3.11 Latin Square Designs

- The experimenter is concerned with a single factor having p levels. However, variability from two other sources can be controlled in the experiment.

- If we can control the effect of these other two variables by grouping experimental units into blocks having the same number of treatment levels as the factor of interest, then a *latin square design* may be appropriate.

- Consider a square with p rows and p columns corresponding to the p levels of each blocking variable. If we assign the p treatments to the rows and columns so that each treatment appears exactly once in each row and in each column, then we have a $p \times p$ **latin square design**.

- It is called a "latin" square because we assign "latin" letters A, B, C, ... to the treatments. Examples of a 4×4 and a 6×6 latin square designs are

		Column			
		1	2	3	4
Rows	1	A	C	B	D
	2	D	A	C	B
	3	B	D	A	C
	4	C	B	D	A

		Column					
		1	2	3	4	5	6
Rows	1	A	B	C	D	E	F
	2	B	C	D	E	F	A
	3	C	D	E	F	A	B
	4	D	E	F	A	B	C
	5	E	F	A	B	C	D
	6	F	A	B	C	D	E

- By blocking in two directions, the MSE will (in general) be reduced. This makes detection of significant results for the factor of interest more likely.

- The experimental units should be arranged so that differences among row and columns represent anticipated/potential sources of variability.

  - In industrial experiments, one blocking variable is often based on units of time. The other blocking variable may represent an effect such as machines or operators.

		Machines			
		1	2	3	4
Six-Hour Work Shift	1	A	C	B	D
	2	D	A	C	B
	3	B	D	A	C
	4	C	B	D	A

		Operator				
		1	2	3	4	5
Day	M	A	B	C	D	E
	Tu	B	C	D	E	A
	W	C	D	E	A	B
	Th	D	E	A	B	C
	F	E	A	B	C	D

  - In agricultural experiments, the experimental units are subplots of land. We would then have the subplots laid out so that soil fertility, moisture, and other sources of variation in two directions are controlled.

  - In greenhouse experiments, the subplots are often laid out in a continuous line. In this case, the rows may be blocks of p adjacent subplots and the columns specify the order within each row block.

Rep. 1							Rep. 2							Rep. 3							Rep. 4							Rep. 5							Rep. 6							Rep. 7						
A	D	G	B	E	C	F	G	F	B	C	A	E	D	B	C	D	G	F	A	E	E	G	A	D	C	F	B	C	B	F	E	D	G	A	F	E	C	A	B	D	G	D	A	E	F	G	B	C

After converting Rep. 1 to Rep. 7 into row blocks, we get

		Column						
		1	2	3	4	5	6	7
	1	A	D	G	B	E	C	F
	2	G	F	B	C	A	E	D
	3	B	C	D	G	F	A	E
Reps	4	E	G	A	D	C	F	B
	5	C	B	F	E	D	G	A
	6	F	E	C	A	B	D	G
	7	D	A	E	F	G	B	C

- Like the RCBD, the latin square design is another design with restricted randomization. Randomization occurs with the initial selection of the latin square design from the set of all possible latin square designs of dimension p and then randomly assigning the treatments to the letters A, B, C, ... The following notation will be used:

  $p =$ the number of treatment levels, row blocks and column blocks.

  $y_{ijk} =$ the observation for the j^{th} treatment within the i^{th} row and k^{th} column.

