Solutions to Selected Exercises

Problem Set 1.1, page 8

1 The combinations give (a) a line in $\mathbf{R}^3$ (b) a plane in $\mathbf{R}^3$ (c) all of $\mathbf{R}^3$.

4 $3\boldsymbol{v} + \boldsymbol{w} = (7, 5)$ and $c\boldsymbol{v} + d\boldsymbol{w} = (2c + d, c + 2d)$.

6 The components of every $c\boldsymbol{v} + d\boldsymbol{w}$ add to zero. $c = 3$ and $d = 9$ give $(3, 3, -6)$.

9 The fourth corner can be $(4, 4)$ or $(4, 0)$ or $(-2, 2)$.

11 Four more corners $(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)$. The center point is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Centers of faces are $(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, 1)$ and $(0, \frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, 1, \frac{1}{2})$.

12 A four-dimensional cube has $2^4 = 16$ corners and $2 \cdot 4 = 8$ three-dimensional faces and 24 two-dimensional faces and 32 edges in Worked Example **2.4 A**.

13 Sum = zero vector. Sum = $-2{:}00$ vector = $8{:}00$ vector. $2{:}00$ is $30°$ from horizontal $= (\cos\frac{\pi}{6}, \sin\frac{\pi}{6}) = (\sqrt{3}/2, 1/2)$.

16 All combinations with $c + d = 1$ are on the line that passes through $\boldsymbol{v}$ and $\boldsymbol{w}$. The point $V = -\boldsymbol{v} + 2\boldsymbol{w}$ is on that line but it is beyond $\boldsymbol{w}$.

17 All vectors $c\boldsymbol{v} + c\boldsymbol{w}$ are on the line passing through $(0, 0)$ and $\boldsymbol{u} = \frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$. That line continues out beyond $\boldsymbol{v} + \boldsymbol{w}$ and back beyond $(0, 0)$. With $c \geq 0$, half of this line is removed, leaving a *ray* that starts at $(0, 0)$.

20 (a) $\frac{1}{3}\boldsymbol{u} + \frac{1}{3}\boldsymbol{v} + \frac{1}{3}\boldsymbol{w}$ is the center of the triangle between $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$; $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{w}$ lies between $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) To fill the triangle keep $c \geq 0$, $d \geq 0$, $e \geq 0$, and $c + d + e = \mathbf{1}$.

22 The vector $\frac{1}{2}(\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w})$ is *outside* the pyramid because $c + d + e = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 1$.

25 (a) For a line, choose $\boldsymbol{u} = \boldsymbol{v} = \boldsymbol{w}$ = any nonzero vector (b) For a plane, choose $\boldsymbol{u}$ and $\boldsymbol{v}$ in different directions. A combination like $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$ is in the same plane.

Problem Set 1.2, page 19

3 Unit vectors $\boldsymbol{v}/\|\boldsymbol{v}\| = (\frac{3}{5}, \frac{4}{5}) = (.6, .8)$ and $\boldsymbol{w}/\|\boldsymbol{w}\| = (\frac{4}{5}, \frac{3}{5}) = (.8, .6)$. The cosine of θ is $\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = \frac{24}{25}$. The vectors $\boldsymbol{w}, \boldsymbol{u}, -\boldsymbol{w}$ make $0°, 90°, 180°$ angles with $\boldsymbol{w}$.

4 (a) $\boldsymbol{v} \cdot (-\boldsymbol{v}) = -1$ (b) $(\boldsymbol{v} + \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{w} = 1 + (\quad) - (\quad) - 1 = 0$ so $\theta = 90°$ (notice $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$) (c) $(\boldsymbol{v} - 2\boldsymbol{w}) \cdot (\boldsymbol{v} + 2\boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - 4\boldsymbol{w} \cdot \boldsymbol{w} = 1 - 4 = -3$.

6 All vectors $\boldsymbol{w} = (c, 2c)$ are perpendicular to $\boldsymbol{v}$. All vectors (x, y, z) with $x+y+z = 0$ lie on a *plane*. All vectors perpendicular to $(1, 1, 1)$ and $(1, 2, 3)$ lie on a *line*.

9 If $v_2 w_2 / v_1 w_1 = -1$ then $v_2 w_2 = -v_1 w_1$ or $v_1 w_1 + v_2 w_2 = \boldsymbol{v} \cdot \boldsymbol{w} = 0$: perpendicular!

