8.6 Jonckheere-Terpstra Test for Ordered Alternatives

In some applications of parametric statistical procedures, it is appropriate to test the null hypothesis of equality among population means (μ_j's)

$$H_0: \mu_1 = \mu_2 = \cdots = \mu_k$$

against an alternative in which order is specified—say,

$$H_1: \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k$$

(where at least one of the inequalities is strict; that is, at least one population mean is less than at least one of the other population means). This alternative is sometimes more meaningful than the alternative

H_1: Not all μ's are equal

which merely negates at least one of the equalities in H_0. Analogous situations arise when the application of nonparametric statistical procedures is appropriate and the location parameters of interest are medians.

In a study of the efficacy of some drug, for example, the investigator may wish to know whether the sample data indicate that increased response accompanies increased dosage. An educator may wish to know whether levels of distraction varying from none to moderate to excessive during an examination result in scores in the reverse order of magnitude. A sociologist may be interested in knowing whether people in low, middle, and high socioeconomic groups possess low, middle, and high knowledge of certain current events. Alternative hypotheses of this type are referred to as *ordered alternatives.*

In the two-sample case involving location parameters, we achieve the objectives of an ordered alternative by using a one-sided alternative instead of a two-sided alternative. When the data available for analysis consist of three or more samples of observations, however, the distinction between one-sided and two-sided tests is not maintained. Consequently we need a procedure that specifically allows for ordered alternatives in the k-sample case.

Terpstra (T36) and Jonckheere (T37) have proposed a test that can be used when an ordered alternative is appropriate.

Assumptions

A. The data for analysis consist of k random samples of sizes $n_1, n_2, \ldots, n_k$ from populations $1, 2, \ldots, k$, with unknown medians $M_1, M_2, \ldots, M_k$. or unknown means $\mu_1, \mu_2, \ldots, \mu_k$

B. The observations are independent, both within and among samples.

C. The variable of interest is continuous.

D. The measurement scale is at least ordinal.

E. The sampled populations are identical except for a possible difference location parameters.

Hypotheses

H_0: $M_1 = M_2 = \cdots = M_k$

H_1: $M_1 \le M_2 \le \cdots \le M_k$, with at least one strict inequality.

OR $H_0: \mu_1 = \mu_2 = \cdots = \mu_k$

$H_1: \mu_1 \le \mu_2 \le \cdots \le \mu_k$ with at least one strict inequality

If the expected direction of inequality is not as specified in this alternative hypothesis, relabel and reorder the samples to achieve conformity.

Test Statistic

The test statistic is $J = \sum_{i<j} U_{ij}$

where U_{ij} is the number of pairs of observations (a, b) for which X_{ia} is less than X_{jb}. In other words, we compare observations in all pairs of samples. We compare each observation in the first sample in the pair of samples with each observation in the second sample in the pair, and if the observation from the first sample is less than the observation in the second sample, we record a score of 1. We record a score of 0 if the observation from the first sample is greater than the observation from the second sample.

Decision Rule

Reject H_0 at the α level of significance if the computed J is greater than or equal to the critical value of J for α, k, and $n_1, n_2, \ldots, n_k$ given in Table A.13. The critical values for J given in Table A.13 for $k = 3$ are tabulated for sample sizes of $n_1 \le n_2 \le n_3$. Because the distribution of J has certain symmetry properties, however, we may obtain critical values for configurations not in that order by rearranging the three sample sizes so that they are in order of increasing size before we enter the table. For example, if we wish critical values for sample sizes $n_1 = 5$, $n_2 = 7$, $n_3 = 3$, we enter Table A.13 at $n_1 = 3$, $n_2 = 5$, $n_3 = 7$.

Ties In computing U_{ij}, record a score of 1/2 for each case where $X_{ia} = X_{jb}$. In other words, each time a tie is encountered when comparing observations, record a score of 1/2 rather than 1.

Large-Sample Approximation For large sample sizes, J is approximately normally distributed with mean 0 and variance 1. When we use the normal approximation, we compute

$$z = \frac{J - [(N^2 - \Sigma_{i=1}^{k} n_i^2)/4]}{\sqrt{[N^2(2N + 3) - \Sigma_{i=1}^{k} n_i^2(2n_i + 3)]/72}} \tag{6.6}$$

and compare it for significance with tabulated values of the standard normal distribution

TABLE A.13(a)

Critical values of *J*, the Jonckheere–Terpstra test statistic (for nominal values of α shown); exact significance levels in parentheses

