3.13 Balanced Incomplete Block Designs (BIBD)

- In many experiments containing blocks, it is not possible to assign all a treatments within each block. The goal is to assign a subset of k ($k < a$) treatments within each block.
- If we assign k ($k < a$) treatments within each block then we have an **incomplete block design**. Several types of incomplete block designs are
  - Balanced Incomplete Block Designs (BIBD)
  - Partially Balanced Incomplete Block Designs (PBIBD)
  - Lattice Designs, Youden Squares

  The BIBD because it is the most commonly-used incomplete block design. In a BIBD:
  - Treatments are applied in each block in a balanced manner so that any two treatments appear together in the same number (λ) of blocks.
  - Treatments are randomized within each block.
- **Notation**:

  $a =$ the number of treatments in the experiment

  $b =$ the number of blocks in the experiment

  $k =$ the number of treatments per block

  $r =$ the number of blocks in which any treatment appears

  $\lambda =$ the number of blocks for which any each pair of treatments appears together
- For any BIBD, $\lambda = \dfrac{r(k-1)}{a-1}$ and $N = bk = ra$.
- The following tables contain a BIBD example with $a =$, $b =$,$k =$, $r =$, and $\lambda =$

  If a cell in the left table is blank, then that treatment is missing in that 'incomplete' block.

	Treatment			
Block	A	B	C	D
1	X	X	X	
2	X		X	X
3	X	X		X
4		X	X	X

$\Longrightarrow$

Block	Treatments		
1	A	B	C
2	A	C	D
3	A	B	D
4	B	C	D

3.13.1 ANOVA for a BIBD

- Assume μ is the overall mean, τ_i is the i^{th} treatment effect, β_j is the j^{th} block effect, and ϵ_{ij} is the random error for y_{ij}. The statistical model for a BIBD is

$$y_{ij} = \qquad\qquad \text{and} \qquad \epsilon_{ij} \sim IIDN(0, \sigma^2). \tag{20}$$

- Because the blocks are incomplete, the Type I and Type III sums of squares will be different. That is, the missing treatments in each block represent missing observations (but not missing 'at random').
- For both Type I and Type III analyses: $RSS_1 = SS_{total} = \sum_i \sum_j y_{ij}^2 - \dfrac{y_{\cdot\cdot}^2}{N}$.

- We can use the sequential **Type I (V2) sum of squares**. The steps are the same for the RCBD with a missing observation (see Section 3.6).

Step	V2 Source	Fit	*df*	Type I SS for V2
1	Total	μ	$N-1$	RSS_1
$2'$	Block	β_j	$b-1$	$R(\beta\|\mu) = RSS_1 - RSS_2^*$
3	Trt(adj)	τ_i	$a-1$	$R(\tau\|\beta,\mu) = RSS_2^* - RSS_3^*$
3	Error	ϵ_{ij}	$N-a-b+1$	RSS_3

- We say "treatment sum of squares is adjusted for blocks". The goal is to account for the block variability before we assess the variability among the treatments.

- For a BIBD, there are formulas for Type I (V2) sums of squares for blocks and (adjusted) treatments:

$$SS_{block} = \sum_{j=1}^{b} \frac{y_{\cdot j}^2}{k} - \frac{y_{\cdot\cdot}^2}{N} \qquad SS_{trt(adj)} = \frac{k\sum_{i=1}^{a} Q_i^2}{\lambda a} \qquad SS_E = SS_{total} - SS_{block} - SS_{trt(adj)}$$

where $Q_i = y_{i\cdot} - \frac{1}{k}\sum_{j=1}^{b} n_{ij} y_{\cdot j}$ and $n_{ij} = 1$ if treatment i appears in block j, and $n_{ij} = 0$ otherwise.

$\frac{1}{k}\sum_{j=1}^{b} n_{ij} y_{\cdot j}$ = average of the block totals across blocks that contain treatment i.