11 $\boldsymbol{v} \cdot \boldsymbol{w} < 0$ means angle $> 90°$; these $\boldsymbol{w}$'s fill half of 3-dimensional space.

12 $(1, 1)$ perpendicular to $(1, 5) - c(1, 1)$ if $6 - 2c = 0$ or $c = 3$; $\boldsymbol{v} \cdot (\boldsymbol{w} - c\boldsymbol{v}) = 0$ if $c = \boldsymbol{v} \cdot \boldsymbol{w} / \boldsymbol{v} \cdot \boldsymbol{v}$. Subtracting $c\boldsymbol{v}$ is the key to perpendicular vectors.

15 $\frac{1}{2}(x + y) = (2 + 8)/2 = 5$; $\cos\theta = 2\sqrt{16}/\sqrt{10}\sqrt{10} = 8/10$.

17 $\cos\alpha = 1/\sqrt{2}$, $\cos\beta = 0$, $\cos\gamma = -1/\sqrt{2}$. For any vector $\boldsymbol{v}$, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = (v_1^2 + v_2^2 + v_3^2)/\|\boldsymbol{v}\|^2 = 1$.

21 $2\boldsymbol{v} \cdot \boldsymbol{w} \le 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|$ leads to $\|\boldsymbol{v} + \boldsymbol{w}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v} + 2\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{w} \le \|\boldsymbol{v}\|^2 + 2\|\boldsymbol{v}\|\|\boldsymbol{w}\| + \|\boldsymbol{w}\|^2$. This is $(\|\boldsymbol{v}\| + \|\boldsymbol{w}\|)^2$. Taking square roots gives $\|\boldsymbol{v} + \boldsymbol{w}\| \le \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$.

22 $v_1^2 w_1^2 + 2v_1 w_1 v_2 w_2 + v_2^2 w_2^2 \le v_1^2 w_1^2 + v_1^2 w_2^2 + v_2^2 w_1^2 + v_2^2 w_2^2$ is true (cancel 4 terms) because the difference is $v_1^2 w_2^2 + v_2^2 w_1^2 - 2v_1 w_1 v_2 w_2$ which is $(v_1 w_2 - v_2 w_1)^2 \ge 0$.

23 $\cos\beta = w_1/\|\boldsymbol{w}\|$ and $\sin\beta = w_2/\|\boldsymbol{w}\|$. Then $\cos(\beta - a) = \cos\beta\cos\alpha + \sin\beta\sin\alpha = v_1 w_1/\|\boldsymbol{v}\|\|\boldsymbol{w}\| + v_2 w_2/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = \boldsymbol{v} \cdot \boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\|$. This is $\cos\theta$ because $\beta - \alpha = \theta$.

24 Example 6 gives $|u_1||U_1| \le \frac{1}{2}(u_1^2 + U_1^2)$ and $|u_2||U_2| \le \frac{1}{2}(u_2^2 + U_2^2)$. The whole line becomes $.96 \le (.6)(.8) + (.8)(.6) \le \frac{1}{2}(.6^2 + .8^2) + \frac{1}{2}(.8^2 + .6^2) = 1$. True: $.96 < 1$.

28 Three vectors in the plane could make angles $> 90°$ with each other: $(1, 0), (-1, 4), (-1, -4)$. Four vectors could not do this (360° total angle). How many can do this in $\mathbf{R}^3$ or $\mathbf{R}^n$?

29 Try $\boldsymbol{v} = (1, 2, -3)$ and $\boldsymbol{w} = (-3, 1, 2)$ with $\cos\theta = \frac{-7}{14}$ and $\theta = 120°$. Write $\boldsymbol{v} \cdot \boldsymbol{w} = xz + yz + xy$ as $\frac{1}{2}(x + y + z)^2 - \frac{1}{2}(x^2 + y^2 + z^2)$. If $x + y + z = 0$ this is $-\frac{1}{2}(x^2 + y^2 + z^2) = -\frac{1}{2}\|\boldsymbol{v}\|\|\boldsymbol{w}\|$. Then $\boldsymbol{v} \cdot \boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = -\frac{1}{2}$.

