Problem Set 1.2

1 Calculate the dot products $\boldsymbol{u} \cdot \boldsymbol{v}$ and $\boldsymbol{u} \cdot \boldsymbol{w}$ and $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w})$ and $\boldsymbol{w} \cdot \boldsymbol{v}$:

$$\boldsymbol{u} = \begin{bmatrix} -.6 \\ .8 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}.$$

2 Compute the lengths $\|\boldsymbol{u}\|$ and $\|\boldsymbol{v}\|$ and $\|\boldsymbol{w}\|$ of those vectors. Check the Schwarz inequalities $|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|\boldsymbol{u}\| \, \|\boldsymbol{v}\|$ and $|\boldsymbol{v} \cdot \boldsymbol{w}| \leq \|\boldsymbol{v}\| \, \|\boldsymbol{w}\|$.

3 Find unit vectors in the directions of $\boldsymbol{v}$ and $\boldsymbol{w}$ in Problem 1, and the cosine of the angle θ. Choose vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ that make 0°, 90°, and 180° angles with $\boldsymbol{w}$.

4 For any *unit* vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, find the dot products (actual numbers) of

(a) $\boldsymbol{v}$ and $-\boldsymbol{v}$ (b) $\boldsymbol{v} + \boldsymbol{w}$ and $\boldsymbol{v} - \boldsymbol{w}$ (c) $\boldsymbol{v} - 2\boldsymbol{w}$ and $\boldsymbol{v} + 2\boldsymbol{w}$

5 Find unit vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in the directions of $\boldsymbol{v} = (3, 1)$ and $\boldsymbol{w} = (2, 1, 2)$. Find unit vectors $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ that are perpendicular to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

6 (a) Describe every vector $\boldsymbol{w} = (w_1, w_2)$ that is perpendicular to $\boldsymbol{v} = (2, -1)$.

(b) The vectors that are perpendicular to $\boldsymbol{V} = (1, 1, 1)$ lie on a _____.

(c) The vectors that are perpendicular to $(1, 1, 1)$ and $(1, 2, 3)$ lie on a _____.

7 Find the angle θ (from its cosine) between these pairs of vectors:

(a) $\boldsymbol{v} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (b) $\boldsymbol{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

(c) $\boldsymbol{v} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$ (d) $\boldsymbol{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$.

8 True or false (give a reason if true or a counterexample if false):

(a) If $\boldsymbol{u}$ is perpendicular (in three dimensions) to $\boldsymbol{v}$ and $\boldsymbol{w}$, those vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel.

(b) If $\boldsymbol{u}$ is perpendicular to $\boldsymbol{v}$ and $\boldsymbol{w}$, then $\boldsymbol{u}$ is perpendicular to $\boldsymbol{v} + 2\boldsymbol{w}$.

(c) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular unit vectors then $\|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{2}$.

9 The slopes of the arrows from $(0, 0)$ to (v_1, v_2) and (w_1, w_2) are v_2/v_1 and w_2/w_1. **Suppose the product $v_2 w_2 / v_1 w_1$ of those slopes is** -1. Show that $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ and the vectors are perpendicular.

10 Draw arrows from $(0, 0)$ to the points $\boldsymbol{v} = (1, 2)$ and $\boldsymbol{w} = (-2, 1)$. Multiply their slopes. That answer is a signal that $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ and the arrows are _____.

11 If $\boldsymbol{v} \cdot \boldsymbol{w}$ is negative, what does this say about the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$? Draw a 3-dimensional vector $\boldsymbol{v}$ (an arrow), and show where to find all $\boldsymbol{w}$'s with $\boldsymbol{v} \cdot \boldsymbol{w} < 0$.

12 With $v = (1, 1)$ and $w = (1, 5)$ choose a number c so that $w - cv$ is perpendicular to v. Then find the formula that gives this number c for any nonzero v and w. (*Note*: cv is the "projection" of w onto v.)

13 Find two vectors v and w that are perpendicular to $(1, 0, 1)$ and to each other.

14 Find nonzero vectors u, v, w that are perpendicular to $(1, 1, 1, 1)$ and to each other.

15 The geometric mean of $x = 2$ and $y = 8$ is $\sqrt{xy} = 4$. The arithmetic mean is larger: $\frac{1}{2}(x + y) =$ ____. This would come in Example 6 from the Schwarz inequality for $v = (\sqrt{2}, \sqrt{8})$ and $w = (\sqrt{8}, \sqrt{2})$. Find $\cos\theta$ for this v and w.

16 **How long is the vector $v = (1, 1, \ldots, 1)$ in 9 dimensions?** Find a unit vector u in the same direction as v and a unit vector w that is perpendicular to v.

17 What are the cosines of the angles α, β, θ between the vector $(1, 0, -1)$ and the unit vectors i, j, k along the axes? Check the formula $\cos^2\alpha + \cos^2\beta + \cos^2\theta = 1$.

Problems 18–31 lead to the main facts about lengths and angles in triangles.

18 The parallelogram with sides $v = (4, 2)$ and $w = (-1, 2)$ is a rectangle. Check the Pythagoras formula $a^2 + b^2 = c^2$ which is for ***right triangles only***:

$$(\text{length of } v)^2 + (\text{length of } w)^2 = (\text{length of } v + w)^2.$$

19 (Rules for dot products) These equations are simple but useful:

(1) $v \cdot w = w \cdot v$ (2) $u \cdot (v + w) = u \cdot v + u \cdot w$ (3) $(cv) \cdot w = c(v \cdot w)$

Use (2) with $u = v + w$ to prove $\|v + w\|^2 = v \cdot v + 2v \cdot w + w \cdot w$.