Type I (V2) Analysis of Variance (ANOVA) Table for a BIBD

Source of Variation	Sum of Squares	d.f.	Mean Square	F Ratio
Blocks	SS_{block}	$b-1$		—
Treatment	$SS_{trt(adj)}$	$a-1$	$MS_{trt(adj)}$	$F_0 = MS_{trt(adj)}/MS_E$
Error	SS_E	$N-a-b+1$	MS_E	—
Total	SS_{total}	$N-1$	—	—

where $N = bk = ar$.

- For the **Type III sum of squares**, the steps are:

Step	Source	Fit	*df*	Type III SS
1	Total	μ	$N-1$	RSS_1
2	Trt(adj)	τ_i	$a-1$	$R(\tau\|\beta,\mu) = RSS_2^* - RSS_3$
3	Block(adj)	β_j	$b-1$	$R(\beta\|\tau,\mu) = RSS_2 - RSS_3$
1	Error	ϵ_{ij}	$N-a-b+1$	RSS_3

- The Type III and the Type I (V2) sums of squares will always produce the same F-statistic for testing the equality of treatment effects. The sums of squares for blocks will be different.

- To test $H_0 : \tau_1 = \tau_2 = \cdots = \tau_a$ and $H_1 : \tau_i \neq \tau_j$ *for some* $i \neq j$:

  Compare F_0 to the critical value $F_\alpha(a-1, N-a-b+1)$.

  If $F_0 > F_\alpha(a-1, N-a-b+1)$ then **reject** $H_0 : \tau_1 = \tau_2 = \cdots = \tau_a$.

  If $F_0 \leq F_\alpha(a-1, N-a-b+1)$ then **fail to reject** $H_0 : \tau_1 = \tau_2 = \cdots = \tau_a$.

- Or, more simply, compare the p-value to α.

  If p-value $\leq \alpha$ then **reject** H_0. If p-value $> \alpha$ then **fail to reject** H_0.

3.13.2 BIBD Example (Fabric Wearability)

This example can be found in the 1978 text by Box, Hunter, and Hunter. The experiment involves a Martindale wear tester which is a machine used to test the wearing quality of types of cloth or other fabrics. It possesses the feature that four pieces of cloth may be processed simultaneously in one run of the machine. The responses is weight loss in tenths of a milligram suffered by the test fabric when it is rubbed against a standard grade of emery paper for 1000 revolutions of the machine. The wearing quality of specimens of seven different types of cloth (treatments) A, B, C, D, E, F, G are to be compared. Four treatments are randomized and mounted in the four Martindale wear tester specimen holders. The layout and results are given below:

	Cloth Type							
Block	A	B	C	D	E	F	G	$y_{\cdot j}$
1		627		248		563	252	1690
2	344		233			442	226	1245
3			251	211	160		297	919
4	337	537			195		300	1369
5		520	278		199	595		1592
6	369			196	185	606		1356
7	396	602	240	273				1511
$y_{i\cdot}$	1446	2286	1002	928	739	2206	1075	9682 $= y_{\cdot\cdot}$

$$\sum\sum y_{ij}^2 = 3974162$$

$a =$ $\qquad b =$ $\qquad k =$ $\qquad r =$ $\qquad \lambda = \dfrac{r(k-1)}{a-1} =$

Estimation of Model Effects

- μ, τ_i $(i = 1, 2, \ldots, a)$, and β_j $(j = 1, 2, \ldots, b)$ are not uniquely estimable. Constraints must be imposed. To be able to calculate estimates $\widehat{\mu}$, $\widehat{\tau}_i$, and $\widehat{\beta}_j$, we need to impose <u>two</u> constraints.

- We will assume constraints $\displaystyle\sum_{i=1}^{a} \tau_i = 0$ and $\displaystyle\sum_{j=1}^{b} \beta_j = 0$

- Applying these constraints will yield

$$\widehat{\mu} = \qquad\qquad \widehat{\tau}_i = \tag{21}$$

  where $Q_i = y_{i\cdot} - \dfrac{1}{k}\displaystyle\sum_{j=1}^{b} n_{ij} y_{\cdot j}$.

- The derivation of these formulas is presented in the Montgomery text.

