§2.2 Solutions

1. $\ell_{21} = 5$. The Δ-system is

$$\begin{aligned} 2x + 3y &= 1 \\ -6y &= 6. \end{aligned}$$

2. The solution is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. for RHS $= \begin{pmatrix} 1 \\ 11 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \end{pmatrix}$ for RHS $= \begin{pmatrix} 4 \\ 44 \end{pmatrix}$

5. $$\begin{aligned} 3x + 2y &= 10 \\ 6x + 4y &= 20 \end{aligned}$$ has infinitely many solutions. e.g. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1/2 \end{pmatrix}$ are solutions.

$$\begin{aligned} 3x + 2y &= 10 \\ 6x + 4y &= a \end{aligned}$$ for $a \neq 20$ has 0 solutions!

12. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

13. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$.

14. (A) The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 4 & d & 1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

Applying GE with $\underline{R_2 \leftarrow -2R_1 + R_2}$ gives

$$\left[\begin{array}{cccc} 2 & 5 & 1 & 0 \\ 0 & d-10 & -1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

(A) this shows that if $\boxed{d=10}$, the second pivot is 0, which would force a row exchange.

(B) Switching rows 2 & 3 for $d=10$ gives the triangular system

$$\left[\begin{array}{cccc} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 2 \end{array}\right]$$

which shows that, the single solution to this system is

$$\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3/2 \\ 1 \\ -2 \end{pmatrix}.$$

(C) If $\boxed{d = 11}$, then the augmented matrix in part (A) is:

$$\left[\begin{array}{cccc} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

Apply GE with $R_3 \leftarrow -R_2 + R_3$ gives

$$\left[\begin{array}{cccc} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Back substitution implies that

$$0 = 1,$$

ABSURD. Thus there are no solutions, and so the coefficient matrix

$$A = \begin{bmatrix} 2 & 5 & 1 \\ 4 & 11 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$ is SINGULAR AND NOT INVERTIBLE.

15. (A) the augmented matrix is:

$$\begin{bmatrix} 1 & b & 0 & 0 \\ 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$R_2 \leftarrow -R_1 + R_2$ gives:

$$\begin{bmatrix} 1 & b & 0 & 0 \\ 0 & -2-b & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

So if $\boxed{b = -2}$, the 2nd pivot is 0 & exchanging rows 2 & 3 is required:

$$\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

which has solution $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

(B) If $b = -1$ then the augmented matrix is:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Applying $R_3 \leftarrow -R_2 + R_3$ gives

$$\left[\begin{array}{cccc} 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$\uparrow$ zero or <u>missing pivot.</u>

Since the last row indicates that $0 = 0$, there are an infinite number of solutions. the solutions are of the form:

(c) $\boxed{\text{Let } z \text{ be free.}}$

the second row of ref (A) shows that

$$-y - z = 0$$

$$\boxed{y = -z}$$

the 1st row of ref (A) shows:

$$x - y = 0$$

$$x - (-z) = 0$$

$$\boxed{x = -z}$$

So the solution is

$$\boxed{\underline{x} = \begin{pmatrix} -z \\ -z \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.}$$

For any $z \in \mathbb{R}$.

21. the only solution to $A\underline{x} = \underline{b}$ is

$$\underline{x} = \begin{pmatrix} -1 \\ 2 \\ -3 \\ 4 \end{pmatrix}.$$

the only solution to $K\underline{x} = \underline{b}$ is:

$$\underline{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

25: There are 3 values of a that fail to give 3 pivots.

$a = 0$ gives a 0 where the 1st pivot should be.

$a = 2$ will give a 0 where the 2nd pivot should when applying $R_2 \leftarrow -R_1 + R_2$.

To see what the 3rd value of a is to get a missing 3rd pivot, apply $R_2 \leftarrow -R_1 + R_2$:
$R_3 \leftarrow -R_1 + R_3$

$$\begin{bmatrix} a & 2 & 3 \\ 0 & a-2 & 1 \\ 0 & a-2 & a-3 \end{bmatrix}$$

Now apply $R_3 \leftarrow -R_2 + R_3$

$$\begin{bmatrix} a & 2 & 3 \\ 0 & a-2 & 1 \\ 0 & 0 & a-4 \end{bmatrix}.$$

thus, $a = 4$ yields a missing 3rd pivot.

27. the augmented matrix is $\begin{bmatrix} 3 & 0 & 0 & 3 \\ 6 & 2 & 0 & 8 \\ 9 & -2 & 1 & 9 \end{bmatrix}$.

Applying $\begin{matrix} R_2 \leftarrow -2R_1 + R_2 \\ R_3 \leftarrow -3R_1 + R_3 \end{matrix}$ gives $\begin{bmatrix} 3 & 0 & 0 & 3 \\ 0 & 2 & 0 & 2 \\ 0 & -2 & 1 & 0 \end{bmatrix}$

Applying $R_3 \leftarrow +R_1 + R_2$ gives $\begin{bmatrix} 3 & 0 & 0 & 3 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

thus the solution is $\underline{x} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.