20 The "Law of Cosines" comes from $(v - w) \cdot (v - w) = v \cdot v - 2v \cdot w + w \cdot w$:

$$\textbf{Cosine Law} \qquad \|v - w\|^2 = \|v\|^2 - 2\|v\|\,\|w\|\cos\theta + \|w\|^2.$$

If $\theta < 90°$ show that $\|v\|^2 + \|w\|^2$ is larger than $\|v - w\|^2$ (the third side).

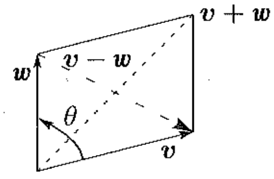
21 The ***triangle inequality*** says: (length of $v + w$) $\leq$ (length of v) + (length of w). Problem 19 found $\|v + w\|^2 = \|v\|^2 + 2v \cdot w + \|w\|^2$. Use the Schwarz inequality $v \cdot w \leq \|v\|\,\|w\|$ to show that $\|$**side 3**$\|$ can not exceed $\|$**side 1**$\| + \|$**side 2**$\|$:

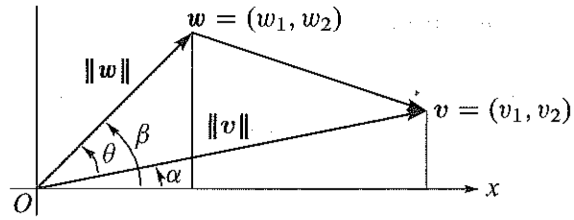
$$\textbf{Triangle inequality} \qquad \|v + w\|^2 \leq (\|v\| + \|w\|)^2 \quad \textbf{or} \quad \|v + w\| \leq \|v\| + \|w\|.$$

22 The Schwarz inequality $|v \cdot w| \leq \|v\|\,\|w\|$ by algebra instead of trigonometry:

(a) Multiply out both sides of $(v_1w_1 + v_2w_2)^2 \leq (v_1^2 + v_2^2)(w_1^2 + w_2^2)$.

(b) Show that the difference between those two sides equals $(v_1w_2 - v_2w_1)^2$. This cannot be negative since it is a square—so the inequality is true.





23 The figure shows that $\cos\alpha = v_1/\|v\|$ and $\sin\alpha = v_2/\|v\|$. Similarly $\cos\beta$ is _____ and $\sin\beta$ is _____. The angle θ is $\beta - \alpha$. Substitute into the trigonometry formula $\cos\beta\cos\alpha + \sin\beta\sin\alpha$ for $\cos(\beta - \alpha)$ to find $\cos\theta = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$.

24 One-line proof of the Schwarz inequality $|\boldsymbol{u}\cdot\boldsymbol{U}| \le 1$ for unit vectors:

$$|\boldsymbol{u}\cdot\boldsymbol{U}| \le |u_1|\,|U_1| + |u_2|\,|U_2| \le \frac{u_1^2 + U_1^2}{2} + \frac{u_2^2 + U_2^2}{2} = \frac{1+1}{2} = 1.$$

Put $(u_1, u_2) = (.6, .8)$ and $(U_1, U_2) = (.8, .6)$ in that whole line and find $\cos\theta$.

25 Why is $|\cos\theta|$ never greater than 1 in the first place?

26 If $\boldsymbol{v} = (1, 2)$ draw all vectors $\boldsymbol{w} = (x, y)$ in the xy plane with $\boldsymbol{v}\cdot\boldsymbol{w} = x + 2y = 5$. Which is the shortest $\boldsymbol{w}$?

27 (*Recommended*) If $||\boldsymbol{v}|| = 5$ and $||\boldsymbol{w}|| = 3$, what are the smallest and largest values of $||\boldsymbol{v} - \boldsymbol{w}||$? What are the smallest and largest values of $\boldsymbol{v}\cdot\boldsymbol{w}$?

Challenge Problems

28 Can three vectors in the xy plane have $\boldsymbol{u}\cdot\boldsymbol{v} < 0$ and $\boldsymbol{v}\cdot\boldsymbol{w} < 0$ and $\boldsymbol{u}\cdot\boldsymbol{w} < 0$? I don't know how many vectors in xyz space can have all negative dot products. (Four of those vectors in the plane would certainly be impossible ...).

29 Pick any numbers that add to $x + y + z = 0$. Find the angle between your vector $\boldsymbol{v} = (x, y, z)$ and the vector $\boldsymbol{w} = (z, x, y)$. Challenge question: Explain why $\boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|$ is always $-\frac{1}{2}$.

30 How could you prove $\sqrt[3]{xyz} \le \frac{1}{3}(x+y+z)$ (geometric mean $\le$ arithmetic mean)?

31 Find four perpendicular unit vectors with all components equal to $\frac{1}{2}$ or $-\frac{1}{2}$.

32 Using $\boldsymbol{v} = \mathsf{randn}(3, 1)$ in MATLAB, create a random unit vector $\boldsymbol{u} = \boldsymbol{v}/\|\boldsymbol{v}\|$. Using $V = \mathsf{randn}(3, 30)$ create 30 more random unit vectors $\boldsymbol{U}_j$. What is the average size of the dot products $|\boldsymbol{u}\cdot\boldsymbol{U}_j|$? In calculus, the average $\int_0^{\pi} |\cos\theta| d\theta/\pi = 2/\pi$.