1 Solving equations

Throughout the calculus sequence we have limited our discussion to real valued solutions to equations. We know the equation $x^2 - 1 = 0$ has distinct real roots $x = 1$ and $x = -1$. The equation $(x - 1)^2 = 0$ has a repeated real root of $x = 1$. However, the equation $x^2 + 1 = 0$ has no real-valued roots.

It will turn out that solutions to the equation $x^2 + 1 = 0$, although not real-valued, will be very important to the development of this subject and in many other applications. To that end, we make the following definition.

Definition 1.1. Define i by $i^2 := -1$ and choose $i = \sqrt{-1}$. We will call i the imaginary unit.

With this expanded version of what we mean by a solution to an equation we can solve equations that previously had no solution. For example,

$$\begin{aligned} x^2 + 1 &= 0 \\ x^2 &= -1 \\ x &= \pm\sqrt{-1} \\ x &= \pm i. \end{aligned}$$

Similarly, we can apply the quadratic formula to solve the equation $x^2 + 3x + 3 = 0$ as follows

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{9 - 12}}{2} \\ &= \frac{-3 \pm \sqrt{-3}}{2} \\ &= \frac{-3 \pm i\sqrt{3}}{2}. \end{aligned}$$

Although we could stop at this point and make use of the algebraic mechanics of i to help us solve differential equations, there are some useful and beautiful results that we will consider in the following sections.

2 Algebra of Complex Numbers

Definition 2.1. A **complex number** z is a a number of the form $z = a + ib$ where $a, b \in \mathbb{R}$. In this case we write $z \in \mathbb{C}$. The **modulus** or **magnitude** is

$$|z| = \sqrt{a^2 + b^2},$$

the **conjugate** is

$$\bar{z} = a - ib,$$

the **real part** is

$$\mathrm{Re}\,(z) = a \in \mathbb{R},$$

and the **imaginary part** is

$$\mathrm{Im}\,(z) = b \in \mathbb{R}.$$

It is important to note that $\mathrm{Im}\,(z)$ is a real number, i.e., $\mathrm{Im}(z) = b \neq ib$. Additionally, two complex numbers z and w are equal if and only if $\mathrm{Re}\,(z) = \mathrm{Re}\,(w)$ and $\mathrm{Im}\,(z) = \mathrm{Im}\,(w)$. In particular, a complex number z is equal to zero if and only if $\mathrm{Re}\,(z) = \mathrm{Im}\,(z) = 0$. It is traditional to use x for a generic real variable and z to represent a generic complex variable.

Basic Operations.

Let $z = a + ib$ and $w = c + id$.

Addition is defined by adding the real parts and the imaginary parts,

$$z + w = (a + ib) + (c + id) = (a + c) + i(b + d).$$

Multiplication is performed by using the standard distribution process, grouping, and $i^2 = -1$, so

$$zw = (a + ib)(c + id) = ac + iad + ibc + i^2 bd = (ac - bd) + i(ad + bc).$$

With this in mind, multiplication by the complex conjugate has the curious and very useful property of always giving a real value,

$$z\bar{z} = (a + ib)(a - ib) = a^2 + b^2 \in \mathbb{R}.$$

Division by a complex number is a little tricky to think about, but the preceding result points us in the correct direction. Multiplying the numerator and denominator by the complex conjugate of the denominator will convert the denominator into a real value,

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}}.$$

Example. It is useful to to see an example of this last result,

$$\frac{1}{-1+i} = \frac{1}{-1+i} \cdot \frac{-1-i}{-1-i} = \frac{-1-i}{(-1)^2 - (i^2)} = \frac{-1-i}{2} = \frac{-1}{2} - i\frac{1}{2}. \tag{1}$$

3 Polar Representation

Since complex numbers are composed from two real numbers, it is appropriate to think of them graphically in a plane. The horizontal axis representing the real axis, the vertical representing the imaginary axis.

