1 Cauchy Condensation Test

Theorem 1.1. *Suppose* $a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots \geq 0$. *Then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges if and only if*

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$$

converges.

Proof. Since the terms in both series are nonnegative, the sequences of partial sums are increasing. We will show that if either sequence of partial sums is bounded, then the other is as well. Applying the Bounded Monotonic Sequence Theorem (Theorem 6, p.544 in the text) we will conclude that either both sequences, and hence both series, converge or both sequences diverge. To that end, let

$$\begin{aligned} s_n &= a_1 + a_2 + a_3 + a_4 + \cdots + a_n \\ t_k &= a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots + 2^k a_{2^k}. \end{aligned}$$

Now, for any $n \in \mathbb{N}$, choose $k \in \mathbb{Z}$ such that $2^k \leq n < 2^{k+1}$.
Since $n < 2^{k+1}$,

$$\begin{aligned} s_n &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \cdots + a_n \\ &\leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\ &\leq a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + \cdots + (a_{2^k} + \cdots + a_{2^k}) \\ &= a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k} \\ &= t_k \end{aligned}$$

so that

$$s_n \leq t_k. \tag{1.1}$$

Since $2^k \leq n$,

$$\begin{aligned} t_k &= a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots + 2^k a_{2^k} \\ &= 2\left(a_1/2 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{k-1} a_{2^k}\right) \\ &\leq 2\left(a_1 + a_2 + (a_4 + a_4) + (a_8 + a_8 + a_8 + a_8) + \cdots + (a_{2^k} + \cdots + a_{2^k}) + a_{2^k+1} + \cdots + a_n\right) \\ &\leq 2\left(a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \cdots + a_n\right) \\ &= 2s_n \end{aligned}$$

so that

$$t_k \leq 2s_n. \tag{1.2}$$

Finally, by (1.1) and (1.2) we have $s_n \leq t_k \leq 2s_n$. It is then clear that the sequence $\{s_n\}$ is bounded if and only if the sequence $\{t_k\}$ is bounded. Applying the Bounded Monotonic Sequence Theorem we see that one series converges if and only if the other sequence converges. □

Theorem 1.2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if and only if* $p > 1$.

Proof. If $p \leq 0$ then the series is divergent by the Divergence Test (Theorm 3, p.553 in the text). If $p > 0$, then Theorem 1.1 applies and we consider the series

$$\sum_{k=0}^{\infty} 2^k \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} 2^{(1-p)k} = \sum_{k=0}^{\infty} \left[2^{(1-p)}\right]^k$$

Since the series above is gemotric, it converges if and only if $2^{(1-p)} < 1$, i.e. $p > 1$. □