  $N = p^2 =$ total number of observations

  $y_{\cdots} =$ the sum of all p^2 observations

  $R_i =$ the sum for row block i

  $C_k =$ the sum for column block k

  $T_j =$ the sum for treatment j

  $\overline{y}_{\cdot\cdot} =$ the grand mean of all observations $= y_{\cdots}/p^2$

  $\overline{y}_{i\cdot\cdot} =$ the i^{th} row block mean

  $\overline{y}_{\cdot\cdot k} =$ the k^{th} column block mean

  $\overline{y}_{\cdot j\cdot} =$ the j^{th} treatment mean

- There are three subscripts (i, j, k), but we only need to sum over two subscripts.
- The standard statistical model associated with a latin square design is:

$$y_{ijk} \ = \ \mu \ + \ \alpha_i \ + \ \tau_j \ + \ \beta_k + \epsilon_{ijk} \tag{18}$$

  where μ is the baseline mean, α_i is the block effect associated with row i , β_k is the block effect associated with column k , τ_j is the j^{th} treatment effect, and ϵ_{ijk} is a random error which is assumed to be $IIDN(0, \sigma^2)$.

- To get estimates for the parameters in (18), we need to impose <u>three</u> constraints. If we assume $\sum_{i=1}^{p} \alpha_i = 0$, $\sum_{j=1}^{p} \tau_j = 0$, and $\sum_{k=1}^{p} \beta_k = 0$, then the least squares estimates are

  $\widehat{\mu} =$ $\qquad$ $\widehat{\alpha}_i =$ $\qquad$ $\widehat{\beta}_k =$ $\qquad$ $\widehat{\tau}_j =$

- Substitution of the estimates into (18) yields

$$\begin{aligned} y_{ijk} &= \widehat{\mu} + \widehat{\alpha}_i + \widehat{\tau}_j + \widehat{\beta}_k + e_{ijk} \\ &= \overline{y}_{\cdots} \ + \ (\overline{y}_{i\cdot\cdot} - \overline{y}_{\cdots}) \ + \ (\overline{y}_{\cdot j\cdot} - \overline{y}_{\cdots}) \ + \ (\overline{y}_{\cdot\cdot k} - \overline{y}_{\cdots}) \ + \ e_{ijk} \end{aligned} \tag{19}$$

  where e_{ijk} is the ijk^{th} residual from a latin square design and

$$e_{ijk} \ =$$

3.11.1 The ANOVA for a Latin Square Design

- Degrees of freedom (df): (Treatment df) = (Row df) = (Column df) = $p-1$
- $SStrt$ = the treatment sum of squares

  MS_{trt} = the treatment mean square = $SS_{trt}/(p-1)$
- SS_{row} = the sum of squares for rows

  MS_{row} = the mean square for rows = $SS_{row}/(p-1)$
- SS_{col} = the sum of squares for columns

  MS_{col} = the mean square for columns = $SS_{col}/(p-1)$
- SS_E = the error sum of squares

  The SS_E degrees of freedom = $(p-1)(p-2)$

  MS_E = the mean square error = $SS_E/[(p-1)(p-2)]$
- SS_{total} = the corrected total sum of squares

  The SS_{total} degrees of freedom = $N-1 = p^2-1$

  The total sum of squares for the latin square design is partitioned into 4 components:

$$SS_{total} \;=\; SS_{row} \;+\; SS_{trt} \;+\; SS_{col} \;+\; SS_E$$

- Formulas to calculate SS_{total}, SS_{row}, SS_{trt} and SS_{col}:

$$SS_{total} = \sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ijk} - \overline{y}_{..})^2 = \sum_{i=1}^{p}\sum_{j=1}^{p} y_{ijk}^2 - \frac{y_{...}^2}{p^2} \qquad SS_{row} = \sum_{i=1}^{p} p(\overline{y}_{i..} - \overline{y}_{..})^2 = \sum_{i=1}^{p}\frac{R_i^2}{p} - \frac{y_{...}^2}{p^2}$$

$$SS_{trt} = \sum_{j=1}^{p} p(\overline{y}_{.j.} - \overline{y}_{..})^2 = \sum_{j=1}^{p}\frac{T_j^2}{p} - \frac{y_{...}^2}{p^2} \qquad SS_{col} = \sum_{k=1}^{p} p(\overline{y}_{..k} - \overline{y}_{..})^2 = \sum_{k=1}^{p}\frac{C_k^2}{p} - \frac{y_{...}^2}{p^2}$$

$$SS_E \;=\; SS_{total} \;-\; SS_{row} \;-\; SS_{trt} \;-\; SS_{col} \qquad \frac{y_{...}^2}{p^2} = \frac{y_{...}^2}{N} = \text{the correction factor.}$$