n_1	n_2	n_3	$\alpha = 0.5$	$\alpha = 0.2$	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$	$\alpha = 0.005$
2	2	2	6 (.57778)	8 (.28889)	9 (.16667)	10 (.08889)	11 (0.3333)	12 (.01111)	12 (.01111)
			7 (.42222)	9 (.16667)	10 (.08889)	11 (.03333)	12 (.01111)	—	—
2	2	3	8 (.56190)	11 (.21905)	12 (.13810)	13 (.07619)	14 (.03810)	15 (.01429)	15 (.01429)
			9 (.43810)	12 (.13810)	13 (.07619)	14 (.03810)	15 (.01429)	16 (.00476)	16 (.00476)
2	2	4	10 (.55238)	13 (.25714)	15 (.11667)	16 (.07143)	17 (.03810)	18 (.01905)	19 (.00714)
			11 (.44762)	14 (.18095)	16 (.07143)	17 (.03810)	18 (.01905)	19 (.00714)	20 (.00238)
2	2	5	12 (.54497)	16 (.21561)	18 (.10450)	19 (.06614)	20 (.03968)	22 (.01058)	22 (.01058)
			13 (.45503)	17 (.15344)	19 (.06614)	20 (.03968)	21 (.02116)	23 (.00397)	23 (.00397)
2	2	6	14 (.53968)	18 (.24444)	20 (.13571)	22 (.06349)	23 (.03968)	25 (.01270)	26 (.00635)
			15 (.46032)	19 (.18492)	21 (.09444)	23 (.03968)	24 (.02381)	26 (.00635)	27 (.00238)
2	2	7	16 (.53535)	21 (.21212)	23 (.12172)	25 (.06061)	27 (.02525)	28 (.01515)	29 (.00808)
			17 (.46465)	22 (.16364)	24 (.08788)	26 (.04040)	28 (.01515)	29 (.00808)	30 (.00404)
2	2	8	18 (.53199)	23 (.23535)	26 (.11178)	28 (.05892)	30 (.02694)	32 (.01010)	33 (.00539)
			19 (.46801)	24 (.18855)	27 (.08215)	29 (.04040)	31 (.01684)	33 (.00539)	34 (.00269)
2	3	3	11 (.50000)	14 (.22143)	15 (.15179)	17 (.05714)	18 (.03036)	19 (.01429)	20 (.00536)
			12 (.40000)	15 (.15179)	16 (.09643)	18 (.03036)	19 (.01429)	20 (.00536)	21 (.00179)
2	3	4	13 (.54286)	17 (.22222)	19 (.11190)	20 (.07381)	22 (.02619)	23 (.01349)	24 (.00635)
			14 (.45714)	18 (.16190)	20 (.07381)	21 (.04524)	23 (.01349)	24 (.00635)	25 (.00238)
2	3	5	16 (.50000)	20 (.22302)	22 (.12421)	24 (.05913)	25 (.03810)	27 (.01310)	28 (.00675)
			17 (.42500)	21 (.16944)	23 (.08770)	25 (.03810)	26 (.02302)	28 (.00675)	29 (.00317)
2	3	6	18 (.53355)	23 (.22338)	25 (.13398)	27 (.07143)	29 (.03290)	31 (.01255)	32 (.00714)
			19 (.46645)	24 (.17554)	26 (.09957)	28 (.04957)	30 (.02100)	32 (.00714)	33 (.00368)
2	3	7	21 (.50000)	26 (.22374)	29 (.10960)	31 (.06023)	33 (.02929)	35 (.01225)	36 (.00732)
			22 (.44003)	27 (.18030)	30 (.08232)	32 (.04268)	34 (.01032)	36 (.00732)	37 (.00417)
2	3	8	23 (.52727)	29 (.22393)	32 (.11826)	35 (.05198)	37 (.02650)	39 (.01189)	40 (.00754)
			24 (.47273)	30 (.18430)	33 (.09192)	36 (.03768)	38 (.01810)	40 (.00754)	41 (.00451)
2	4	4	16 (.53746)	20 (.25587)	23 (.10794)	25 (.05016)	26 (.03206)	28 (.01079)	29 (.00540)
			17 (.46254)	21 (.19810)	24 (.07556)	26 (.03206)	27 (.01905)	29 (.00540)	30 (.00254)
2	4	5	19 (.53261)	24 (.22872)	27 (.10491)	29 (.05397)	30 (.03680)	32 (.01501)	33 (.00880)
			20 (.46739)	25 (.18095)	28 (.07662)	30 (.03680)	31 (.02395)	33 (.00880)	34 (.00491)
2	4	6	22 (.52929)	28 (.20859)	31 (.10245)	33 (.05685)	35 (.02821)	37 (.01219)	38 (.00758)
			23 (.47071)	29 (.16797)	32 (.07742)	34 (.04076)	36 (.01898)	38 (.00758)	39 (.00440)
2	4	7	25 (.52634)	31 (.23209)	35 (.10047)	37 (.05921)	39 (.03193)	42 (.01033)	43 (.00660)
			26 (.47366)	32 (.19305)	36 (.07797)	38 (.04406)	40 (.02261)	43 (.00660)	44 (.00408)
2	4	8	28 (.52410)	35 (.21496)	38 (.12266)	41 (.06112)	44 (.02593)	46 (.01310)	48 (.00593)
			29 (.47590)	36 (.18077)	39 (.09879)	42 (.04686)	45 (.01863)	47 (.00892)	49 (.00377)
2	5	5	23 (.50000)	28 (.23274)	31 (.11935)	34 (.05014)	35 (.03565)	38 (.01046)	39 (.00643)
			24 (.44228)	29 (.19000)	32 (.09157)	35 (.03565)	36 (.02453)	39 (.00643)	40 (.00373)
2	5	6	26 (.52597)	32 (.23596)	36 (.10462)	38 (.06277)	40 (.03469)	43 (.01179)	44 (.00777)
			27 (.47403)	33 (.19708)	37 (.08178)	39 (.04715)	41 (.02486)	44 (.00777)	45 (.00491)
2	5	7	30 (.50000)	37 (.20292)	40 (.11588)	43 (.05821)	46 (.02507)	48 (.01290)	50 (.00601)
			31 (.45303)	38 (.17057)	41 (.09355)	44 (.04477)	47 (.01820)	49 (.00894)	51 (.00393)
2	5	8	33 (.52151)	41 (.20773)	45 (.10400)	48 (.05459)	51 (.02519)	53 (.01383)	55 (.00701)
			34 (.47849)	42 (.17764)	46 (.08500)	49 (.04283)	52 (.01885)	54 (.00996)	56 (.00482)
2	6	6	30 (.52338)	37 (.22198)	41 (.10607)	44 (.05260)	46 (.03031)	49 (.01139)	51 (.00526)
			31 (.47662)	38 (.18816)	42 (.08528)	45 (.04027)	47 (.02235)	50 (.00786)	52 (.00343)

