**2.2 C** Use elimination to reach upper triangular matrices $U$. Solve by back substitution or explain why this is impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the $-x$ in the last equation.

$$\begin{array}{rll} \textbf{Success} & x+y+z=7 & x+y+z=7 \\ \textbf{then} & x+y-z=5 & x+y-z=5 \\ \textbf{Failure} & x-y+z=3 & -x-y+z=3 \end{array}$$

**Solution** For the first system, subtract equation 1 from equations 2 and 3 (the multipliers are $\ell_{21}=1$ and $\ell_{31}=1$). The 2, 2 entry becomes zero, so exchange equations:

$$\textbf{Success} \quad \begin{aligned} x+y+z &= \ \ 7 \\ \mathbf{0}y-2z &= -2 \\ -2y+0z &= -4 \end{aligned} \quad \text{exchanges into} \quad \begin{aligned} x+y+z &= \ \ 7 \\ \mathbf{-2}y+0z &= -4 \\ \mathbf{-2}z &= -2 \end{aligned}$$

Then back substitution gives $z=1$ and $y=2$ and $x=4$. The pivots are $1, -2, -2$.

For the second system, subtract equation 1 from equation 2 as before. Add equation 1 to equation 3. This leaves zero in the 2, 2 entry *and also below*:

$$\textbf{Failure} \quad \begin{aligned} x+y+z &= \ \ 7 \\ \mathbf{0}y-2z &= -2 \\ \mathbf{0}y+2z &= \ \ 10 \end{aligned} \quad \begin{array}{l} \text{There is } \textit{no pivot} \text{ in column 2 (it was } - \text{ column 1)} \\ \text{A further elimination step gives } 0z = 8 \\ \text{The three planes don't meet} \end{array}$$

Plane 1 meets plane 2 in a line. Plane 1 meets plane 3 in a parallel line. *No solution.*

If we change the "3" in the original third equation to "$-5$" then elimination would lead to $0=0$. There are infinitely many solutions! *The three planes now meet along a whole line.*

Changing 3 to $-5$ moved the third plane to meet the other two. The second equation gives $z=1$. Then the first equation leaves $x+y=6$. **No pivot in column 2 makes $y$ free** (it can have any value). Then $x=6-y$.

## Problem Set 2.2

**Problems 1–10 are about elimination on 2 by 2 systems.**

**1** What multiple $\ell_{21}$ of equation 1 should be subtracted from equation 2?

$$\begin{aligned} 2x+3y &= 1 \\ 10x+9y &= 11. \end{aligned}$$

After this elimination step, write down the upper triangular system and circle the two pivots. The numbers 1 and 11 have no influence on those pivots.

**2** Solve the triangular system of Problem 1 by back substitution, $y$ before $x$. Verify that $x$ times $(2, 10)$ plus $y$ times $(3, 9)$ equals $(1, 11)$. If the right side changes to $(4, 44)$, what is the new solution?

**3** What multiple of equation 1 should be *subtracted* from equation 2?

$$\begin{aligned} 2x - 4y &= 6 \\ -x + 5y &= 0. \end{aligned}$$

After this elimination step, solve the triangular system. If the right side changes to $(-6, 0)$, what is the new solution?

**4** What multiple $\ell$ of equation 1 should be subtracted from equation 2 to remove $c$?

$$\begin{aligned} ax + by &= f \\ cx + dy &= g. \end{aligned}$$

The first pivot is $a$ (assumed nonzero). Elimination produces what formula for the second pivot? What is $y$? The second pivot is missing when $ad = bc$: singular.

**5** Choose a right side which gives no solution and another right side which gives infinitely many solutions. What are two of those solutions?

$$\textbf{Singular system} \qquad \begin{aligned} 3x + 2y &= 10 \\ 6x + 4y &= \end{aligned}$$

**6** Choose a coefficient $b$ that makes this system singular. Then choose a right side $g$ that makes it solvable. Find two solutions in that singular case.

$$\begin{aligned} 2x + by &= 16 \\ 4x + 8y &= g. \end{aligned}$$

**7** For which numbers $a$ does elimination break down (1) permanently (2) temporarily?

$$\begin{aligned} ax + 3y &= -3 \\ 4x + 6y &= \ \ 6. \end{aligned}$$

Solve for $x$ and $y$ after fixing the temporary breakdown by a row exchange.

