Theorem: Comparison Test

Assume $0 \le a_n \le b_n$ for large n.

(i) If $\sum b_n$ converges, then $\sum a_n$ converges. (ii) If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Notes regarding the Comparison Test.

1. You should make a statement of the form
$$0 \le a_n \le b_n.$$
NOTE: We are comparing the terms of the series, not the series themselves, i.e. do **NOT** write $0 \le \sum a_n \le \sum b_n$.

2. The series you compare to should be one of the two types of series that we know about, p-series or geometric series. For a constant $c \neq 0$, a p-series is a series of **exactly** the form
$$\sum \frac{c}{n^p}$$
and a geometric series is a series of **exactly** the form
$$\sum cr^n.$$
NOTES: The Harmonic series is a p-series with $p = 1$, and is frequently used for comparisons. When making a claim about the convergence or divergence of a known series, make sure you are discussing the series, i.e. do **NOT** write $\frac{1}{n}$ diverges; write what you mean, $\sum \frac{1}{n}$ diverges.

3. You should draw a conclusion about the original series based on the Comparison Test.

4. You should be logically consistent. If the larger series diverges, it tells us nothing about the smaller series. Vice versa, if the smaller series converges, it tells us nothing about the larger series.

5. Notation and Language are important. Among other things, if the question posed included indices, you should use them throughout. If the question posed left them off (see above and A. below), you should as well. You should use enough words and punctuation to craft a complete argument.

Examples.

A. Use the Comparison Test to show $\sum \left|\frac{\sin n \cos 2n}{n^2+1}\right|$ converges.

Solution.

Since $0 \le \left|\frac{\sin n \cos 2n}{n^2+1}\right| \le \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$), by the Comparison Test, $\sum \left|\frac{\sin n \cos 2n}{n^2+1}\right|$ also converges.

QUESTION: Why can't we apply the comparison test to $\sum \frac{\sin n \cos 2n}{n^2+1}$?

B. Use the Comparison Test to show $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^5}$ diverges.

Solution.

Initially we note that for large n, $(\ln n)^5 < \left(\sqrt[5]{n}\right)^5 = n$, so that $0 \le \frac{1}{n} \le \frac{1}{(\ln n)^5}$. Since the Harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, by the Comparison Test, $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^5}$ also diverges.

Theorem: Limit Comparison Test
Assume $0 \leq a_n, b_n$ and $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.

(i) If $0 < L < \infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.

(ii) If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

(iii) If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Notes regarding the Limit Comparison Test.

1. You should identify two **positive** series - the series to be tested and the series you intend to compare to - and then compute the limit.

2. The series you compare to should be a known series - see the previous notes.

3. Based on the case the limit found above applies to, you should draw a conclusion about the original series based on the Limit Comparison Test.

4. You should be logically consistent. $L = 0$ and $\sum b_n$ diverges tells us nothing. $L = \infty$ and $\sum b_n$ converges tells us nothing.

5. Notation and language are important - see the previous notes.

Examples.
A. Use the Limit Comparison Test to show $\sum \frac{1}{\sqrt{n^2+1}}$ diverges.
Solution.
First, since the series $\sum \frac{1}{\sqrt{n^2+1}}$ and $\sum \frac{1}{n}$ are both positive, we can apply the Limit Comparison Test. To that end, we compute the limit

$$L = \lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} \Big/ \frac{1}{n} \right) = \lim_{n\to\infty} \left(\frac{n}{\sqrt{n^2+1}} \right) = 1.$$

Since the limit is positive and finite, i.e. $0 < L < \infty$, both series converge, or both series diverge. Since the Harmonic series diverges, $\sum \frac{1}{\sqrt{n^2+1}}$ also must diverge.

B. Use the Limit Comparison Test to show $\sum_{n=3}^{\infty} \frac{\sqrt[3]{2n^2+7}}{n^2+3}$ converges.
Solution.
We would like to apply the Limit Comparison Test to compare the given positive series $\sum_{n=3}^{\infty} \frac{\sqrt[3]{2n^2+7}}{n^2+3}$ and the positive p-series $\sum_{n=3}^{\infty} \frac{1}{n^{4/3}}$. To start with we will compute the limit

$$L = \lim_{n\to\infty} \left(\frac{\sqrt[3]{2n^2+7}}{n^2+3} \Big/ \frac{1}{n^{4/3}} \right) = \lim_{n\to\infty} \left(\frac{n^2 \sqrt[3]{2+7/n^2}}{n^2+3} \right) = \sqrt[3]{2}.$$

The series both behave the same way since the limit is positive and finite. We know $\sum_{n=3}^{\infty} \frac{1}{n^{4/3}}$ is a convergent p-series ($p = 4/3 > 1$), so $\sum_{n=3}^{\infty} \frac{\sqrt[3]{2n^2+7}}{n^2+3}$ also converges.