TABLE A.13(a) *(continued)*

n_1	n_2	n_3	$\alpha = 0.5$	$\alpha = 0.2$	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$	$\alpha = 0.005$
2	6	7	34 (.52125)	42 (.21088)	46 (.10721)	49 (.05720)	52 (.02703)	55 (.01103)	57 (.00551)
			35 (.47875)	43 (.18087)	47 (.08803)	50 (.04521)	53 (.02040)	56 (.00789)	58 (.00376)
2	6	8	38 (.51949)	47 (.20176)	51 (.10804)	54 (.06118)	57 (.03135)	61 (.01070)	63 (.00569)
			39 (.48051)	48 (.17491)	52 (.09031)	55 (.04953)	58 (.02449)	62 (.00788)	64 (.00404)
2	7	7	39 (.50000)	47 (.21740)	52 (.10029)	55 (.05628)	58 (.02858)	61 (.01293)	64 (.00509)
			40 (.46130)	48 (.18948)	53 (.08358)	56 (.04543)	59 (.02225)	62 (.00964)	65 (.00360)
2	7	8	43 (.51781)	52 (.22285)	57 (.11128)	61 (.05543)	64 (.02987)	68 (.01127)	70 (.00642)
			44 (.48219)	53 (.19675)	58 (.09468)	62 (.04555)	65 (.02381)	69 (.00857)	71 (.00474)
2	8	8	48 (.51641)	58 (.21616)	63 (.11392)	68 (.05085)	71 (.02858)	75 (.01170)	78 (.00537)
			49 (.48359)	59 (.19248)	64 (.09833)	69 (.04231)	72 (.02319)	76 (.00913)	79 (.00404)
3	3	3	14 (.50000)	17 (.25952)	19 (.13869)	21 (.06131)	22 (.03690)	24 (.01071)	24 (.01071)
			15 (.41548)	18 (.19405)	20 (.09464)	22 (.03690)	23 (.02083)	25 (.00476)	25 (.00476)
3	3	4	17 (.50000)	21 (.22833)	23 (.13000)	25 (.06405)	27 (.02643)	28 (.01548)	29 (.00857)
			18 (.42667)	22 (.17500)	24 (.09310)	26 (.04214)	28 (.01548)	29 (.00857)	30 (.00429)
3	3	5	20 (.50000)	25 (.20584)	27 (.12348)	29 (.06623)	31 (.03106)	33 (.01234)	34 (.00714)
			21 (.43528)	26 (.16147)	28 (.09177)	30 (.04621)	32 (.02002)	34 (.00714)	35 (.00390)
3	3	6	23 (.50000)	28 (.23193)	31 (.11845)	33 (.06791)	35 (.03506)	38 (.01017)	39 (.00622)
			24 (.44210)	29 (.18912)	32 (.09075)	34 (.04946)	36 (.02408)	39 (.00622)	40 (.00357)
3	3	7	26 (.50000)	32 (.21323)	35 (.11451)	38 (.05219)	40 (.02768)	42 (.01320)	44 (.00551)
			27 (.44761)	33 (.17619)	36 (.08989)	39 (.03849)	41 (.01941)	43 (.00868)	45 (.00335)
3	3	8	29 (.50000)	35 (.23428)	39 (.11131)	42 (.05446)	44 (.03092)	47 (.01122)	48 (.00759)
			30 (.45216)	36 (.19843)	40 (.08914)	43 (.04144)	45 (.02259)	48 (.00759)	49 (.00498)
3	4	4	20 (.53221)	25 (.23247)	28 (.10926)	30 (.05758)	32 (.02649)	34 (.01030)	35 (.00589)
			21 (.46779)	26 (.18528)	29 (.08043)	31 (.03974)	33 (.01688)	35 (.00589)	36 (.00320)
3	4	5	24 (.50000)	29 (.23579)	32 (.12269)	35 (.05281)	37 (.02648)	39 (.01169)	40 (.00732)
			25 (.44304)	30 (.19325)	33 (.09481)	36 (.03791)	38 (.01789)	40 (.00732)	41 (.00440)
3	4	6	27 (.52566)	33 (.23834)	37 (.10723)	39 (.06505)	42 (.02642)	44 (.01284)	46 (.00553)
			28 (.47434)	34 (.19973)	38 (.08432)	40 (.04923)	43 (.01865)	45 (.00856)	47 (.00343)
3	4	7	31 (.50000)	38 (.20504)	41 (.11810)	44 (.06003)	47 (.02633)	49 (.01379)	51 (.00657)
			32 (.45344)	39 (.17279)	42 (.09566)	45 (.04644)	48 (.01926)	50 (.00963)	52 (.00435)
3	4	8	34 (.52137)	42 (.20952)	46 (.10583)	49 (.05607)	52 (.02624)	55 (.01058)	57 (.00522)
			35 (.47863)	43 (.17947)	47 (.08672)	50 (.04419)	53 (.01974)	56 (.00752)	58 (.00354)
3	5	5	28 (.50000)	34 (.22029)	37 (.12200)	40 (.05823)	42 (.03227)	45 (.01116)	46 (.00740)
			29 (.44913)	35 (.18365)	38 (.09706)	41 (.04382)	43 (.02324)	46 (.00740)	47 (.00475)
3	5	6	32 (.50000)	39 (.20820)	42 (.12137)	45 (.06278)	48 (.02822)	51 (.01071)	53 (.00500)
			33 (.45405)	40 (.17607)	48 (.09882)	46 (.04890)	49 (.02085)	52 (.00741)	54 (.00328)
3	5	7	36 (.50000)	43 (.22963)	48 (.10022)	51 (.05332)	54 (.02518)	57 (.01033)	59 (.00519)
			37 (.45809)	44 (.19851)	49 (.08220)	52 (.04211)	55 (.01903)	58 (.00740)	60 (.00356)
3	5	8	40 (.50000)	48 (.21844)	53 (.10138)	56 (.05718)	59 (.02926)	63 (.01001)	65 (.00534)
			41 (.46147)	49 (.19057)	54 (.08461)	57 (.04627)	60 (.02284)	64 (.00737)	66 (.00380)
3	6	6	36 (.52087)	44 (.21513)	48 (.11162)	51 (.06089)	54 (.02965)	57 (.01264)	59 (.00656)
			37 (.47913)	45 (.18533)	49 (.09226)	52 (.04855)	55 (.02267)	58 (.00919)	60 (.00459)
3	6	7	41 (.50000)	49 (.22091)	54 (.10392)	57 (.05931)	60 (.03081)	64 (.01085)	66 (.00590)
			42 (.46187)	50 (.19315)	55 (.08704)	58 (.04821)	61 (.02420)	65 (.00807)	67 (.00425)
3	6	8	45 (.51759)	54 (.22580)	59 (.11440)	63 (.05797)	67 (.02551)	70 (.01238)	73 (.00539)
			46 (.48241)	55 (.19983)	60 (.09770)	64 (.04788)	68 (.02027)	71 (.00950)	74 (.00397)
3	7	7	46 (.50000)	55 (.21371)	60 (.10697)	64 (.05366)	67 (.02919)	71 (.01125)	73 (.00651)
			47 (.46502)	56 (.18868)	61 (.09112)	65 (.04421)	68 (.02337)	72 (.00861)	74 (.00486)