Problem Set 1.3, page 29

1 $2\boldsymbol{s}_1 + 3\boldsymbol{s}_2 + 4\boldsymbol{s}_3 = (2, 5, 9)$. The same vector $\boldsymbol{b}$ comes from S times $\boldsymbol{x} = (2, 3, 4)$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} (\text{row } 1) \cdot \boldsymbol{x} \\ (\text{row } 2) \cdot \boldsymbol{x} \\ (\text{row } 2) \cdot \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}.$$

2 The solutions are $y_1 = 1$, $y_2 = 0$, $y_3 = 0$ (right side = column 1) and $y_1 = 1$, $y_2 = 3$, $y_3 = 5$. That second example illustrates that the first n odd numbers add to n^2.

4 The combination $0\boldsymbol{w}_1 + 0\boldsymbol{w}_2 + 0\boldsymbol{w}_3$ always gives the zero vector, but this problem looks for other *zero* combinations (then the vectors are *dependent*, they lie in a plane): $\boldsymbol{w}_2 = (\boldsymbol{w}_1 + \boldsymbol{w}_3)/2$ so one combination that gives zero is $\frac{1}{2}\boldsymbol{w}_1 - \boldsymbol{w}_2 + \frac{1}{2}\boldsymbol{w}_3$.

5 The rows of the 3 by 3 matrix in Problem 4 must also be *dependent*: $\boldsymbol{r}_2 = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_3)$. The column and row combinations that produce $\mathbf{0}$ are the same: this is unusual.

7 All three rows are perpendicular to the solution $\boldsymbol{x}$ (the three equations $\boldsymbol{r}_1 \cdot \boldsymbol{x} = 0$ and $\boldsymbol{r}_2 \cdot \boldsymbol{x} = 0$ and $\boldsymbol{r}_3 \cdot \boldsymbol{x} = 0$ tell us this). Then the whole plane of the rows is perpendicular to $\boldsymbol{x}$ (the plane is also perpendicular to all multiples $c\boldsymbol{x}$).

9 The cyclic difference matrix C has a line of solutions (in 4 dimensions) to $C\boldsymbol{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ when } \boldsymbol{x} = \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix} = \text{any constant vector.}$$

11 The forward differences of the squares are $(t+1)^2 - t^2 = t^2 + 2t + 1 - t^2 = 2t + 1$. Differences of the nth power are $(t+1)^n - t^n = t^n - t^n + nt^{n-1} + \cdots$. The leading term is the derivative nt^{n-1}. The binomial theorem gives all the terms of $(t+1)^n$.

12 Centered difference matrices of *even* size seem to be invertible. Look at eqns. 1 and 4:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \begin{matrix} \text{First} \\ \text{solve} \\ x_2 = b_1 \\ -x_3 = b_4 \end{matrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -b_2 - b_4 \\ b_1 \\ -b_4 \\ b_1 + b_3 \end{bmatrix}$$

13 *Odd size*: The five centered difference equations lead to $b_1 + b_3 + b_5 = 0$.

$$\begin{aligned} x_2 \quad &= b_1 \\ x_3 - x_1 &= b_2 \\ x_4 - x_2 &= b_3 \\ x_5 - x_3 &= b_4 \\ -x_4 &= b_5 \end{aligned}$$

Add equations 1, 3, 5
The left side of the sum is zero
The right side is $b_1 + b_3 + b_5$
There cannot be a solution unless $b_1 + b_3 + b_5 = 0$.

14 An example is $(a, b) = (3, 6)$ and $(c, d) = (1, 2)$. The ratios a/c and b/d are equal. Then $ad = bc$. Then (when you divide by bd) the ratios a/b and c/d are equal!

Problem Set 2.1, page 40

1 The columns are $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{j} = (0, 1, 0)$ and $\boldsymbol{k} = (0, 0, 1)$ and $\boldsymbol{b} = (2, 3, 4) = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$.

2 The planes are the same: $2x = 4$ is $x = 2$, $3y = 9$ is $y = 3$, and $4z = 16$ is $z = 4$. The solution is the same point $\boldsymbol{X} = \boldsymbol{x}$. The columns are changed; but same combination.

4 If $z = 2$ then $x + y = 0$ and $x - y = z$ give the point $(1, -1, 2)$. If $z = 0$ then $x + y = 6$ and $x - y = 4$ produce $(5, 1, 0)$. Halfway between those is $(3, 0, 1)$.

