KEY

Exam 1

MATH 221-02

February 20, 2015 Chapters 1, 2, and 5

1. (28 pts) Circle **T** or **F** below indicating whether the following statements are True or False. You do not need to justify your answers for this problem.

(#4, 1.3) (a) T F $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(#1a, 1.1) (b) T F Let $\boldsymbol{u} = [1 \;\; 2 \;\; 3]$ and $\boldsymbol{v} = [3 \;\; 6 \;\; 9]$. All linear combinations $a\boldsymbol{u} + b\,\boldsymbol{v}$ (a and b are scalars) form a plane in three dimensions.

(#11, 1.2) (c) T F Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be $n \times 1$ vectors. If $\boldsymbol{u} \cdot \boldsymbol{v} = 1$ then the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is between 0 ~~than~~ and 90 degrees.

(#2, 1.2) (d) T F If $\boldsymbol{u}$ and $\boldsymbol{v}$ are $n \times 1$ vectors, then $\boldsymbol{u} \cdot \boldsymbol{v} \leq ||\boldsymbol{u}||\;||\boldsymbol{v}||$.

(e) T F If $\boldsymbol{u}$ and $\boldsymbol{v}$ are $n \times 1$ vectors, then $||\boldsymbol{u}|| + ||\boldsymbol{v}|| \leq ||\boldsymbol{u} + \boldsymbol{v}||$.

(#1, 1.3) (f) T F Let S be a 3×3 matrix with columns $\boldsymbol{s}_1$, $\boldsymbol{s}_2$, and $\boldsymbol{s}_3$. Then the linear combination $\boldsymbol{s}_1 - \boldsymbol{s}_2 + 5\boldsymbol{s}_3$ is equal to the matrix vector product $S \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$.

(#17, 2.1) (g) T F $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(h) T F You find that an $n \times n$ matrix A has no solutions to $A\boldsymbol{x} = \boldsymbol{b}$ for one RHS $\boldsymbol{b}$. Then $A\boldsymbol{x} = \boldsymbol{c}$ has no solutions for any other RHS $\boldsymbol{c}$.

(i) T F $(A^{-1}B^{-1})^{-1} = AB$.

(#6, 2.4) (j) T F If A and B are $n \times n$ square matrices then $(A - B)^2 = A^2 - 2AB + B^2$.

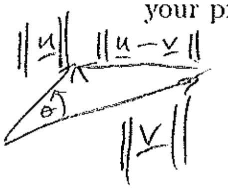
(k) T F If A and B are $n \times n$ square matrices, and $\det(A) \neq 0$ and $\det(B) \neq 0$, then the matrix $C = AB$ is invertible.

(l) T F If $\det(A) = 5$, then $\det(10A) = 50$.

(m) T F If A is an $m \times n$ matrix, then $A^T A$ is symmetric.

(n) T F If A is invertible and symmetric, then A^{-1} is symmetric.

2. (8 pts) Prove that $\boldsymbol{u} \cdot \boldsymbol{v} = ||\boldsymbol{u}||||\boldsymbol{v}|| \cos(\theta)$ where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$. Justify each step in your proof.



By the law of cosines:

$$||\underline{u} - \underline{v}||^2 \leq ||\underline{u}||^2 + ||\underline{v}||^2 - 2||\underline{u}||\,||\underline{v}|| \cos\theta$$

$$(\underline{u} - \underline{v}) \cdot (\underline{u} - \underline{v}) \leq \underline{u} \cdot \underline{u} + \underline{v} \cdot \underline{v} - 2||\underline{u}|| \times ||\underline{v}|| \cos\theta$$

$$\underline{u} \cdot \underline{u} - 2\,\underline{u} \cdot \underline{v} + \underline{v} \cdot \underline{v} \leq \underline{u} \cdot \underline{u} + \underline{v} \cdot \underline{v} - 2||\underline{u}|| \times ||\underline{v}|| \cos\theta$$

$$-2\,\underline{u} \cdot \underline{v} \leq -2\,||\underline{u}||\,||\underline{v}|| \cos\theta$$

$$\underline{u} \cdot \underline{v} \leq ||\underline{u}|| \times ||\underline{v}|| \cos\theta$$

3. (14 pts) In each of the following, circle an appropriate statement from the options *statement1/statement2*. You do not need to justify your answers.