TABLE A.13(a) *(continued)*

n_1	n_2	n_3	$\alpha = 0.5$	$\alpha = 0.2$	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$	$\alpha = 0.005$
3	7	8	51 (.50000) 52 (.46769)	61 (.20768) 62 (.18490)	66 (.10953) 67 (.09460)	70 (.05853) 71 (.04917)	74 (.02783) 75 (.02265)	78 (.01156) 79 (.00907)	81 (.00540) 82 (.00410)
3	8	8	56 (.51497) 57 (.48503)	67 (.21440) 68 (.19289)	73 (.10544) 74 (.09197)	78 (.05022) 79 (.04251)	81 (.02986) 82 (.02477)	86 (.01089) 87 (.00869)	89 (.00539) 90 (.00418)
4	4	4	24 (.52840) 25 (.47160)	30 (.21573) 31 (.17558)	33 (.10993) 34 (.08439)	35 (.06323) 36 (0.4632)	37 (.03296) 38 (.02286)	39 (.01530) 40 (.00993)	41 (.00615) 42 (.00367)
4	4	5	28 (.52535) 29 (.47465)	35 (.20291) 36 (.16825)	38 (.11051) 39 (.08738)	41 (.05178) 42 (.03873)	43 (.02833) 44 (.02027)	45 (.01412) 46 (.00959)	47 (.00630) 48 (.00402)
4	4	6	32 (.52292) 33 (.47708)	39 (.22651) 40 (.19294)	43 (.11087) 44 (.08984)	46 (.05649) 47 (.04376)	48 (.03336) 49 (.02497)	51 (.01321) 52 (.00931)	53 (.00639) 54 (.00429)
4	4	7	36 (.52091) 37 (.47909)	44 (.21471) 45 (.18488)	48 (.11118) 49 (.09184)	51 (.06052) 52 (.04822)	54 (.02939) 55 (.02244)	57 (.01248) 58 (.00906)	59 (.00645) 60 (.00450)
4	4	8	40 (.51923) 41 (.48077)	49 (.20504) 50 (.17830)	53 (.11139) 54 (.09353)	57 (.05216) 58 (.04204)	60 (.02636) 61 (.02049)	63 (.01188) 64 (.00885)	65 (.00649) 66 (.00468)
4	5	5	33 (.50000) 34 (.45453)	40 (.21074) 41 (.17872)	44 (.10139) 45 (.08177)	47 (.05094) 48 (.03928)	49 (.02980) 50 (.02220)	52 (.01162) 53 (.00815)	54 (.00557) 55 (.00371)
4	5	6	37 (.52068) 38 (.47932)	45 (.21719) 46 (.18750)	49 (.11377) 50 (.09435)	53 (.05021) 54 (.03970)	55 (.03096) 56 (.02382)	58 (.01346) 59 (.00987)	61 (.00502) 62 (.00347)
4	5	7	42 (.50000) 43 (.46215)	50 (.22261) 51 (.19494)	55 (.10570) 56 (.08875)	58 (.06081) 59 (.04959)	62 (.02519) 63 (.01963)	65 (.01147) 66 (.00858)	67 (.00633) 68 (.00459)
4	5	8	46 (.51748) 47 (.48252)	56 (.20134) 57 (.17722)	60 (.11594) 61 (.09919)	64 (.05923) 65 (.04905)	68 (.02636) 69 (.02102)	71 (.01294) 72 (.00998)	74 (.00572) 75 (.00425)
4	6	6	42 (.51886) 43 (.48114)	51 (.20965) 52 (.18307)	55 (.11612) 56 (.09810)	59 (.05592) 60 (.04548)	62 (.02909) 63 (.02287)	66 (.01031) 67 (.00769)	68 (.00565) 69 (.00408)
4	6	7	47 (.51733) 48 (.48267)	57 (.20342) 58 (.17938)	62 (.10126) 63 (.08619)	66 (.05067) 67 (.04174)	69 (.02756) 70 (.02208)	73 (.01066) 74 (.00818)	75 (.00619) 76 (.00463)
4	6	8	52 (.51603) 53 (.48397)	62 (.22166) 63 (.19820)	68 (.10397) 69 (.08972)	72 (.05539) 73 (.04651)	76 (.02631) 77 (.02141)	80 (.01095) 81 (.00859)	83 (.00513) 84 (.00390)
4	7	7	53 (.50000) 54 (.46809)	63 (.21068) 64 (.18800)	68 (.11261) 69 (.09759)	73 (.05154) 74 (.04318)	76 (.02963) 77 (.02426)	81 (.01000) 82 (.00783)	83 (.00607) 84 (.00465)
4	7	8	58 (.51481) 59 (.48519)	69 (.21695) 70 (.19552)	75 (.10806) 76 (.09450)	80 (.05226) 81 (.04441)	84 (.02621) 85 (.02170)	88 (.01177) 89 (.00946)	91 (.00595) 92 (.00466)
4	8	8	64 (.51376) 65 (.48624)	76 (.21292) 77 (.19320)	82 (.11160) 83 (.09869)	87 (.05754) 88 (.04966)	92 (.02610) 93 (.02191)	97 (.01023) 98 (.00831)	100 (.00538) 101 (.00428)
5	5	5	38 (.50000) 39 (.45888)	46 (.20318) 47 (.17478)	50 (.10490) 51 (.08666)	53 (.05715) 54 (.04558)	56 (.02788) 57 (.02136)	59 (.01196) 60 (.00873)	61 (.00626) 62 (.00440)
5	5	6	43 (.50000) 44 (.46248)	51 (.22463) 52 (.19706)	56 (.10781) 57 (.09078)	60 (.05124) 61 (.04151)	63 (.02637) 64 (.02066)	66 (.01222) 67 (.00921)	69 (.00501) 70 (.00360)