6 Equation 1 + equation 2 − equation 3 is now $0 = -4$. Line misses plane; *no solution*.

8 Four planes in 4-dimensional space normally meet at a *point*. The solution to $A\boldsymbol{x} = (3, 3, 3, 2)$ is $\boldsymbol{x} = (0, 0, 1, 2)$ if A has columns $(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1)$. The equations are $x + y + z + t = 3$, $y + z + t = 3$, $z + t = 3$, $t = 2$.

11 $A\boldsymbol{x}$ equals $(14, 22)$ and $(0, 0)$ and $(9, 7)$.

14 $2x + 3y + z + 5t = 8$ is $A\boldsymbol{x} = \boldsymbol{b}$ with the 1 by 4 matrix $A = [2 \;\; 3 \;\; 1 \;\; 5]$. The solutions $\boldsymbol{x}$ fill a 3D "plane" in 4 dimensions. It could be called a *hyperplane*.

16 90° rotation from $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, 180° rotation from $R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$.

18 $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ subtract the first component from the second.

22 The dot product $Ax = [1 \;\; 4 \;\; 5]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (1 \text{ by } 3)(3 \text{ by } 1)$ is zero for points (x, y, z) on a plane in three dimensions. The columns of A are one-dimensional vectors.

23 $A = [1 \;\; 2 \;\; ; \;\; 3 \;\; 4]$ and $\boldsymbol{x} = [5 \;\; -2]'$ and $\boldsymbol{b} = [1 \;\; 7]'$. $\boldsymbol{r} = \boldsymbol{b} - A * \boldsymbol{x}$ prints as zero.

25 $\textbf{ones}(4, 4) * \textbf{ones}(4, 1) = [4 \;\; 4 \;\; 4 \;\; 4]'$; $B * \boldsymbol{w} = [10 \;\; 10 \;\; 10 \;\; 10]'$.

28 The row picture shows four *lines* in the 2D plane. The column picture is in *four-dimensional* space. No solution unless the right side is a combination of *the two columns*.

29 $\boldsymbol{u}_7, \boldsymbol{v}_7, \boldsymbol{w}_7$ are all close to $(.6, .4)$. Their components still add to 1.

30 $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix} =$ *steady state* $\boldsymbol{s}$. No change when multiplied by $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$.

31 $M = \begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 5+u & 5-u+v & 5-v \\ 5-u-v & 5 & 5+u+v \\ 5+v & 5+u-v & 5-u \end{bmatrix}$; $M_3(1,1,1) = (15, 15, 15)$; $M_4(1,1,1,1) = (34, 34, 34, 34)$ because $1 + 2 + \cdots + 16 = 136$ which is $4(34)$.

32 A is singular when its third column $\boldsymbol{w}$ is a combination $c\boldsymbol{u} + d\boldsymbol{v}$ of the first columns. A typical column picture has $\boldsymbol{b}$ outside the plane of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$. A typical row picture has the intersection line of two planes parallel to the third plane. *Then no solution.*

33 $\boldsymbol{w} = (5, 7)$ is $5\boldsymbol{u} + 7\boldsymbol{v}$. Then $A\boldsymbol{w}$ equals 5 times $A\boldsymbol{u}$ plus 7 times $A\boldsymbol{v}$.

34 $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ has the solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 8 \\ 6 \end{bmatrix}$.

35 $\boldsymbol{x} = (1, \ldots, 1)$ gives $S\boldsymbol{x} =$ sum of each row $= 1 + \cdots + 9 = 45$ for Sudoku matrices. 6 row orders $(1,2,3)$, $(1,3,2)$, $(2,1,3)$, $(2,3,1)$, $(3,1,2)$, $(3,2,1)$ are in Section 2.7. The same 6 permutations of *blocks* of rows produce Sudoku matrices, so $6^4 = 1296$ orders of the 9 rows all stay Sudoku. (And also 1296 permutations of the 9 columns.)

Problem Set 2.2, page 51

3 Subtract $-\frac{1}{2}$ (or add $\frac{1}{2}$) times equation 1. The new second equation is $3y = 3$. Then $y = 1$ and $x = 5$. If the right side changes sign, so does the solution: $(x, y) = (-5, -1)$.

4 Subtract $\ell = \frac{c}{a}$ times equation 1. The new second pivot multiplying y is $d - (cb/a)$ or $(ad - bc)/a$. Then $y = (ag - cf)/(ad - bc)$.

6 Singular system if $b = 4$, because $4x + 8y$ is 2 times $2x + 4y$. Then $g = 32$ makes the lines become the *same*: infinitely many solutions like $(8, 0)$ and $(0, 4)$.