Latin Square Design ANOVA Table

Source of Variation	Sum of Squares	d.f.	Mean Square	F Ratio
Treatments	SS_{trt}	$p-1$	MS_{trt}	$\frac{MS_{trt}}{MS_E}$
Rows	SS_{row}	$p-1$	MS_{row}	
Columns	SS_{col}	$p-1$	MS_{col}	
Error	SS_E	$(p-1)(p-2)$	MS_E	
Total	SS_{total}	p^2-1		

- Note that there are two **restrictions on randomization** with latin square designs: (i) a row restriction that all treatments must appear in each row and (ii) a column restriction that all treatments must appear in each column.

- Because of these restrictions on randomization, there is no F-test for the equality of row block effects and no F-test for the equality of column block effects. This is not a problem, because the factor of interest is the treatment, and there is an F-test for treatments.

3.11.2 Latin Square Example (Peanut Varieties)

Example: A plant biologist conducted an experiment to compare the yields of 4 varieties of peanuts (A, B, C, D). A plot of land was divided into 16 subplots (4 rows and 4 columns) The following latin square design was run. The responses are given in the table to the right.

Treatment (peanut variety)

	Column			
Row	E	EC	WC	W
N	C	A	B	D
NC	A	B	D	C
SC	B	D	C	A
S	D	C	A	B

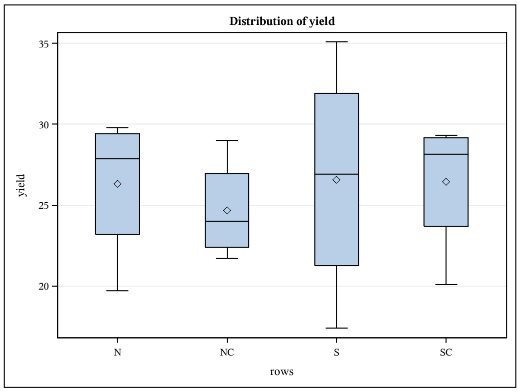
Response (yield)

	Column			
Row	E	EC	WC	W
N	26.7	19.7	29.0	29.8
NC	23.1	21.7	24.9	29.0
SC	29.3	20.1	29.0	27.3
S	25.1	17.4	28.7	35.1

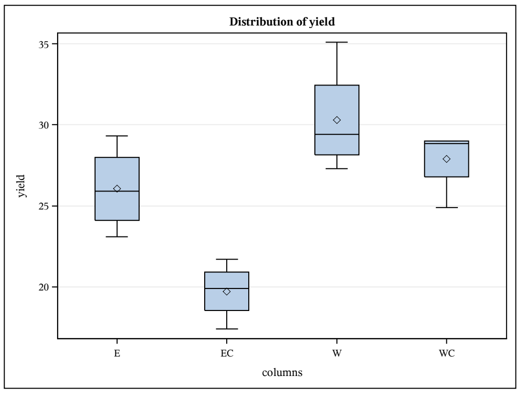
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	9	298.0056250	33.1117361	8.28	0.0091
Error	6	23.9837500	3.9972917		
Corrected Total	15	321.9893750			

R-Square	Coeff Var	Root MSE	yield Mean
0.925514	7.691552	1.999323	25.99375

Source	DF	Type III SS	Mean Square	F Value	Pr > F
columns	3	245.9118750	81.9706250	20.51	0.0015
rows	3	9.4268750	3.1422917	0.79	0.5439
peanut	3	42.6668750	14.2222917	3.56	0.0870