5	5	7	48 (.50000) 49 (.46549)	57 (.21690) 58 (.19200)	62 (.11026) 63 (.09430)	66 (.05631) 67 (.04665)	70 (.02514) 71 (.02008)	73 (.01241) 74 (.00960)	76 (.00554) 77 (.00413)
5	5	8	53 (.50000) 54 (.46806)	63 (.21043) 64 (.18774)	68 (.11235) 69 (.09734)	73 (.05135) 74 (.04300)	76 (.02948) 77 (.02413)	80 (.01256) 81 (.00992)	83 (.00601) 84 (.00461)
5	6	6	48 (.51720) 49 (.48280)	58 (.20518) 59 (.18118)	63 (.10301) 64 (.08787)	67 (.05205) 68 (.04301)	70 (.02859) 71 (.02299)	74 (.01125) 75 (.00868)	76 (.00661) 77 (.00498)
5	6	7	54 (.50000) 55 (.46829)	64 (.21215) 65 (.18952)	69 (.11412) 70 (.09906)	74 (.05272) 75 (.04427)	78 (.02507) 79 (.02042)	82 (.01048) 83 (.00824)	84 (.00641) 85 (.00494)
5	6	8	59 (.51473) 60 (.48527)	70 (.21820) 71 (.19681)	76 (.10935) 77 (.09575)	81 (.05328) 82 (.04535)	85 (.02694) 86 (.02235)	89 (.01223) 90 (.00985)	92 (.00624) 93 (.00490)
5	7	7	60 (.50000) 61 (.47066)	71 (.20814) 72 (.18741)	77 (.10319) 78 (.09019)	81 (.05828) 82 (.04981)	85 (.02998) 86 (.02499)	90 (.01125) 91 (.00904)	93 (.00571) 94 (.00447)
5	7	8	66 (.50000) 67 (.47271)	78 (.20471) 79 (.18559)	84 (.10680) 85 (.09438)	89 (.05493) 90 (.04739)	93 (.02948) 94 (.02489)	98 (.01193) 99 (.00077)	102 (.00515) 103 (.00410)
5	8	8	72 (.51273) 73 (.48727)	85 (.21165) 86 (.19344)	92 (.10461) 93 (.09319)	97 (.05635) 98 (.04917)	102 (.02724) 103 (.02322)	107 (.01166) 108 (.00968)	111 (.00536) 112 (.00434)
6	6	6	54 (.51582) 55 (.48418)	65 (.20145) 66 (.17959)	70 (.10721) 71 (.09285)	74 (.05805) 75 (.04897)	78 (.02816) 79 (.02306)	82 (.01206) 83 (.00954)	85 (.00581) 86 (.00447)
6	6	7	60 (.51464) 61 (.48536)	71 (.21964) 72 (.19831)	77 (.11084) 78 (.09721)	82 (.05446) 83 (.04645)	86 (.02778) 87 (.02311)	91 (.01031) 92 (.00827)	94 (.00520) 95 (.00406)
6	6	8	66 (.51362) 67 (.48638)	78 (.21527) 79 (.19561)	85 (.10104) 86 (.08914)	90 (.05151) 91 (.04436)	94 (.02745) 95 (.02313)	99 (.01100) 100 (.00899)	102 (.00588) 103 (.00471)
6	7	7	67 (.50000) 68 (.47285)	79 (.20598) 80 (.18689)	85 (.10808) 86 (.09563)	90 (.05595) 91 (.04835)	95 (.02559) 96 (.02153)	100 (.01016) 101 (.00829)	103 (.00541) 104 (.00432)
6	7	8	73 (.51267) 74 (.48733)	86 (.21274) 87 (.19456)	93 (.10571) 94 (.09426)	99 (.05002) 100 (.04351)	103 (.02787) 104 (.02380)	109 (.01002) 110 (.00829)	112 (.00558) 113 (.00454)
6	8	8	80 (.51184) 81 (.48816)	94 (.21055) 95 (.19364)	101 (.10987) 102 (.09885)	107 (.05532) 108 (.04873)	112 (.02822) 113 (.02437)	118 (.01098) 119 (.00923)	122 (.00533) 123 (.00439)
7	7	7	74 (.50000) 75 (.47473)	87 (.20413) 88 (.18643)	94 (.10045) 95 (.08944)	99 (.05401) 100 (.04711)	104 (.02609) 105 (.02225)	109 (.01118) 110 (.00929)	113 (.00515) 114 (.00418)
7	7	8	81 (.50000) 82 (.47637)	95 (.20251) 96 (.18602)	102 (.10477) 103 (.09414)	108 (.05235) 109 (.04605)	113 (.02653) 114 (.02288)	119 (.01023) 120 (.00859)	122 (.00597) 123 (.00494)
7	8	8	88 (.51108) 89 (.48892)	103 (.20959) 104 (.19380)	111 (.10393) 112 (.09402)	117 (.05443) 118 (.04834)	123 (.02539) 124 (.02209)	129 (.01041) 130 (.00885)	133 (.00530) 134 (.00443)
8	8	8	96 (.51040) 97 (.48960)	112 (.20874) 113 (.19393)	120 (.10852) 121 (.09891)	127 (.05365) 128 (.04798)	133 (.02629) 134 (.02310)	139 (.01152) 140 (.00992)	144 (.00527) 145 (.00445)

Source: Robert E. Odeh, "On Jonckheere's *k*-Sample Test against Ordered Alternatives," *Technometrics*, 13 (1971). 912–918.