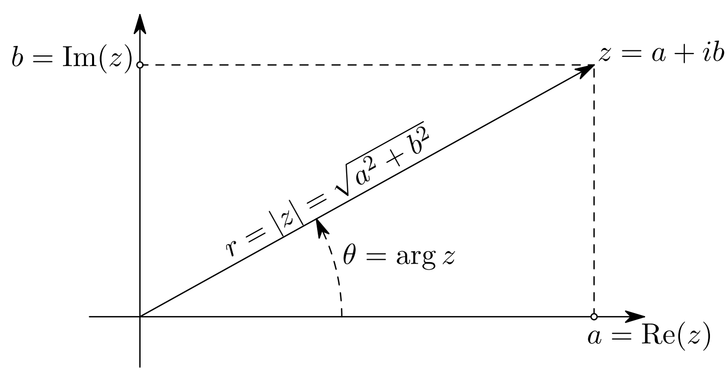

Figure 1: The complex number $z = a + ib$.

In 172 we saw that we could view points in a plane using polar coordinates. We do the same in the complex plane with

$$z = a + ib = r(\cos\theta + i\sin\theta). \tag{2}$$

As before, $r = \sqrt{a^2 + b^2} = |z|$, i.e., the distance from the pole. We define the **argument** of z to be the angle θ between the real axis and the segment connecting the pole to z, and write

$$\arg z = \theta.$$

Although the polar representation is helpful, it becomes even more useful when we introduce **Euler's Formula**. Specifically, for any real number θ we make the following definition

$$e^{i\theta} := \cos\theta + i\sin\theta.$$

A natural question arises, does it make sense to use exponential language? Two important properties of the exponential for us are,

1. $e^x e^y = e^{x+y}$, and
2. $y = e^{at}$ is the unique solution to the initial value problem $y' = ay, y(0) = 1$.

Do these properties hold if we allow the exponents to be of the form $i\theta$? It turns out they do, but are worth looking at closely.

1. The first is a application of the addition laws for sine and cosine. Specifically,

$$\begin{aligned}\sin(\theta+\phi) &= \sin\theta\cos\phi + \cos\theta\sin\phi, \text{ and}\\ \cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi.\end{aligned}$$

   With these in mind we proceed as follows,

$$\begin{aligned} e^{i\theta}e^{i\phi} &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)\\ &= \cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi + \cos\theta\sin\phi)\\ &= \cos(\theta+\phi) + i\sin(\theta+\phi)\\ &= e^{i(\theta+\phi)}.\end{aligned} \tag{3}$$

2. In this case we are interested in if $z = e^{i\theta}$ satisfies the initial value problem $z' = iz, z(0) = 1$. Before we can provide an answer, we need to understand how to differentiate a complex valued function $f(t) : \mathbb{R} \to \mathbb{C}$. For $f(t) = u(t) + iv(t)$ where $u(t)$ and $v(t)$ are real valued the derivative works as you would expect, differentiate each real valued function $u(t)$ and $v(t)$ with i a constant. Specifically,

$$f'(t) = u'(t) + iv'(t).$$

   One last algebraic note before we show $z = e^{i\theta}$ is the unique solution to the initial value problem, since $i^2 = -1$ dividing by i gives

$$i = \frac{-1}{i}.$$

   We use this below to show $z = e^{i\theta}$ satisfies the differential equation $z' = iz$ as follows,

$$\begin{aligned} z' &= \frac{d}{d\theta}e^{i\theta}\\ &= \frac{d}{d\theta}(\cos\theta + i\sin\theta)\\ &= -\sin\theta + i\cos\theta\\ &= i\left(\cos\theta - \frac{1}{i}\sin\theta\right)\\ &= i(\cos\theta + i\sin\theta)\\ &= ie^{i\theta}\\ &= iz.\end{aligned}$$

   Showing $z = e^{i\theta}$ satisfies the initial data $z(0) = 1$ is a trivial trigonometry exercise (assuming we all know the value of $\cos 0$ and $\sin 0$, which we do, right?),

$$\begin{aligned} z(0) &= e^{i0}\\ &= \cos 0 + i\sin 0\\ &= 1.\end{aligned}$$

Exercise for the interested reader. Assuming that Taylor series work with complex numbers the same as they do for real numbers (they do), find the Taylor series representation for $e^{i\theta}$ and show that it can be regrouped into the series representation for $\cos\theta + i\sin\theta$. Hint, write out the first few terms, collect all real valued terms into one series and all imaginary into the other. The Appendix in the text details this argument as well.

Using Euler's formula we can express the polar form in (2) as

$$z = re^{i\theta}.$$

This combined with the result in (3) gives us a very nice way to view multiplication in the complex plane,

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}.$$

In other words, we multiply the lengths, i.e., the moduli, and add the arguments. In particular, since $i = e^{i\pi/2}$ has modulus 1, multiplying any complex number by i just rotates it counterclockwise 90° in the complex plane.