8 If $k = 3$ elimination must fail: no solution. If $k = -3$, elimination gives $0 = 0$ in equation 2: infinitely many solutions. If $k = 0$ a row exchange is needed: one solution.

14 Subtract 2 times row 1 from row 2 to reach $(d-10)y - z = 2$. Equation (3) is $y - z = 3$. If $d = 10$ exchange rows 2 and 3. If $d = 11$ the system becomes singular.

15 The second pivot position will contain $-2-b$. If $b=-2$ we exchange with row 3. If $b=-1$ (singular case) the second equation is $-y-z=0$. A solution is $(1,1,-1)$.

17 If row 1 = row 2, then row 2 is zero after the first step; exchange the zero row with row 3 and there is no *third* pivot. If column 2 = column 1, then column 2 has no pivot.

19 Row 2 becomes $3y-4z=5$, then row 3 becomes $(q+4)z=t-5$. If $q=-4$ the system is singular — no third pivot. Then if $t=5$ the third equation is $0=0$. Choosing $z=1$ the equation $3y-4z=5$ gives $y=3$ and equation 1 gives $x=-9$.

20 Singular if row 3 is a combination of rows 1 and 2. From the end view, the three planes form a triangle. This happens if rows 1+2 = row 3 on the left side but not the right side: $x+y+z=0$, $x-2y-z=1$, $2x-y=4$. No parallel planes but still no solution.

25 $a=2$ (equal columns), $a=4$ (equal rows), $a=0$ (zero column).

28 $A(2,:)=A(2,:)-3*A(1,:)$ will subtract 3 times row 1 from row 2.

29 Pivots 2 and 3 can be arbitrarily large. I believe their averages are infinite! *With row exchanges* in MATLAB's lu code, the averages are much more stable (and should be predictable, also for randn with normal instead of uniform probability distribution).

30 If $A(5,5)$ is 7 not 11, then the last pivot will be 0 not 4.

31 Row j of U is a combination of rows $1,\ldots,j$ of A. If $A\boldsymbol{x}=\mathbf{0}$ then $U\boldsymbol{x}=\mathbf{0}$ (not true if $\boldsymbol{b}$ replaces $\mathbf{0}$). U is the diagonal of A when A is *lower triangular.*

Problem Set 2.3, page 63

1 $E_{21}=\begin{bmatrix}1&0&0\\-5&1&0\\0&0&1\end{bmatrix}$, $E_{32}=\begin{bmatrix}1&0&0\\0&1&0\\0&7&1\end{bmatrix}$, $P=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$.

3 $\begin{bmatrix}1&0&0\\-4&1&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}$ $M=E_{32}E_{31}E_{21}=\begin{bmatrix}1&0&0\\-4&1&0\\10&-2&1\end{bmatrix}$.

5 Changing a_{33} from 7 to 11 will change the third pivot from 5 to 9. Changing a_{33} from 7 to 2 will change the pivot from 5 to *no pivot.*

9 $M=\begin{bmatrix}1&0&0\\0&0&1\\-1&1&0\end{bmatrix}$. After the exchange, we need E_{31} (not E_{21}) to act on the new row 3.

10 $E_{13}=\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$; $\begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix}$; $E_{31}E_{13}=\begin{bmatrix}2&0&1\\0&1&0\\1&0&1\end{bmatrix}$. Test on the identity matrix!

12 The first product is $\begin{bmatrix}9&8&7\\6&5&4\\3&2&1\end{bmatrix}$ rows and also columns reversed. The second product is $\begin{bmatrix}1&2&3\\0&1&-2\\0&2&-3\end{bmatrix}$.

14 E_{21} has $-\ell_{21}=\frac{1}{2}$, E_{32} has $-\ell_{32}=\frac{2}{3}$, E_{43} has $-\ell_{43}=\frac{3}{4}$. Otherwise the E's match I.

18 $EF=\begin{bmatrix}1&0&0\\a&1&0\\b&c&1\end{bmatrix}$, $FE=\begin{bmatrix}1&0&0\\a&1&0\\b+ac&c&1\end{bmatrix}$, $E^2=\begin{bmatrix}1&0&0\\2a&1&0\\2b&0&1\end{bmatrix}$, $F^3=\begin{bmatrix}1&0&0\\0&1&0\\0&3c&1\end{bmatrix}$.