TABLE A.13(b)

Critical values of *J*, the Jonckheere–Terpstra test statistic (for nominal values of α shown and *k* samples all of size *n*); exact significance levels in parentheses

	k	$\alpha = 0.5$	$\alpha = 0.2$	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$	$\alpha = 0.005$
$n = 2$	4	12 (.54921) 13 (.45079)	15 (.26825) 16 (.19286)	17 (.13016) 18 (.08294)	18 (.08294) 19 (.04841)	20 (.02619) 21 (.01230)	21 (.01230) 22 (.00516)	22 (.00516) 23 (.00159)
	5	20 (.53534) 21 (.46466)	25 (.21102) 26 (.16246)	27 (.12133) 28 (.08779)	29 (.06126) 30 (.04116)	31 (.02646) 32 (.01623)	32 (.01623) 33 (.00939)	34 (.00511) 35 (.00257)
	6	30 (.52707) 31 (.47293)	36 (.22650) 37 (.18713)	39 (.12151) 40 (.09533)	42 (.05533) 43 (.04083)	44 (.02944) 45 (.02071)	46 (.01418) 47 (.00944)	48 (.00608) 49 (.00379)
$n = 3$	4	27 (.52760) 28 (.47240)	33 (.22197) 34 (.18229)	36 (.11663) 37 (.09067)	39 (.05145) 40 (.03744)	41 (.02657) 42 (.01834)	43 (.01229) 44 (.00797)	44 (.00797) 45 (.00498)
	5	45 (.51980) 46 (.48020)	53 (.22740) 54 (.19822)	58 (.10487) 59 (.08738)	61 (.05884) 62 (.04752)	64 (.02995) 65 (.02335)	68 (.01023) 69 (.00755)	70 (.00549) 71 (.00392)
	6	68 (.50000) 69 (.46981)	79 (.20145) 80 (.18058)	84 (.11087) 85 (.09686)	89 (.05331) 90 (.04524)	93 (.02262) 94 (.02201)	97 (.01193) 98 (.00958)	100 (.00604) 101 (.00473)
$n = 4$	4	48 (.51826) 49 (.48174)	57 (.21724) 58 (.19096)	62 (.10581) 63 (.08950)	66 (.05142) 67 (.04198)	69 (.02715) 70 (.02150)	72 (.01304) 73 (.00998)	75 (.00562) 76 (.00414)
	5	80 (.51305) 81 (.48695)	93 (.20589) 94 (.18756)	99 (.11129) 100 (.09910)	105 (.05211) 106 (.04523)	109 (.02876) 110 (.02450)	115 (.01016) 116 (.00839)	118 (.00561) 119 (.00455)
	6	120 (.50994) 121 (.49006)	137 (.20490) 138 (.19092)	146 (.10048) 147 (.09181)	153 (.05084) 154 (.04567)	159 (.02572) 160 (.02274)	166 (.01025) 167 (.00888)	170 (.00568) 171 (.00486)
$n = 5$	4	75 (.51321) 76 (.48679)	88 (.20295) 89 (.18455)	94 (.10832) 95 (.09621)	99 (.05735) 100 (.04983)	104 (.02708) 105 (.02296)	109 (.01125) 110 (.00928)	113 (.00502) 114 (.00404)
	5	125 (.50942) 126 (.49058)	143 (.20345) 144 (.19032)	152 (.10385) 153 (.09542)	159 (.05492) 160 (.04970)	166 (.02603) 167 (.02318)	173 (.01095) 174 (.00958)	178 (.00545) 179 (.00470)
	6	188 (.50000) 189 (.48567)	211 (.20386) 212 (.19377)	223 (.10319) 224 (.09679)	233 (.05153) 234 (.04775)	241 (.02701) 242 (.02477)	251 (.01067) 252 (.00964)	258 (.00510) 259 (.00456)
$n = 6$	4	108 (.51013) 109 (.48987)	125 (.20037) 126 (.18631)	133 (.10521) 134 (.09607)	140 (.05287) 141 (.04743)	146 (.02647) 147 (.02336)	153 (.01035) 154 (.00894)	157 (.00565) 158 (.00481)
	5	180 (.50721) 181 (.49279)	203 (.20745) 204 (.19719)	215 (.10494) 216 (.09842)	225 (.05229) 226 (.04844)	234 (.02505) 235 (.02292)	243 (.01072) 244 (.00969)	250 (.00510) 251 (.00456)
	6	270 (.50548) 271 (.49452)	301 (.20070) 302 (.19304)	316 (.10478) 317 (.09982)	329 (.05285) 330 (.04990)	341 (.02523) 342 (.02361)	353 (.01078) 354 (.00999)	362 (.00527) 363 (.00485)

Jonckheere-Terpstra Test Example

Nappi (E12) investigated the changes occurring in the haemocytes of larvae of *Drosophila algonquin* during parasitization by the hymenopterous parasite (parasitoid) *Pseudeucoila bochei.* Twenty-seven hours after parasitization of *Drosophila algonquin* larvae, differential counts (%) of plasmatocytes were made on three groups: host larvae in which reaction was successful (S), those in which the reaction was unsuccessful (U), and those in which there was no visible host reaction (N). The results are shown in Table 6.16. We wish to test the null hypothesis of no difference among the three groups against the alternative that the differential counts of plasmatocytes (%) decrease in the three groups from group N to group S.

TABLE 6.16

Differential plasmatocyte counts, percent from larvae of *Drosophila algonquin* 27 hours after parasitization by *Pseudeucoila bochei* (host age 91 hours when parasitized)

Successful host reactions (*S*)	Unsuccessful host reactions (*U*)	No visible host reactions (*N*)
54.0	79.8	98.6
67.0	82.0	99.5
47.2	88.8	95.8
71.1	79.6	93.3
62.7	85.7	98.9
44.8	81.7	91.1
67.4	88.5	94.5
80.2		

Source: A. J. Nappi, "Cellular Immune Reactions of Larvae of *Drosophila algonquin*," *Parasitology*, 70 (1975), 189–194: published by Cambridge University Press.

EXAMPLE 2

Davis (E14) investigated the performance of hard-of-hearing school children on a task involving knowledge of 50 basic concepts considered necessary for satisfactory academic achievement during kindergarten, first grade, and second grade. The raw scores made by 24 hard-of-hearing school children (kindergarten to third grade) on the Boehm Test of Basic Concepts (E15) are shown in Table 6.18 by age. Do these data provide sufficient evidence to indicate that, on the average, scores tend to increase with age? Find the P value.