Therefore, for the fabric wear example, we have

$$
\begin{aligned}
Q_1 &= \quad -\tfrac{1}{4}\,(\qquad + \qquad + \qquad + \qquad) = \\
Q_2 &= \quad -\tfrac{1}{4}\,(\qquad + \qquad + \qquad + \qquad) = \\
Q_3 &= \quad -\tfrac{1}{4}\,(\qquad + \qquad + \qquad + \qquad) = \\
Q_4 &= \quad -\tfrac{1}{4}\,(\qquad + \qquad + \qquad + \qquad) = \\
Q_5 &= \quad -\tfrac{1}{4}\,(\qquad + \qquad + \qquad + \qquad) = \\
Q_6 &= \quad -\tfrac{1}{4}\,(\qquad + \qquad + \qquad + \qquad) = \\
Q_7 &= \quad -\tfrac{1}{4}\,(\qquad + \qquad + \qquad + \qquad) =
\end{aligned}
$$

$$
\begin{aligned}
SS_{total} &= 3974162 - \frac{9682^2}{28} = \\
SS_{blocks} &= \frac{1690^2 + 1245^2 + 919^2 + 1369^2 + 1592^2 + 1356^2 + 1511^2}{4} - \frac{9682^2}{28} = \\
SS_{cloth(adj)} &= \frac{k \sum Q_i^2}{\lambda a} \\
&= \frac{4[75.75^2 + 745.5^2 + (-314.5)^2 + (-441)^2 + (-570)^2 + 735.25^2 + (-230.75)^2]}{(2)(7)} = \\
SS_E &= 626264.7143 \; - \; 97394.7143 \; - \; 506798.5714 \; =
\end{aligned}
$$

- You can calculate the estimates using the formulas in (21), or you can use the **LSMEANS** statement in SAS. The LSMEANS statement calculates $\widehat{\mu} + \widehat{\tau}_i$. Thus,

  $$\widehat{\tau}_i \;=\; \text{LSMEAN}_i - \widehat{\mu} \;=\; \text{LSMEAN}_i - \overline{y}_{\cdot\cdot}.$$

- For the fabric wearability example, $\widehat{\tau}_i = \text{LSMEAN}_i - 345.786$:

  $$\widehat{\tau}_1 = 21.643 \qquad \widehat{\tau}_2 = 213.000 \qquad \widehat{\tau}_3 = -89.929 \qquad \widehat{\tau}_4 = -126.000$$
  $$\widehat{\tau}_5 = -162.857 \qquad \widehat{\tau}_6 = 210.071 \qquad \widehat{\tau}_7 = -65.929$$

- You can also calculate the estimates directly from ESTIMATE statements using the same structure used for the oneway ANOVA and the RCBD.

- The LSMEANS statement can also be used to perform a multiple comparison procedure (MCP) using the ADJ= option. For example, The **/ ADJ=BON** options creates a Bonferroni MCP table with adjusted p-values for all pairwise comparisons . For any adjusted p-value $< .05$, we would reject $H_0 : \overline{\mu}_i = \overline{\mu}_j$, or equivalently, reject $H_0 : \tau_i = \tau_j$.

  Note: $\overline{\mu}_i$ is the mean for treatment i averaged over the blocks.

- Summarizing the table results indicates:

Treatment	E	D	C	G	A	F	B
LSMEAN	182.9	219.8	255.9	279.9	367.4	555.9	558.8

SAS Code for BIBD Example

```
DM 'LOG; CLEAR; OUT; CLEAR;';

ODS GRAPHICS ON;
ODS PRINTER PDF file='C:\COURSES\ST541\BIBD.PDF';
OPTIONS NODATE NONUMBER;

***************************************;
*** Balanced Incomplete Block Design ***;
*** From Box, Hunter, and Hunter     ***;
***************************************;

DATA bibd; INPUT cloth $ block wear @@; CARDS;
F 1 563   D 1 248   G 1 252   B 1 627
C 2 233   A 2 344   G 2 226   F 2 442
G 3 297   D 3 211   E 3 160   C 3 251
E 4 195   G 4 300   B 4 537   A 4 337
B 5 520   E 5 199   C 5 278   F 5 595
D 6 196   A 6 369   E 6 185   F 6 606
D 7 273   C 7 240   B 7 602   A 7 396
 ;
TITLE 'BALANCED INCOMPLETE BLOCK DESIGN (BIBD)';

PROC GLM DATA=bibd PLOTS=(ALL);
     CLASS cloth block ;
     MODEL wear = block cloth ;
     MEANS cloth ;
     LSMEANS cloth / ADJ=BON ;
     RANDOM block / TEST ;
     ESTIMATE 'cloth A effect' cloth  6 -1 -1 -1 -1 -1 -1 / DIVISOR=7;
     ESTIMATE 'cloth B effect' cloth  -1 6 -1 -1 -1 -1 -1 / DIVISOR=7;
     ESTIMATE 'cloth C effect' cloth  -1 -1 6 -1 -1 -1 -1 / DIVISOR=7;
     ESTIMATE 'cloth D effect' cloth  -1 -1 -1 6 -1 -1 -1 / DIVISOR=7;
     ESTIMATE 'cloth E effect' cloth  -1 -1 -1 -1 6 -1 -1 / DIVISOR=7;
     ESTIMATE 'cloth F effect' cloth  -1 -1 -1 -1 -1 6 -1 / DIVISOR=7;
     ESTIMATE 'cloth G effect' cloth  -1 -1 -1 -1 -1 -1 6 / DIVISOR=7;
TITLE2 'BLOCKS FIRST, TREATMENTS SECOND';
RUN;
```

BALANCED INCOMPLETE BLOCK DESIGN (BIBD)
BLOCKS FIRST, TREATMENTS SECOND

The GLM Procedure

Dependent Variable: wear

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	12	604193.2857	50349.4405	34.22	<.0001
Error	15	22071.4286	1471.4286		
Corrected Total	27	626264.7143			

R-Square	Coeff Var	Root MSE	wear Mean
0.964757	11.09335	38.35920	345.7857

Source	DF	Type I SS	Mean Square	F Value	Pr > F
block	6	97394.7143	16232.4524	11.03	<.0001
cloth	6	506798.5714	84466.4286	57.40	<.0001

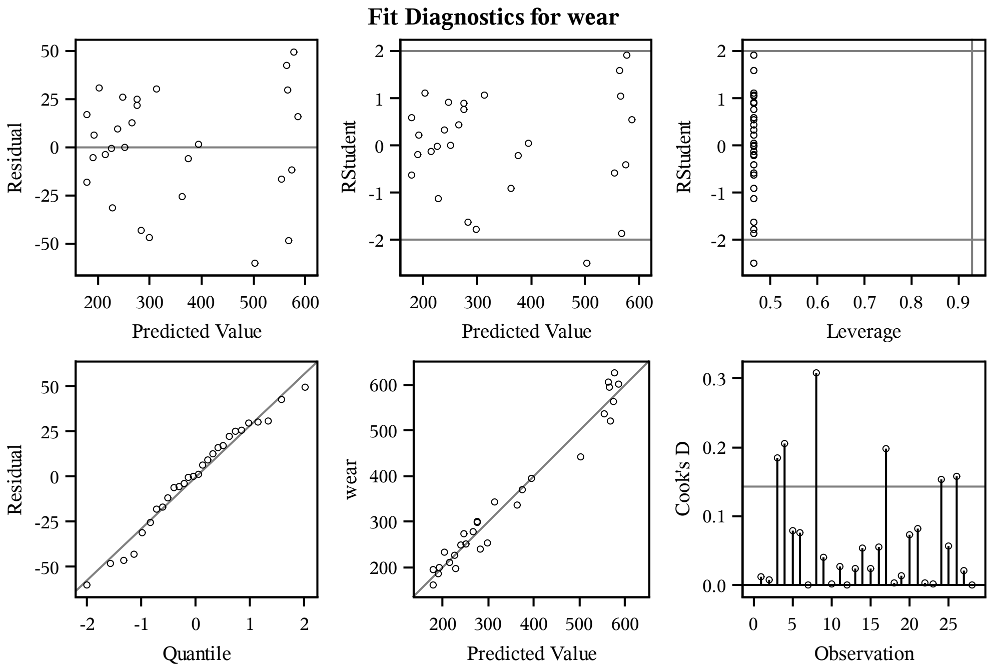
Source	DF	Type III SS	Mean Square	F Value	Pr > F
block	6	14570.0714	2428.3452	1.65	0.2015
cloth	6	506798.5714	84466.4286	57.40	<.0001