**8** For which three numbers $k$ does elimination break down? Which is fixed by a row exchange? In each case, is the number of solutions 0 or 1 or $\infty$?

$$\begin{aligned} kx + 3y &= \ \ 6 \\ 3x + ky &= -6. \end{aligned}$$

**9** What test on $b_1$ and $b_2$ decides whether these two equations allow a solution? How many solutions will they have? Draw the column picture for $\boldsymbol{b} = (1, 2)$ and $(1, 0)$.

$$\begin{aligned} 3x - 2y &= b_1 \\ 6x - 4y &= b_2. \end{aligned}$$

**10** In the $xy$ plane, draw the lines $x + y = 5$ and $x + 2y = 6$ and the equation $y =$ _____ that comes from elimination. The line $5x - 4y = c$ will go through the solution of these equations if $c =$ _____.

**Problems 11–20 study elimination on 3 by 3 systems (and possible failure).**

**11** (Recommended) A system of linear equations can't have exactly two solutions. *Why*?

(a) If $(x, y, z)$ and $(X, Y, Z)$ are two solutions, what is another solution?

(b) If 25 planes meet at two points, where else do they meet?

**12** Reduce this system to upper triangular form by two row operations:

$$\begin{aligned} 2x + 3y + z &= 8 \\ 4x + 7y + 5z &= 20 \\ -2y + 2z &= 0. \end{aligned}$$

Circle the pivots. Solve by back substitution for $z, y, x$.

**13** Apply elimination (circle the pivots) and back substitution to solve

$$\begin{aligned} 2x - 3y \quad\;\; &= 3 \\ 4x - 5y + \;\; z &= 7 \\ 2x - \;\; y - 3z &= 5. \end{aligned}$$

List the three row operations: Subtract _____ times row _____ from row _____.

**14** Which number $d$ forces a row exchange, and what is the triangular system (not singular) for that $d$? Which $d$ makes this system singular (no third pivot)?

$$\begin{aligned} 2x + 5y + z &= 0 \\ 4x + dy + z &= 2 \\ y - z &= 3. \end{aligned}$$

**15** Which number $b$ leads later to a row exchange? Which $b$ leads to a missing pivot? In that singular case find a nonzero solution $x, y, z$.

$$\begin{aligned} x + by \quad\;\; &= 0 \\ x - 2y - z &= 0 \\ y + z &= 0. \end{aligned}$$

**16** (a) Construct a 3 by 3 system that needs two row exchanges to reach a triangular form and a solution.

(b) Construct a 3 by 3 system that needs a row exchange to keep going, but breaks down later.

**17** If rows 1 and 2 are the same, how far can you get with elimination (allowing row exchange)? If columns 1 and 2 are the same, which pivot is missing?

$$\begin{array}{llll} \textbf{Equal} & 2x - y + z = 0 & 2x + 2y + z = 0 & \textbf{Equal} \\ \textbf{rows} & 2x - y + z = 0 & 4x + 4y + z = 0 & \textbf{columns} \\ & 4x + y + z = 2 & 6x + 6y + z = 2. & \end{array}$$

**18** Construct a 3 by 3 example that has 9 different coefficients on the left side, but rows 2 and 3 become zero in elimination. How many solutions to your system with $\boldsymbol{b} = (1, 10, 100)$ and how many with $\boldsymbol{b} = (0, 0, 0)$?

**19** Which number $q$ makes this system singular and which right side $t$ gives it infinitely many solutions? Find the solution that has $z = 1$.

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 6z &= 6 \\ 3y + qz &= t. \end{aligned}$$

**20** Three planes can fail to have an intersection point, even if no planes are parallel. The system is singular if row 3 of $A$ is a _____ of the first two rows. Find a third equation that can't be solved together with $x + y + z = 0$ and $x - 2y - z = 1$.