Raw scores for 24 hearing-impaired children by age groups

Age 6												
Raw scores	17	20	24	34	34	38						
Age 7												
Raw scores	23	25	27	34	38	47						
Age 8												
Raw scores	22	23	26	32	34	34	36	38	38	42	48	50

Source: Julia Davis, "Performance of Young Hearing-Impaired Children on a Test of Basic Concepts," *J. Speech Hear. Res.*, 17 (1974), 342–351.

R code for Jonckheere-Terpstra Test

```
# Jonckheere-Terpstra Test

library(clinfun)

# Example 1
group <- c(rep(3,8),rep(2,7),rep(1,7))

space <-c(54.0,67.0,47.2,71.1,62.7,44.8,67.4,80.2,
79.8,82.0,88.8,79.6,85.7,81.7,88.5,
98.6,99.5,95.8,93.3,98.9,91.1,94.5)

jonckheere.test(space,group,alternative="decreasing")

# Example 2
age <- c(rep(1,6),rep(2,6),rep(3,12))

score <-c(17,20,24,34,34,38,23,25,27,34,38,47,
22,23,26,32,34,34,36,38,38,42,48,50)

jonckheere.test(score,age,alternative="increasing")
```

R output for Jonckheere-Terpstra Test

```
> # Example 1

        Jonckheere-Terpstra test

data:
JT = 2, p-value = 7.29e-09

alternative hypothesis: decreasing



> # Example 2

        Jonckheere-Terpstra test

data:
JT = 118, p-value = 0.06416

alternative hypothesis: increasing
```

8.7 A Permutation (Randomization) F-Test

- The data setup is the same as the Kruskal-Wallis Test. That is, we have k groups with n_i observations from the i^{th} group.

	Groups				
Levels	1	2	3	$\cdots$	k
	x_{11}	x_{21}	x_{31}	$\cdots$	x_{a1}
	x_{12}	x_{22}	x_{32}	$\cdots$	x_{a2}
	x_{13}	x_{23}	x_{33}	$\cdots$	x_{a3}
	.	.	.	.	.
	x_{1n_1}	x_{2n_2}	x_{3n_3}	$\cdots$	x_{kn_k}
means	$\overline{x}_1$	$\overline{x}_2$	$\overline{x}_3$	$\cdots$	$\overline{x}_k$
variances	s_1^2	s_2^2	s_3^2	$\cdots$	s_k^2

Let $N = n_1 + n_2 + \cdots + n_k$ and $\overline{x} = (1/N)\sum_{i=1}^{k}\sum_{j=1}^{n_i} x_{ij}$ is the mean of the entire data set.

- If the k groups represent samples from k populations in an observational study, then we will be performing a Permutation F-Test.
- If the k groups represent k treatments from a randomized experiment, then we will be performing a Randomization F-Test.
- Without loss of generality (WLOG), in the notes I will just refer to a Permutation F-test.
- In the traditional analysis of variance (ANOVA) F-test we are testing the null hypothesis of equality (no differences) in treatment means

$$H_0 : \ \mu_1 \ = \ \mu_2 \ = \cdots \ = \mu_k$$

against the alternative hypothesis

$$\begin{array}{ll} H_1 : \text{ not all means are equal} & \qquad \text{or, equivalently,} \\ H_1 : \ \mu_i \neq \mu_j \quad \text{for some } i \neq j & \end{array}$$

- To compare $k \geq 3$ treatment means, the test statistic is

$$F \ = \ \frac{SS_{groups}/(k-1)}{SSE/(N-k)} \ = \ \frac{MS_{groups}}{MSE}$$

k = the number of groups,

N = the total number of observations in the data set, and

n_i = the number of observations for group i.

– The *sums of squares for groups*

$$SS_{groups} \ = \ \sum_{i=1}^{k}\sum_{j=1}^{n_i}(\overline{x}_i - \overline{x})^2 \ = \ \sum_{i=1}^{k} n_i(\overline{x}_i - \overline{x})^2$$

and MS_{groups} is called the *mean square for groups.*

- The *sums of squares for error*

  $$SSE \;=\; \sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij}-\overline{x}_i)^2 \;=\; \sum_{i=1}^{k}(n_i-1)s_i^2$$

  and MSE is called the *mean squared error.*

- If the data within each group were samples from normal distributions with equal variances, then the test statistic F has an F-distribution with $k-1$ degrees of freedom for the numerator and $N-k$ degrees of freedom for the denominator.

- In this case, the experimenter compares the F-statistic to the $F(k-1, N-k)$ distribution to determine a p-value for the test.

- However, if the assumptions are violated, (for example, the data within each group were not samples from normal distributions with equal variances), then a Permutation F-test may be appropriate.

The Steps in the Permutation F-Test (Monte-Carlo Approach)

- Calculate the F-statistic from the original data. Call this F_{obs}.

- Generate a large number P_{rep} of permutations of the data with respect to the groups. Thus, a permutation is a random assignment of the N observations to the k groups while preserving the group sample sizes $n_1, n_2, \ldots, n_k$.

- For each permutation, calculate the F-statistic.

- Find the proportion of this set of P_{rep} permutation F-statistics that are $\geq F_{obs}$. This is the p-value for the Permutation F-test.

Example 1 from *Introduction to Modern Nonparametric Statistics*, J. Higgins.

The data consist of three groups with $n_1 = n_2 = n_3 = 5$ observations per group.

Group 1	Group 2	Group 3
6.08	30.45	32.04
22.29	22.71	28.03
7.51	44.52	32.74
34.36	31.47	23.84
23.68	36.81	29.64

- The test can be performed using the `perm` package in R.

- In this example, I set the number of permutations $P_{rep} = 50000$.

- The seed is set to 109285 which means that every time you run this R program you will get the same random sample of permutations. If you change the seed, it will generate a different set of permutations.

- The actual F-statistic calculated from the data is $F_{obs} = 3.781445$. In this example, $2530/50000 = .0506$ of the permutations produced F-values $\geq F_{obs}$. Thus, the p-value = .0506.