Assume that A is $n \times n$ and that the matrix system $Ax = b$ has one solution.

(a) A is singular/nonsingular.
(b) A is invertible/(not invertible).
(c) $Ax = b$ has/(does not have) non-zero homogeneous (nullspace) solutions.
(d) $\det(A)$ is/(is not) equal to zero.
(e) b is/(is not) a linear combination of the columns of A.
(f) The columns of A are linearly dependent/independent.
(g) When performing Gauss-Elimination to put the augmented matrix into row echelon form (assuming there are no row switches), there are/(are not) n non-zero pivot(s).

4. (6 pts) Fill in the blanks with the number of solutions. You do not need to justify your answers.

(a) The system $Ax = b$ with augmented matrix $\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 10 & 1 & 1 \\ 0 & 10 & 1 & 1 \end{array}\right)$ has $\underline{\infty}$ solution(s).

(b) The system $Ax = b$ with augmented matrix $\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 7 & 0 & 0 \end{array}\right)$ has $\underline{1}$ solution(s).

5. (8 pts) The system $Ax = b$ has augmented matrix $\left(\begin{array}{ccc|c} 1 & 6 & 3 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$.

(a) Find the solution(s) to the system $Ax = b$. Identify any free variables if there are any. SHOW YOUR WORK!

Let $\boxed{z \text{ be free.}}$ Then $\boxed{y = 1 - z}$ and $x = -6(1-z) - 3z$

$\boxed{x = 3z - 6}$

OR Let $\boxed{y}$ be free then $\boxed{z = -y+1}$, $x = -6y - 3(-y+1)$

$\boxed{x = -3y - 3}$

(b) Write the solution(s) as a particular solution plus a homogeneous (or nullspace) solution, carefully identifying each.

$$\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3z - 6 \\ 1 - z \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}.$$

$$= \underline{x}_p + \underline{x}_n$$

(#169, 2.3)

6. Xavier is x years old. Yolande is y years old. Xavier is twice as old as Yolande; and their ages add to 33.

   (a) (4 pts) Write out the linear system $Ax = b$ that must be solved in order to find x and y. Carefully identify A, x and b in your answer.

$x = 2y$ $\qquad\qquad$ $x + y = 33$

$$\therefore \quad \underbrace{\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 0 \\ 33 \end{bmatrix}}_{\underline{b}}$$

   (b) (8 pts) Give the augmented matrix then use Gauss Elimination to put the augmented matrix into row echelon form and solve for x and y. SHOW YOUR WORK!

the augmented matrix is

$$\begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & 33 \end{bmatrix} \xrightarrow{R_2 = -R_1 + R_2} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & 33 \end{bmatrix}$$

$$\therefore \quad 3y = 33 \quad \boxed{y = 11} \implies \boxed{x = 2(11) = 22}$$

   (c) (4 pts) Write the RHS b as a linear combination of the columns of the coefficient matrix A.

$$\begin{bmatrix} 0 \\ 33 \end{bmatrix} = 22 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 11 \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

(#27, 2.5)

7. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) (8 pts) Without actually trying to find the inverse of A, show that it is invertible. SHOW YOUR WORK and explain why your work proves invertibility.

Expanding along the last row

$$\boxed{\det(A) = 1 \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1}$$

Expanding along the 2nd column

$$\boxed{\det(A) = 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1}$$

Since $\det(A) = 1 \neq 0$, then A is invertible.

(b) (12 pts) Find A^{-1}. SHOW YOUR WORK!

The augmented matrix is

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow -2R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow +R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Check it!

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$