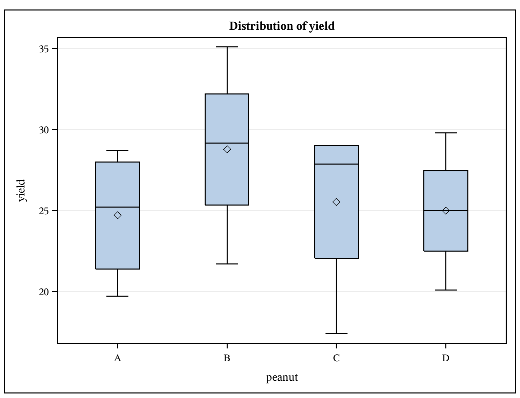


Level of rows	N	yield Mean	Std Dev
N	4	26.3000000	4.59202207
NC	4	24.6750000	3.16688596
S	4	26.5750000	7.38348382
SC	4	26.4250000	4.30764824



Level of columns	N	yield Mean	Std Dev
E	4	26.0500000	2.61979643
EC	4	19.7250000	1.77458915
W	4	30.3000000	3.36551135
WC	4	27.9000000	2.00499377

Tukey's Studentized Range (HSD) Test for reaction



Alpha	0.1
Error Degrees of Freedom	6
Error Mean Square	3.997292
Critical Value of Studentized Range	4.06509
Minimum Significant Difference	4.0637

Means with the same letter are not significantly different.				
Tukey Grouping		**Mean**	**N**	**peanut**
	A	28.775	4	B
	A			
B	A	25.525	4	C
B	A			
B	A	24.975	4	D
B				
B		24.700	4	A

ANALYSIS FOR LATIN SQUARE DESIGN

Obs	rows	columns	peanut	yield	pred	resid
1	N	E	C	26.7	25.8875	0.8125
2	N	EC	A	19.7	18.7375	0.9625
3	N	WC	B	29.0	30.9875	-1.9875
4	N	W	D	29.8	29.5875	0.2125
5	NC	E	A	23.1	23.4375	-0.3375
6	NC	EC	B	21.7	21.1875	0.5125
7	NC	WC	D	24.9	25.5625	-0.6625
8	NC	W	C	29.0	28.5125	0.4875
9	SC	E	B	29.3	29.2625	0.0375
10	SC	EC	D	20.1	19.1375	0.9625
11	SC	WC	C	29.0	27.8625	1.1375
12	SC	W	A	27.3	29.4375	-2.1375
13	S	E	D	25.1	25.6125	-0.5125
14	S	EC	C	17.4	19.8375	-2.4375
15	S	WC	A	28.7	27.1875	1.5125
16	S	W	B	35.1	33.6625	1.4375

SAS Code for Latin Square Design Example

```
DM 'LOG; CLEAR; OUT; CLEAR;';

ODS GRAPHICS ON;
ODS PRINTER PDF file='C:\COURSES\ST541\LATIN.PDF';
OPTIONS NODATE NONUMBER;

DATA in;
   DO rows    = 'N ', 'NC', 'SC', 'S ';
   DO columns = 'E ', 'EC', 'WC', 'W ';
      INPUT peanut $ yield @@; output;
   END; END;
LINES;
C 26.7  A 19.7  B 29.0  D 29.8
A 23.1  B 21.7  D 24.9  C 29.0
B 29.3  D 20.1  C 29.0  A 27.3
D 25.1  C 17.4  A 28.7  B 35.1
;
PROC GLM DATA=in PLOTS=(ALL);
     CLASS columns rows peanut;
     MODEL yield = columns rows peanut / SS3;
     MEANS rows columns;
     MEANS peanut / TUKEY ALPHA=.10;
     OUTPUT OUT=diag P=pred R=resid;
TITLE 'ANALYSIS FOR LATIN SQUARE DESIGN';

PROC PRINT DATA=diag;
RUN;
```

3.12 Matrix Forms for the Latin Square Design

- Consider the 4×4 Peanut Variety Latin Square Design Example
- Model: $y_{ijk} = \mu + \alpha_i + \tau_j + \beta_k + \epsilon_{ijk}$ for $i, j, k = 1, 2, 3, 4$ $\epsilon_{ijk} \sim N(0, \sigma^2)$
- Assume (i) $\sum_{i=1}^4 \alpha_i = 0$, (ii) $\sum_{j=1}^4 \tau_j = 0$ and (iii) $\sum_{k=1}^4 \beta_k = 0$.