4 Three Applications

Often by converting a problem into the complex plane we can simplify the computations. Three examples of this type of argument follow.

Example 1. Prove[1] the identity $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

$$\begin{aligned} \sin(\alpha+\beta) &= \operatorname{Im}\left(e^{i(\alpha+\beta)}\right) \\ &= \operatorname{Im}\left(e^{i\alpha}e^{i\beta}\right) \\ &= \operatorname{Im}\left((\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)\right) \\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{aligned}$$

Example 2. Prove $A\cos\theta + B\sin\theta = C\cos(\theta - \phi)$ where $C = \sqrt{A^2+B^2}$ and $\tan\phi = B/A$.

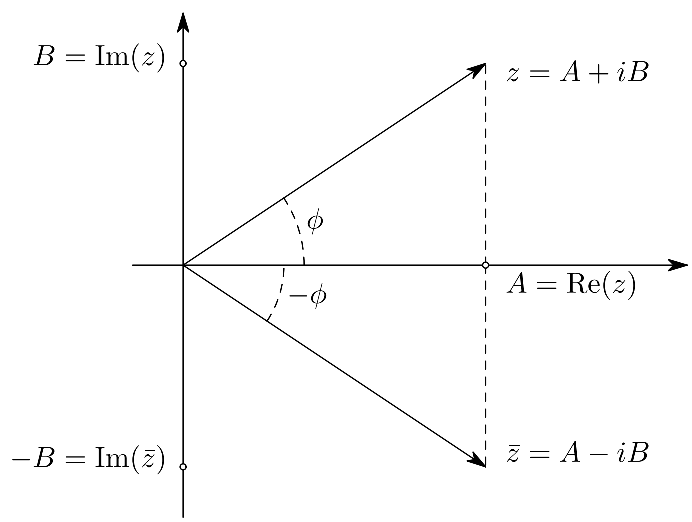

Figure 2: Example 2

Let $u = \cos\theta + i\sin\theta$ and $z = A + iB$ so that $\bar{z} = A - iB$. The product $u\bar{z}$ is easy to compute,

$$\begin{aligned} u\bar{z} &= (\cos\theta + i\sin\theta)(A - iB) \\ &= (A\cos\theta + B\sin\theta) + i(A\sin\theta - B\cos\theta). \end{aligned} \tag{4}$$

[1]The attentive reader will note that we used this identity to show $e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}$ in (3). It is circular to write this 'proof.' However, there will likely come a day when you forget the addition law for sine and having a convenient way to compute may come in handy.

Now, consider the exponential form of each; $u = e^{i\theta}$ and $\bar{z} = Ce^{i(-\phi)}$. We again consider the product,

$$\begin{aligned} u\bar{z} &= e^{i\theta} C e^{i(-\phi)} \\ &= Ce^{i(\theta-\phi)} \\ &= C(\cos(\theta - \phi) + i \sin(\theta - \phi)). \end{aligned} \tag{5}$$

Equating the real parts of (4) and (5) gives the desired result,

$$A\cos\theta + B\sin\theta = C\cos(\theta - \phi).$$

Example 3. In 172 we computed the integral $\int e^{-x} \cos x \, dx$ using integration by parts twice. With the machinery we now have available we can consider a related complex valued integral and then take the real part. Since $\cos x = \mathrm{Re}(e^{ix})$, we can integrate as follows

$$\begin{aligned} \int e^{-x} \cos x \, dx &= \mathrm{Re} \int e^{-x} e^{ix} \, dx \\ &= \mathrm{Re} \int e^{(-1+i)x} \, dx \\ &= \mathrm{Re} \left(\frac{1}{-1+i} e^{(-1+i)x} + C \right) \end{aligned}$$

It is worth pausing briefly to note the integration was about as simple as it gets. However, we need to figure out what dividing by $(-1 + i)$ means. Luckily we already did in (1). Additionally, we have to compute the real part, which can take some work. So, to finish we have,

$$\begin{aligned} \int e^{-x} \cos x \, dx &= \mathrm{Re} \left(\frac{-1-i}{2} e^{-x} e^{ix} \right) + c \\ &= \mathrm{Re} \left(\frac{e^{-x}}{2} (-1 - i)(\cos x + i \sin x) \right) + c \\ &= \mathrm{Re} \left(\frac{e^{-x}}{2} (-\cos(x) + \sin x + i(\text{stuff we don't care about})) \right) + c \\ &= \frac{e^{-x}}{2} (-\cos(x) + \sin x) + c. \end{aligned}$$

One final note, the C in the first part is complex valued, when we take the real part we have a real valued constant c in the lower part.