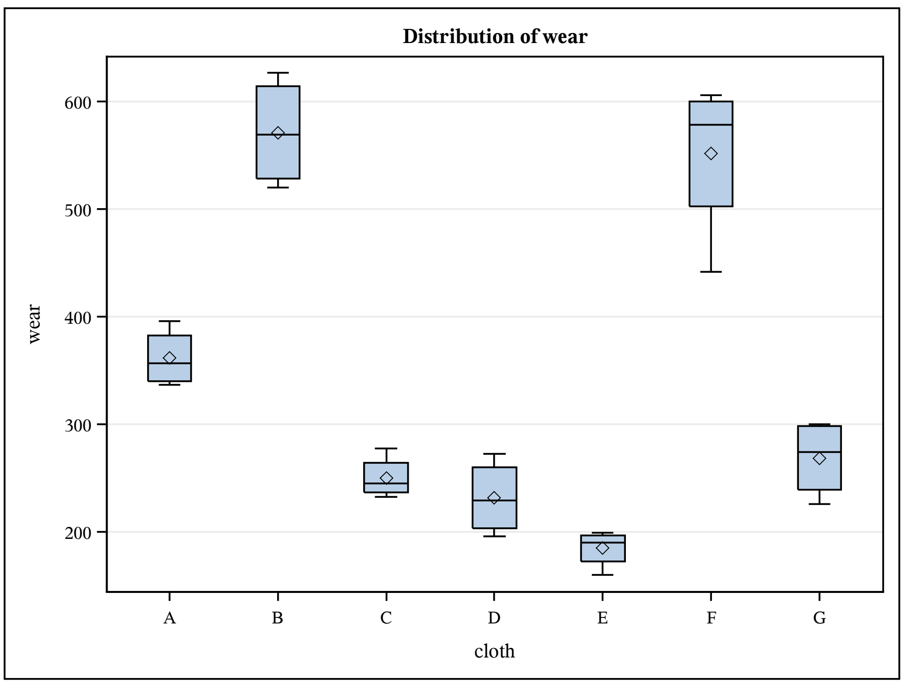
Parameter	Estimate	Standard Error	t Value	Pr > |t|
cloth A effect	21.642857	18.9828832	1.14	0.2721
cloth B effect	213.000000	18.9828832	11.22	<.0001
cloth C effect	-89.928571	18.9828832	-4.74	0.0003
cloth D effect	-126.000000	18.9828832	-6.64	<.0001
cloth E effect	-162.857143	18.9828832	-8.58	<.0001
cloth F effect	210.071429	18.9828832	11.07	<.0001
cloth G effect	-65.928571	18.9828832	-3.47	0.0034

Tests of Hypotheses for Mixed Model Analysis of Variance

Variable: wear

Source	DF	Type III SS	Mean Square	F Value	Pr > F
block	6	14570	2428.345238	1.65	0.2015
cloth	6	506799	84466	57.40	<.0001
Error: MS(Error)	15	22071	1471.428571		