Using the Alternate Approach: Keeping $3p + 1$ Columns

$$
X = \begin{array}{c}
\begin{array}{ccccccccccccc} \mu & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \tau_1 & \tau_2 & \tau_3 & \tau_4 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{array} \\
\left[\begin{array}{c|cccc|cccc|cccc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{array}\right]
\end{array}
\qquad
y = \left[\begin{array}{r}
26.7 \\ 19.7 \\ 29.0 \\ 29.8 \\ 23.1 \\ 21.7 \\ 24.9 \\ 29.0 \\ 29.3 \\ 20.1 \\ 29.0 \\ 27.3 \\ 25.1 \\ 17.4 \\ 28.7 \\ 35.1 \\ 0 \\ 0 \\ 0
\end{array}\right]
$$

$$
X'X = \left[\begin{array}{rrrrrrrrrrrrr}
16 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 1 & 5 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 & 5 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 5 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 5 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 5 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 5 & 1 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 5
\end{array}\right]
$$

$$(X'X)^{-1} = \frac{1}{16}\begin{bmatrix} 4 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

$$X'y = \begin{bmatrix} 415.9 \\ 105.2 \\ 98.7 \\ 105.7 \\ 106.3 \\ 98.8 \\ 115.1 \\ 102.1 \\ 99.9 \\ 104.2 \\ 78.9 \\ 111.6 \\ 121.2 \end{bmatrix} \qquad (X'X)^{-1}X'y = \begin{bmatrix} 25.9938 \\ 0.3062 \\ -1.3188 \\ 0.4312 \\ 0.5813 \\ -1.2938 \\ 2.7812 \\ -0.4688 \\ -1.0188 \\ 0.0562 \\ -6.2688 \\ 1.9063 \\ 4.3062 \end{bmatrix} = \begin{bmatrix} \widehat{\mu} \\ \widehat{\alpha}_1 \\ \widehat{\alpha}_2 \\ \widehat{\alpha}_3 \\ \widehat{\alpha}_4 \\ \widehat{\tau}_1 \\ \widehat{\tau}_2 \\ \widehat{\tau}_3 \\ \widehat{\tau}_4 \\ \widehat{\beta}_1 \\ \widehat{\beta}_2 \\ \widehat{\beta}_3 \\ \widehat{\beta}_4 \end{bmatrix}$$

3.12.1 Selection of a Latin Square Design

- Randomization with a latin square design occurs by (i) randomly selecting a generating design, (ii) randomly permuting the p columns and (iii) randomly permuting the last $p-1$ rows.
- For some values of p, the number of unique designs that can be generated from permutations of a generating design are not all equal. This will be true for $p=5$ and $p=6$.
- For $p=3$, there is only one generating design

A	B	C
B	C	A
C	A	B

  - Randomly permute the three columns.
  - Randomly permute rows 2 and 3.
- For $p=4$, there are 4 generating designs

(a)				(b)				(c)				(d)			
A	B	C	D	A	B	C	D	A	B	C	D	A	B	C	D
B	A	D	C	B	C	D	A	B	D	A	C	B	A	D	C
C	D	B	A	C	D	A	B	C	A	D	B	C	D	A	B
D	C	A	B	D	A	B	C	D	C	B	A	D	C	B	A

  - Randomly select generating design (a), (b), (c), or (d).
  - Randomly permute the four columns.
  - Randomly permute rows 2 to 4.
- For $p=5$, there are 2 generating designs

(a) $(1-50)$					(b) $(51-56)$				
A	B	C	D	E	A	B	C	D	E
B	A	D	E	C	B	C	D	E	A
C	E	A	B	D	C	D	E	A	B
D	C	E	A	B	D	E	A	B	C
E	D	B	C	A	E	A	B	C	D

  - Randomly select a number D between 1 and 56.