5 Homework

1. Solve the following equations. Your solutions may be complex.

   (a) $(x+1)^2 - 4 = 0$

   (b) $(x-1)^2 + 4 = 0$

   (c) $x^2 + 2x + 2 = 0$

   (d) $x^2 + 4x + 7 = 0$

2. Perform the indicated operation; express your solution in the form $a + ib$ with $a, b \in \mathbb{R}$.

   (a) $(2 - 5i) - (3 + 4i)$

   (b) $\dfrac{3+i}{4-2i}$

   (c) Solve $(2+i)z = 2$.

3. Evaluate.

   (a) $e^{i\pi} + 1$

   (b) $\text{Re}\left((2-3i)e^{2\pi i/3}\right)$

4. For $\alpha, \beta, v_1, v_2 \in \mathbb{R}$ evaluate.

   (a) $\text{Re}\left(e^{(\alpha+i\beta)t}\right)$

   (b) $\text{Im}\left(e^{(\alpha+i\beta)t}\right)$

   (c) $\text{Re}\left(e^{(\alpha+i\beta)t}(v_1 + iv_2)\right)$

   (d) $\text{Im}\left(e^{(\alpha+i\beta)t}(v_1 + iv_2)\right)$

5. Let $z(t) = u(t) + iv(t)$ be a solution to

$$ay'' + by' + cy = 0 \tag{6}$$

   with $a, b, c, u(t), v(t) \in \mathbb{R}$. Show $u(t)$ and $v(t)$ are real-valued solutions to (6).

6. Verify the identity $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$.

7. For s a positive constant, evaluate the integral

$$\int e^{-st}\sin(bt)\,dt.$$

   Hint: $\sin(bt) = \text{Im}(e^{ibt})$.

6 Solutions

1. (a) $x = 1$ or $x = -3$
   (b) $x = 1 \pm 2i$
   (c) $x = -1 \pm i$
   (d) $x = -2 \pm i\sqrt{3}$

2. (a) $-1 - 9i$
   (b) $\frac{1+i}{2} = \frac{1}{2} + i\frac{1}{2}$
   (c) $z = \frac{4-2i}{5} = \frac{4}{5} - i\frac{2}{5}$.

3. (a) $e^{i\pi} + 1 = 0$. This is known as **Euler's Identity** and is often argued to be one of the most beautiful equations in mathematics.
   (b) $\frac{3\sqrt{3}}{2} - 1$

4. (a) $e^{\alpha t} \cos \beta t$
   (b) $e^{\alpha t} \sin \beta t$
   (c) $e^{\alpha t} (v_1 \cos \beta t - v_2 \sin \beta t)$
   (d) $e^{\alpha t} (v_1 \sin \beta t + v_2 \cos \beta t)$

5. By assumption, $z(t) = u(t) + iv(t)$ is a solution to (6) so

$$\begin{aligned} az'' + bz' + cz &= 0, \\ a(u'' + iv'') + b(u' + iv') + c(u + iv) &= 0, \\ (au'' + bu' + cu) + i(av'' + bv' + cv) &= 0. \end{aligned}$$

   Since a complex number is zero if and only if the real and imaginary parts are zero we have

$$au'' + bu' + cu = 0 \quad \text{and} \quad av'' + bv' + cv = 0.$$

   Hence, both $u(t)$ and $v(t)$ are real-valued solutions of (6).

6. Verify the identity $\cos(\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi$.

$$\begin{aligned} \cos(\theta + \phi) &= \text{Re}(e^{i(\theta+\phi)}) \\ &= \text{Re}\left(e^{i\theta} e^{i\phi}\right) \\ &= \text{Re}\left((\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)\right) \\ &= \cos\theta\cos\phi - \sin\theta\sin\phi \end{aligned}$$

7. For s a positive constant,

$$\int e^{-st} \sin(bt)\, dt = -\frac{e^{-st}}{s^2 + b^2}(b\cos(bt) + s\sin(bt)) + c.$$