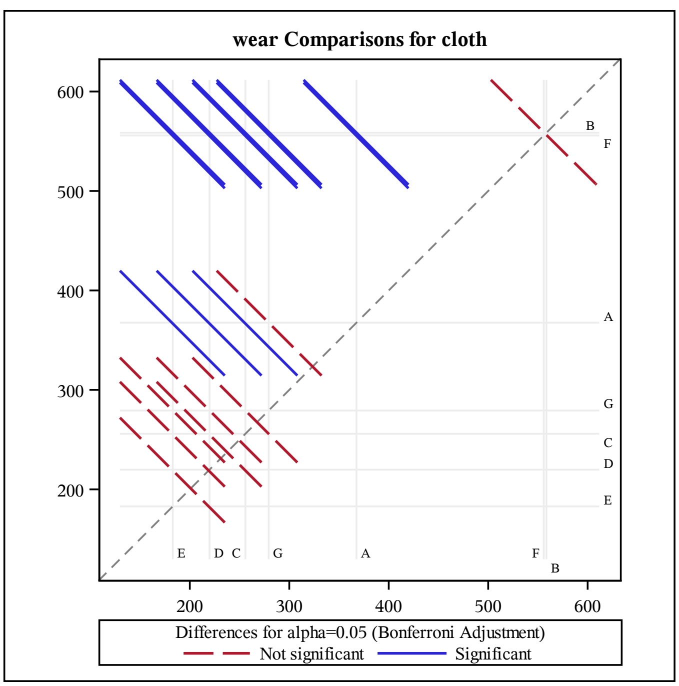


Level of cloth	N	wear	
		Mean	**Std Dev**
A	4	361.500000	26.7893013
B	4	571.500000	51.1631378
C	4	250.500000	19.7737199
D	4	232.000000	34.9952378
E	4	184.750000	17.5190373
F	4	551.500000	75.2440474
G	4	268.750000	35.9756862

The GLM Procedure
Least Squares Means
Adjustment for Multiple Comparisons: Bonferroni

cloth	wear LSMEAN	LSMEAN Number
A	367.428571	1
B	558.785714	2
C	255.857143	3
D	219.785714	4
E	182.928571	5
F	555.857143	6
G	279.857143	7

Least Squares Means for effect cloth Pr > |t| for H0: LSMean(i)=LSMean(j) Dependent Variable: wear							
i/j	1	2	3	4	5	6	7
1		0.0002	0.0332	0.0028	0.0003	0.0002	0.1809
2	0.0002		<.0001	<.0001	<.0001	1.0000	<.0001
3	0.0332	<.0001		1.0000	0.4995	<.0001	1.0000
4	0.0028	<.0001	1.0000		1.0000	<.0001	1.0000
5	0.0003	<.0001	0.4995	1.0000		<.0001	0.0935
6	0.0002	1.0000	<.0001	<.0001	<.0001		<.0001
7	0.1809	<.0001	1.0000	1.0000	0.0935	<.0001	



3.13.3 Matrix Forms for BIBD

- Model: $y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij}$ for $i = 1, 2, 3, 4, 5, 6, 7$ and $j = 1, 2, 3, 4, 5, 6, 7$

$$\epsilon_{ij} \sim N(0, \sigma^2) \qquad \beta_j \sim N(0, \sigma^2_\beta)$$

- Assume (i) $\sum_{i=1}^{7} \tau_i = 0$ and (ii) $\sum_{j=1}^{7} \beta_j = 0$.

- Goal: Estimate $[\, \mu, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7 \,]$.

Alternate Approach: Keeping $a + b + 1$ Columns

$$\begin{array}{ccccccccccccccc} \mu & \tau_1 & \tau_2 & \tau_3 & \tau_4 & \tau_5 & \tau_6 & \tau_7 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 \end{array}$$

$$X = \left[\begin{array}{c|ccccccc|ccccccc}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right] \qquad y = \left[\begin{array}{c}
627 \\ 248 \\ 563 \\ 252 \\ 344 \\ 233 \\ 442 \\ 226 \\ 251 \\ 211 \\ 160 \\ 297 \\ 337 \\ 537 \\ 195 \\ 300 \\ 520 \\ 278 \\ 199 \\ 595 \\ 369 \\ 196 \\ 185 \\ 606 \\ 396 \\ 602 \\ 240 \\ 273 \\ \hline 0 \\ 0
\end{array}\right]$$

$$
X'X \;=\; \left[\begin{array}{r|rrrrrrr|rrrrrrr}
28 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ \hline
4 & 5 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
4 & 1 & 5 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
4 & 1 & 1 & 5 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
4 & 1 & 1 & 1 & 5 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 5 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
4 & 1 & 1 & 1 & 1 & 1 & 5 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 & 5 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline
4 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 5 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 5 & 1 & 1 & 1 & 1 & 1 \\
4 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 5 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 5 & 1 & 1 & 1 \\
4 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 5 & 1 & 1 \\
4 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 5 & 1 \\
4 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 5
\end{array}\right]
$$