    If $1 \leq D \leq 50$, then select generating design (a).

    If $51 \leq D \leq 56$, then select generating design (b).
  - Randomly permute the five columns.
  - Randomly permute rows 2 to 5.
- For $p=6$, there are 17 generating designs (I, II, ..., XVII). These generating designs are shown on the next page.
  - Randomly select a number D between 1 and 9408.
  - Find the generating design with interval that contains D.
  - Randomly permute the six columns.
  - Randomly permute rows 2 to 6.

I

A	B	C	D	E	F
B	C	F	A	D	E
C	F	B	E	A	D
D	E	A	B	F	C
E	A	D	F	C	B
F	D	E	C	B	A

0001–1080
1081–2160

II

A	B	C	D	E	F
B	C	F	E	A	D
C	F	B	A	D	E
D	E	A	B	F	C
E	A	D	F	C	B
F	D	E	C	B	A

2161–3240

III

A	B	C	D	E	F
B	A	F	E	C	D
C	F	B	A	D	E
D	C	E	B	F	A
E	D	A	F	B	C
F	E	D	C	A	B

3241–4320

IV

A	B	C	D	E	F
B	A	E	F	C	D
C	F	B	A	D	E
D	E	A	B	F	C
E	D	F	C	B	A
F	C	D	E	A	B

4321–5400

V

A	B	C	D	E	F
B	A	E	C	F	D
C	F	B	A	D	E
D	E	F	B	C	A
E	D	A	F	B	C
F	C	D	E	A	B

5401–5940
5941–6480

VI

A	B	C	D	E	F
B	A	F	E	C	D
C	F	B	A	D	E
D	E	A	B	F	C
E	C	D	F	B	A
F	D	E	C	A	B

6481–7020

VII

A	B	C	D	E	F
B	C	D	E	F	A
C	E	A	F	B	D
D	F	B	A	C	E
E	D	F	B	A	C
F	A	E	C	D	B

7021–7560

VIII

A	B	C	D	E	F
B	A	E	F	C	D
C	F	A	E	D	B
D	C	B	A	F	E
E	D	F	C	B	A
F	E	D	B	A	C

7561–7920
7921–8280

IX

A	B	C	D	E	F
B	A	E	F	C	D
C	F	A	B	D	E
D	E	B	A	F	C
E	D	F	C	B	A
F	C	D	E	A	B

8281–8640

X

A	B	C	D	E	F
B	C	F	A	D	E
C	F	B	E	A	D
D	A	E	B	F	C
E	D	A	F	C	B
F	E	D	C	B	A

8641–8820

XI

A	B	C	D	E	F
B	C	A	F	D	E
C	A	B	E	F	D
D	F	E	B	A	C
E	D	F	C	B	A
F	E	D	A	C	B

8821–8940
8941–9060

XII

A	B	C	D	E	F
B	C	A	E	F	D
C	A	B	F	D	E
D	E	F	B	A	C
E	F	D	A	C	B
F	D	E	C	B	A

9061–9180

XIII

A	B	C	D	E	F
B	C	A	F	D	E
C	A	B	E	F	D
D	F	E	B	A	C
E	D	F	A	C	B
F	E	D	C	B	A

9181–9240

XIV

A	B	C	D	E	F
B	C	A	E	F	D
C	A	B	F	D	E
D	F	E	B	A	C
E	D	F	C	B	A
F	E	D	A	C	B

9241–9280

XV

A	B	C	D	E	F
B	A	F	E	D	C
C	D	A	B	F	E
D	F	E	A	C	B
E	C	B	F	A	D
F	E	D	C	B	A

9281–9316
