&1&1&1&1&0 \\ 1&1&1&0&0&0&0&0&0&0 \\ 0&0&0&1&1&1&0&0&0&0 \\ 0&0&0&0&0&0&1&1&1&1 \end{bmatrix} \begin{bmatrix} 1&1&0&0 \\ 1&1&0&0 \\ 1&1&0&0 \\ 1&0&1&0 \\ 1&0&1&0 \\ 1&0&1&0 \\ 1&0&0&1 \\ 1&0&0&1 \\ 1&0&0&1 \\ 0&0&0&1 \end{bmatrix} = \begin{bmatrix} 9&3&3&3 \\ 3&3&0&0 \\ 3&0&3&0 \\ 3&0&0&4 \end{bmatrix} \qquad (X_2'X_2)^{-1} = \frac{1}{3}\begin{bmatrix} 4&-4&-4&-3 \\ -4&5&4&3 \\ -4&4&5&3 \\ -3&3&3&3 \end{bmatrix}$$

$$X_2'y_2 = \begin{bmatrix} 1&1&1&1&1&1&1&1&1&0 \\ 1&1&1&0&0&0&0&0&0&0 \\ 0&0&0&1&1&1&0&0&0&0 \\ 0&0&0&0&0&0&1&1&1&1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \\ 10 \\ 12 \\ 14 \\ 7 \\ 8 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 78 \\ 15 \\ 36 \\ 27 \end{bmatrix} = \begin{bmatrix} y_{\cdot\cdot} \\ y_{1\cdot} \\ y_{2\cdot} \\ y_{3\cdot} \end{bmatrix}$$

$$\widehat{\theta} = (X_2'X_2)^{-1}X_2'y_2 = \frac{1}{3}\begin{bmatrix} 4&-4&-4&-3 \\ -4&5&4&3 \\ -4&4&5&3 \\ -3&3&3&3 \end{bmatrix} \begin{bmatrix} 78 \\ 15 \\ 36 \\ 27 \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \\ 13 \\ 0 \end{bmatrix} = \begin{bmatrix} \widehat{\mu} \\ \widehat{\tau}_2 \\ \widehat{\tau}_2 \\ \widehat{\tau}_3 \end{bmatrix}$$