**21** Find the pivots and the solution for both systems ($A\boldsymbol{x} = \boldsymbol{b}$ and $K\boldsymbol{x} = \boldsymbol{b}$):

$$\begin{aligned} 2x + y \qquad\qquad &= 0 \\ x + 2y + z \qquad &= 0 \\ y + 2z + t &= 0 \\ z + 2t &= 5 \end{aligned} \qquad \begin{aligned} 2x - y \qquad\qquad &= 0 \\ -x + 2y - z \qquad &= 0 \\ -y + 2z - t &= 0 \\ -z + 2t &= 5. \end{aligned}$$

**22** If you extend Problem 21 following the $1, 2, 1$ pattern or the $-1, 2, -1$ pattern, what is the fifth pivot? What is the $n$th pivot? $K$ is my favorite matrix.

**23** If elimination leads to $x + y = 1$ and $2y = 3$, find three possible original problems.

**24** For which two numbers $a$ will elimination fail on $A = \begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$?

**25** For which three numbers $a$ will elimination fail to give three pivots?

$$A = \begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix} \text{ is singular for three values of } a.$$

**26** Look for a matrix that has row sums 4 and 8, and column sums 2 and $s$:

$$\text{Matrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{aligned} a + b &= 4 & a + c &= 2 \\ c + d &= 8 & b + d &= s \end{aligned}$$

The four equations are solvable only if $s =$ _____. Then find two different matrices that have the correct row and column sums. *Extra credit*: Write down the 4 by 4 system $A\boldsymbol{x} = \boldsymbol{b}$ with $\boldsymbol{x} = (a, b, c, d)$ and make $A$ triangular by elimination.

**27** Elimination in the usual order gives what matrix $U$ and what solution to this "lower triangular" system? We are really solving by *forward substitution*:

$$\begin{aligned} 3x \qquad\qquad &= 3 \\ 6x + 2y \qquad &= 8 \\ 9x - 2y + z &= 9. \end{aligned}$$

**28** Create a MATLAB command A(2, : ) = ... for the new row 2, to subtract 3 times row 1 from the existing row 2 if the matrix $A$ is already known.

## Challenge Problems

**29** Find experimentally the average 1st and 2nd and 3rd pivot sizes from MATLAB 's $[L, U] = \mathbf{lu}(\mathbf{rand}(3))$. The average size $\mathbf{abs}(U(1,1))$ is above $\frac{1}{2}$ because **lu** picks the largest available pivot in column 1. Here $A = \mathbf{rand}(3)$ has random entries between 0 and 1.

**30** If the last corner entry is $A(5,5) = 11$ and the last pivot of $A$ is $U(5,5) = 4$, what different entry $A(5,5)$ would have made $A$ singular?

**31** Suppose elimination takes $A$ to $U$ without row exchanges. Then row $j$ of $U$ is a combination of which rows of $A$? If $A\boldsymbol{x} = \mathbf{0}$, is $U\boldsymbol{x} = \mathbf{0}$? If $A\boldsymbol{x} = \boldsymbol{b}$, is $U\boldsymbol{x} = \boldsymbol{b}$? If $A$ starts out lower triangular, what is the upper triangular $U$?

**32** Start with 100 equations $A\boldsymbol{x} = \mathbf{0}$ for 100 unknowns $\boldsymbol{x} = (x_1, \ldots, x_{100})$. Suppose elimination reduces the 100th equation to $0 = 0$, so the system is "singular".

(a) Elimination takes linear combinations of the rows. So this singular system has the singular property: Some linear combination of the 100 **rows** is _____.

(b) Singular systems $A\boldsymbol{x} = \mathbf{0}$ have infinitely many solutions. This means that some linear combination of the 100 ***columns*** is _____.

(c) Invent a 100 by 100 singular matrix with no zero entries.

(d) For your matrix, describe in words the row picture and the column picture of $A\boldsymbol{x} = \mathbf{0}$. Not necessary to draw 100-dimensional space.