9317–9352

XVI

A	B	C	D	E	F
B	A	E	C	F	D
C	E	A	F	D	B
D	C	F	A	B	E
E	F	D	B	A	C
F	D	B	E	C	A

9353–9388

XVII

A	B	C	D	E	F
B	C	A	F	D	E
C	A	B	E	F	D
D	E	F	A	B	C
E	F	D	C	A	B
F	D	E	B	C	A

9389–9408

FIG. 6. The 6×6 Latin squares.

No enumeration has as yet been made of squares larger than 6×6. In Fig. 7 we give six squares, with sides from 7 to 12, from which any square of the transformation sets which contain them may be generated by the permutation of all rows, columns, and letters amongst themselves. These transformation sets, or even the smaller sets generated by the permutation of rows and columns, or either and letters, will give sets of squares amply large enough to serve all agricultural purposes.

A	B	C	D	E	F	G
B	D	E	F	A	G	C
C	G	F	E	B	A	D
D	E	A	B	G	C	F
E	C	B	G	F	D	A
F	A	G	C	D	E	B
G	F	D	A	C	B	E

7×7

A	B	C	D	E	F	G	H
B	C	A	E	F	D	H	G
C	A	D	G	H	E	F	B
D	F	G	C	A	H	B	E
E	H	B	F	G	C	A	D
F	D	H	A	B	G	E	C
G	E	F	H	C	B	D	A
H	G	E	B	D	A	C	F

8×8

A	B	C	D	E	F	G	H	I
B	C	E	G	D	I	F	A	H
C	D	F	A	H	G	I	E	B
D	H	A	B	F	E	C	I	G
E	G	B	I	C	H	D	F	A
F	I	H	E	B	D	A	G	C
G	F	I	C	A	B	H	D	E
H	E	G	F	I	A	B	C	D
I	A	D	H	G	C	E	B	F

9×9

A	B	C	D	E	F	G	H	I	J
B	G	A	E	H	C	F	I	J	D
C	H	J	G	F	B	E	A	D	I
D	A	G	I	J	E	C	B	F	H
E	F	H	J	I	G	A	D	B	C
F	E	B	C	D	I	J	G	H	A
G	I	F	B	A	D	H	J	C	E
H	C	I	F	G	J	D	E	A	B
I	J	D	A	C	H	B	F	E	G
J	D	E	H	B	A	I	C	G	F

10×10

A	B	C	D	E	F	G	H	I	J	K
B	A	J	I	D	C	F	K	H	G	E
C	K	H	A	B	I	J	F	D	E	G
D	C	G	J	I	K	E	B	F	A	H
E	J	B	G	K	H	D	C	A	I	F
F	E	I	C	G	A	K	J	B	H	D
G	F	D	B	H	J	A	I	E	K	C
H	I	K	F	A	D	B	E	G	C	J
I	D	E	H	J	B	C	G	K	F	A
J	G	A	K	F	E	H	D	C	B	I
K	H	F	E	C	G	I	A	J	D	B

11×11

A	B	C	D	E	F	G	H	I	J	K	L
B	L	G	C	D	J	K	E	H	A	F	I
C	K	A	B	F	L	I	D	G	H	J	E
D	F	I	A	L	E	C	G	J	B	H	K
E	D	F	G	J	K	A	L	C	I	B	H
F	H	K	E	G	C	D	B	A	L	I	J
G	I	D	F	K	H	J	A	L	C	E	B
H	E	L	J	C	A	B	I	K	D	G	F
I	J	B	L	H	G	F	K	D	E	A	C
J	C	E	K	A	I	H	F	B	G	L	D
K	G	J	H	I	B	L	C	E	F	D	A
L	A	H	I	B	D	E	J	F	K	C	G

12×12

FIG. 7.