$$
(X'X)^{-1} \;=\; \frac{1}{196} \left[\begin{array}{r|rrrrrrr|rrrrrrr}
15 & -4 & -4 & -4 & -4 & -4 & -4 & -4 & -4 & -4 & -4 & -4 & -4 & -4 & -4 \\ \hline
\text{-4} & 52 & -4 & -4 & -4 & -4 & -4 & -4 & 8 & -6 & 8 & -6 & 8 & -6 & -6 \\
\text{-4} & -4 & 52 & -4 & -4 & -4 & -4 & -4 & -6 & 8 & 8 & -6 & -6 & 8 & -6 \\
\text{-4} & -4 & -4 & 52 & -4 & -4 & -4 & -4 & 8 & -6 & -6 & 8 & -6 & 8 & -6 \\
\text{-4} & -4 & -4 & -4 & 52 & -4 & -4 & -4 & -6 & 8 & -6 & 8 & 8 & -6 & -6 \\
\text{-4} & -4 & -4 & -4 & -4 & 52 & -4 & -4 & 8 & 8 & -6 & -6 & -6 & -6 & 8 \\
\text{-4} & -4 & -4 & -4 & -4 & -4 & 52 & -4 & -6 & -6 & 8 & 8 & -6 & -6 & 8 \\
\text{-4} & -4 & -4 & -4 & -4 & -4 & -4 & 52 & -6 & -6 & -6 & -6 & 8 & 8 & 8 \\ \hline
\text{-4} & 8 & -6 & 8 & -6 & 8 & -6 & -6 & 52 & -4 & -4 & -4 & -4 & -4 & -4 \\
\text{-4} & -6 & 8 & -6 & 8 & 8 & -6 & -6 & -4 & 52 & -4 & -4 & -4 & -4 & -4 \\
\text{-4} & 8 & 8 & -6 & -6 & -6 & 8 & -6 & -4 & -4 & 52 & -4 & -4 & -4 & -4 \\
\text{-4} & -6 & -6 & 8 & 8 & -6 & 8 & -6 & -4 & -4 & -4 & 52 & -4 & -4 & -4 \\
\text{-4} & 8 & -6 & -6 & 8 & -6 & -6 & 8 & -4 & -4 & -4 & -4 & 52 & -4 & -4 \\
\text{-4} & -6 & 8 & 8 & -6 & -6 & -6 & 8 & -4 & -4 & -4 & -4 & -4 & 52 & -4 \\
\text{-4} & -6 & -6 & -6 & -6 & 8 & 8 & 8 & -4 & -4 & -4 & -4 & -4 & -4 & 52
\end{array}\right]
$$

$$
X'y \;=\; \begin{bmatrix} 9682 \\ 1446 \\ 2286 \\ 1002 \\ 928 \\ 739 \\ 2206 \\ 1075 \\ 1690 \\ 1245 \\ 919 \\ 1369 \\ 1592 \\ 1356 \\ 1511 \end{bmatrix}
\qquad
(X'X)^{-1}X'y \;=\; \begin{bmatrix} 345.7857 \\ 21.6429 \\ 213 \\ -89.9286 \\ -126 \\ -162.8571 \\ 210.0714 \\ -65.9286 \\ 18.9286 \\ -53.5 \\ -4.8571 \\ -5 \\ 9.6429 \\ 7.5 \\ 27.2857 \end{bmatrix}
\;=\; \begin{bmatrix} \widehat{\mu} \\ \widehat{\tau}_1 \\ \widehat{\tau}_2 \\ \widehat{\tau}_3 \\ \widehat{\tau}_4 \\ \widehat{\tau}_5 \\ \widehat{\tau}_6 \\ \widehat{\tau}_7 \\ \widehat{\beta}_1 \\ \widehat{\beta}_2 \\ \widehat{\beta}_3 \\ \widehat{\beta}_4 \\ \widehat{\beta}_5 \\ \widehat{\beta}_6 \\ \widehat{\beta}_7 \end{bmatrix}
$$