R code for Permutation F-test for Example 1

```
library(perm)

# Enter the number of permutations to take
Prep = 50000
Prep

# Enter vector of responses
y <- c(6.08,22.29,7.51,34.36,23.68,
30.45,22.71,44.52,31.47,36.81,32.04,28.03,32.74,23.84,29.64)
y

# Enter the number of observations for each treatment
nvec <- c(5,5,5)

# Create treatment vector
group <- as.factor(c(rep(1,nvec[1]),rep(2,nvec[2]),rep(3,nvec[3])))
group

permControl=permControl(nmc=Prep,seed=109285,p.conf.level=.99)
permKS(y,group,control=permControl)
```

R output for Permutation F-test for Example 2

```
> # Enter the number of permutations to take
[1] 50000
>
> # Enter vector of responses
> y
 [1]  6.08 22.29  7.51 34.36 23.68 30.45 22.71 44.52 31.47 36.81 32.04 28.03
[13] 32.74 23.84 29.64

> # Create treatment vector
> group
 [1] 1 1 1 1 1 2 2 2 2 2 3 3 3 3 3
Levels: 1 2 3


        K-Sample Exact Permutation Test Estimated by Monte Carlo

data:  y and group
p-value = 0.0506

p-value estimated from 50000 Monte Carlo replications

99 percent confidence interval on p-value:
 0.04808852 0.05315745
```

Example 2: Reconsider the data from the study of the spacing in the cells of muscles in the heart using the "tracer method". The data for this study are:

Group 1	Group 2	Group 3
.175	.169	.194
.185	.176	.195
.185	.179	.204
.187	.183	.209
.188	.185	.219
.194	.189	.219
.197	.193	.233
.209	.195	.234

- Warning! In this example, $n_1 = n_2 = n_3 = 8$. For these values, the default in the `perm` package switches to an asymptotic approximation.
- You can request monte-carlo estimation using the `exact.mc` option.
- The actual F-statistic calculated from the data is $F_{obs} = 13.9736$. In this example, $18/50000 = .00036$ of the permutations produced F-values $\geq F_{obs}$. Thus, the p-value = .00036.

R code for Permutation F-test for Example 2

```
library(perm)

# Enter the number of permutations to take
Prep = 50000

# Enter vector of responses
y <- c(.185,.187,.209,.194,.175,.197,.188,.185,
.189,.193,.176,.195,.169,.183,.185,.179,
.219,.204,.219,.234,.233,.194,.209,.195)
y

# Enter the number of observations for each treatment
nvec <- c(8,8,8)

# Create treatment vector
group <- as.factor(
c(rep("Group1",nvec[1]),rep("Group2",nvec[2]),rep("Group3",nvec[3])))
group

# For (8,8,8) the default method is asymptotic approximation
permControl=permControl(nmc=Prep,seed=10928,p.conf.level=.95)
permKS(y,group,control=permControl)

# You can request monte-carlo estimation using the exact.mc option
permKS(y,group,control=permControl,method="exact.mc")
```

R output for Permutation F-test for Example 2

```
> # Enter vector of responses
> y
 [1] 0.185 0.187 0.209 0.194 0.175 0.197 0.188 0.185 0.189 0.193 0.176 0.195
[13] 0.169 0.183 0.185 0.179 0.219 0.204 0.219 0.234 0.233 0.194 0.209 0.195

> # Create treatment vector
> group
 [1] Group1 Group1 Group1 Group1 Group1 Group1 Group1 Group1 Group2 Group2
[11] Group2 Group2 Group2 Group2 Group2 Group2 Group3 Group3 Group3 Group3
[21] Group3 Group3 Group3 Group3
Levels: Group1 Group2 Group3

        K-Sample Asymptotic Permutation Test

data:  y and group
Chi Square = 13.1322, df = 2, p-value = 0.001407


> # You can request monte-carlo estimation using the exact.mc option
> permKS(y,group,control=permControl,method="exact.mc")

        K-Sample Exact Permutation Test Estimated by Monte Carlo

data:  y and group
p-value = 0.00036

p-value estimated from 50000 Monte Carlo replications
95 percent confidence interval on p-value:
 0.0001980746 0.0005443173
```

8.7.1 Alternative (Quicker) Permutation F-Test

- To save computational time, you do not actually have to compute the F-statistic for each permutation. All we need is an "equivalent statistic". That is, find a statistic that has the same order in its permutation distribution as the order for the F-statistic.

- The *total sum of squares*, denoted SS_{total} is defined as

$$SS_{total} \;=\; \sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij}-\overline{x})^2 \;=\; (N-1)s^2$$

  where s^2 is the sample variance of the N responses.

- Note that SS_{total} does not change from one permutation to the next.

- It is known that $SS_{total} = SS_{groups} + SSE$ which implies that $SSE = SS_{total} - SS_{groups}$.

- Thus, we can rewrite the F-statistic as

$$F \;=\; \frac{SS_{groups}/(k-1)}{SSE/(N-a)} \;=\; \frac{SS_{groups}/(k-1)}{(SS_{total}-SS_{groups})/(N-a)}$$

- This makes F an increasing function of SS_{groups}. Thus, a Permutation Test based only on SS_{groups} is equivalent to the Permutation F-Test based on the F-statistic.

- **Alternative 1**: Therefore, you can just calculate SS_{groups} for each permutation and determine the proportion of permutations yielding $SS_{groups} \geq$ observed SS_{groups}.

- To simplify the computational demands even further, it can be shown that the formula for SS_{groups} can be rewritten as

$$SS_{groups} \;=\; \left(\sum_{i=1}^{k} n_i\,\overline{x}_i^2\right) - N\overline{x}^2.$$

  But, $N\overline{x}^2$ does not change from one permutation to the next. So, it can be ignored.

- **Alternative 2**: Therefore, you can just calculate

$$SSX = \sum_{i=1}^{k} n_i\,\overline{x}_i^2$$

  for each permutation and determine the proportion of permutations yielding $SSX \geq$ observed SSX.